Scaling laws for symmetry breaking by blowout bifurcation in chaotic systems

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Recent works have demonstrated that a blowout bifurcation can lead to symmetry breaking in chaotic systems with a simple kind of symmetry. That is, as a system parameter changes, when a chaotic attractor lying in some invariant subspace becomes unstable with respect to perturbations transverse to the invariant subspace, a symmetry-broken attractor can be born. As the parameter varies further, a symmetry-increasing bifurcation can occur, after which the attractor possesses the system symmetry. The purpose of this paper is to present numerical experiments and heuristic arguments for the scaling laws associated with this type of symmetry-breaking and symmetry-increasing bifurcations. Specifically, we investigate (1) the scaling of the average transient time preceding the blowout bifurcation and (2) the scaling of the average switching time after the symmetry-increasing bifurcation. We also study the effect of noise. It is found that small-amplitude noise can restore the symmetry in the attractor after the blowout bifurcation and that the average time for trajectories to switch between the symmetry-broken components of the attractor scales algebraically with the noise amplitude. [S1063-651X(97)11807-4]

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I. INTRODUCTION

In nonlinear dynamics, symmetry-related phenomena have been an interesting subject of study [1,2]. For a dynamical system, if the governing equations are invariant under a symmetric operation to some state variables, we say that the system possesses the corresponding symmetry. Given that the system has a certain symmetry, an interesting question is whether the system symmetry can be seen in the asymptotic state (e.g., attractor) of the system. Depending on the choice of parameters, the system symmetry may or may not exist in the asymptotic attractor. If not, we say that symmetry is broken for the asymptotic attractor. In general, symmetry exists in the attractor for some parameter regimes, say, $p<p_c$. Disappearance of the symmetry occurs when p increases through the critical value p_c. This is referred to as the symmetry-breaking bifurcation. As the parameter changes further, the attractor can gain partial or full symmetry of the system through the so-called symmetry-increasing bifurcation [2,3]. In the past, the question of symmetry and chaos has been extensively studied [1–3]. The phenomenon of increasing symmetry is also believed to be relevant to physical phenomena such as the time-averaged patterns seen in spatio-temporal dynamical systems [4].

As a simple example to illustrate the symmetry-breaking and symmetry-increasing bifurcations, consider the one-dimensional odd-logistic map $x \to ax - x^3$ [2]. This map is invariant under the symmetric operation $x \to -x$ (reflecting symmetry). When $a<1$, the fixed point $x=0$ is the stable attractor. This attractor possesses the reflecting symmetry trivially. For $a>1$, depending on the initial condition, the attractor lies either in $x>0$ (if $x_0>0$) or in $x<0$ (if $x_0<0$). Thus a symmetry-breaking bifurcation occurs at $a_c=1$. The attractor recovers the system symmetry when $p>p_s=3\sqrt{3}/2$, at which the $x>0$ attractor merges with its symmetric component, the $x<0$ attractor. Hence p_s is the symmetry-increasing bifurcation point [2].

In this paper, we consider chaotic systems with a symmetric low-dimensional invariant subspace. Denote the invariant subspace by **S**. Since **S** is invariant, initial conditions in **S** result in trajectories which remain in **S** forever. We restrict our investigation to the situation where there is a chaotic attractor in **S**. In this case, whether the chaotic attractor attracts or repels initial conditions in the vicinity of **S** is determined by the sign of the largest Lyapunov exponent $\Lambda_\perp$ computed for trajectories in **S** with respect to perturbations in the subspace **T** that is *transverse* to **S**. When $\Lambda_\perp$ is negative, **S** attracts trajectories transversely in the phase space and the chaotic attractor in **S** is also an attractor of the whole phase space. When $\Lambda_\perp$ is positive, trajectories in the vicinity of **S** are repelled away from it, and, consequently, the chaotic attractor is transversely unstable and it is hence not an attractor of the whole phase space. The bifurcation from the former to the latter behaviors has been investigated [5–7], and it is called the ''blowout'' bifurcation [5]. It is also known [5–7] that when $\Lambda_\perp$ is slightly positive, dynamical variables in the transverse subspace **T** can exhibit an extreme type of temporarily intermittent bursting behavior: on-off intermittency [8–10].

Recent works have indicated that a blowout bifurcation can lead to symmetry breaking [11–14]. The purpose of this paper is to study the scaling laws of this type of symmetry-breaking bifurcation and the influence of noise. Specifically, we find that preceding the blowout bifurcation, there is a transient on-off intermittent behavior. The average transient time scales algebraically with the system parameter and diverges at the blowout bifurcation point. As the system parameter changes further, a symmetry-increasing bifurcation can occur, after which an orbit switches between the originally symmetry-broken attractors. We find that the average switching time also scales algebraically with parameter and

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diverges at the symmetry-increasing bifurcation point. As a practical issue, the effect of small-amplitude random noise on symmetry-broken attractors is investigated. It is found that noise can restore the system symmetry in the attractor through the mechanism of intermittent switching. The average switching time, or the average transient time for trajectories to stay in the symmetry-broken attractor, scales algebraically with the noise amplitude. Our approach to address these scaling laws is to perform numerical experiments and then to provide heuristic arguments for the scaling laws with a two-dimensional noninvertible map, but similar results are also observed for continuous flows and higher-dimensional coupled map lattices.

The paper is organized as follows. In Sec. II, we present an illustrative example, a two-dimensional map, for which partial analytic understanding of the symmetry-breaking and symmetry-increasing bifurcations can be obtained, and we present scaling results. Section III is devoted to the study of effect of noise and the corresponding scaling law. Discussions are presented in Sec. IV.

II. SCALING RESULTS

We study the following general class of N-dimensional dynamical systems:

$$\mathbf{x}_{n+1}=\mathbf{f}(\mathbf{x}_n),$$

$$\mathbf{y}_{n+1}=F(\mathbf{x}_n,p)\mathbf{G}(\mathbf{y}_n), \tag{1}$$

where $\mathbf{x}\in\mathbf{S}(R^{N_S})$, $\mathbf{y}\in\mathbf{T}\ (R^{N_T})$, $N_S\geqslant 1$, $N_T\geqslant 1$, $N_S+N_T=N$, and p is the bifurcation parameter. The function $\mathbf{G}(\mathbf{y})$ possesses certain symmetry, e.g., $\mathbf{G}(-\mathbf{y})=-\mathbf{G}(\mathbf{y})$, so that $\mathbf{G}(\mathbf{0})=\mathbf{0}$. The symmetric invariant subspace is then defined by $\mathbf{y}=\mathbf{0}$. We assume that both the $\mathbf{x}$ and $\mathbf{y}$ dynamics are bounded. The vector function $\mathbf{f}(\mathbf{x})$ is a map that has a chaotic attractor. Trajectories restricted to the invariant subspace asymptote to this attractor because trajectories starting from $\mathbf{y}_0=\mathbf{0}$ have $\mathbf{y}_n=\mathbf{0}$ for all subsequent iterations. The scalar function $F(\mathbf{x},p)$ can be regarded as a ''driving'' from the $\mathbf{x}$ dynamics in the invariant subspace to the symmetric $\mathbf{y}$ subspace which is transverse to the invariant subspace. The largest transverse Lyapunov exponent $\Lambda_\perp$ is given by

$$\Lambda_\perp=\lim_{L\to\infty}\frac{1}{L}\sum_{n=1}^{L}\ln\left|\left.\frac{\partial\mathbf{y}_{n+1}}{\partial\mathbf{y}_n}\right|_{\mathbf{y}_n=0}\cdot\mathbf{u}\right| =\lim_{L\to\infty}\frac{1}{L}\sum_{n=1}^{L}\ln|F(\mathbf{x}_n,p)\mathbf{DG}(\mathbf{0})\cdot\mathbf{u}|, \tag{2}$$

where

$$\mathbf{DG}(\mathbf{0})\equiv\left.\frac{\partial\mathbf{G}(\mathbf{y}_n)}{\partial\mathbf{y}_n}\right|_{\mathbf{y}_n=0}$$

is the Jacobian matrix of the function $\mathbf{G}(\mathbf{y}_n)$ evaluated at $\mathbf{y}_n=\mathbf{0}$. As p changes through the blowout bifurcation point p_c, $\Lambda_\perp$ crosses zero from the negative side. Note that the largest Lyapunov exponent of the $\mathbf{y}$ subsystem is given by

$$\Lambda_y=\lim_{L\to\infty}\frac{1}{L}\sum_{n=1}^{L}\ln|F(\mathbf{x}_n,p)\mathbf{DG}(\mathbf{y}_n)\cdot\mathbf{u}|, \tag{3}$$

where now $\partial\mathbf{G}(\mathbf{y}_n)/\partial\mathbf{y}_n$ is the Jacobian matrix evaluated along a typical trajectory $(\mathbf{x}_n,\mathbf{y}_n)$. Thus, for $p<p_c$, we have $\Lambda_y=\Lambda_\perp$ because $\mathbf{y}_n\to\mathbf{0}$ asymptotically. But $\Lambda_y\neq\Lambda_\perp$ if $p>p_c$.

As our representative example, we consider the following version of Eq. (1):

$$x_{n+1}=rx_n(1-x_n),$$

$$y_{n+1}=\frac{1}{2\pi}px_n\sin(2\pi y_n), \tag{4}$$

where both the invariant subspace x and the symmetric subspace y are one dimensional, x is restricted to the unit interval [0,1], r is the parameter in the logistic map, and p is the bifurcation parameter. We restrict our study to the case where $p>0$. The y equation is invariant under the reflecting symmetric operation $y\to -y$. Since x is bounded, y is also bounded. We choose r so that the logistic map generates a chaotic attractor in the x subspace. The transverse Lyapunov exponent is

$$\Lambda_\perp=\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\ln(ax_j)=\int_0^1\ln|px|\rho(x)dx, \tag{5}$$

where $\rho(x)$ is the natural invariant density of x for the logistic map. Thus we have

$$p_c=\exp\left[-\int_0^1\ln|x|\rho(x)dx\right], \tag{6}$$

where $\Lambda_\perp\geqslant 0$ for $p\geqslant p_c$ and $\Lambda_\perp<0$ for $p<p_c$.

To illustrate our results, we use $r=3.8$ for numerical experiments. We find $p_c\approx 1.725\,666$. For $p<p_c$, the asymptotic attractor is $y=0$, which is trivially invariant under $y\to -y$. For $p\gtrsim p_c$, numerical computation reveals that the resultant attractor no longer possesses the mirror symmetry about $y=0$: For an initial condition with $y_0>0$ (<0), the resulting trajectory has $y_n>0$ (<0) for subsequent iterations [14]. Thus, for the asymptotic attractor in the phase space, the reflecting symmetry in the y equation is broken immediately after the transverse Lyapunov exponent $\Lambda_\perp$ becomes positive. The y variable after the symmetry-breaking bifurcation exhibits on-off intermittent behavior [14].

Although symmetry breaking and on-off intermittency have been presented briefly in Ref. [14], the analyses below regarding the parameter range in which the symmetry breaking occurs and the characteristics of the Lyapunov exponents have not been published in details. To assess this range, we note that in Eq. (4), if y_n exceeds 0.5, y_{n+1} immediately becomes negative, indicating that the trajectory on the positive-y chaotic component can be reinjected into the basin of the coexisting negative-y chaotic component. Since the positive-y and negative-y chaotic components are completely symmetric with respect to each other, in this case the system symmetry is recovered for the attractor. Let p_s be the symmetry-increasing bifurcation point, which we can compute by requiring that the maximum value of $|y_n|$ be 0.5. We obtain

$$\frac{1}{2\pi} p_s x_{\max} = 0.5, \tag{7}$$

which gives $p_s = \pi/x_{\max}$. For the logistic map in x, we have $x_{\max} = r/4$. For $r = 3.8$, we obtain $p_s = 4\pi/3.8 \approx 3.306\,939\,6$. Symmetry breaking occurs for $p_c < p < p_s$. Numerically, we observe that the y Lyapunov exponent Λ_y becomes positive at $p = p_y \approx 3.245$. Thus the asymptotic attractor of the system possesses one positive Lyapunov exponent (the x Lyapunov exponent Λ_x) for $p \leq p_y$. The attractor becomes hyperchaotic with two positive Lyapunov exponents for $p > p_y$. An interesting observation is that Λ_y remains negative for $p_c < p < p_s$ except when p is very close to p_s. The reason why Λ_y is negative immediately after the symmetry-breaking bifurcation can be understood by noting that

$$\Lambda_y \approx \Lambda_\perp + \int \ln|\cos(2\pi y_n)|\rho(y)dy, \tag{8}$$

where $\rho(y)$ is the probability distribution of y after the symmetry-breaking bifurcation. Since $\ln|\cos(2\pi y_n)| \leq 0$ and $\int\rho(y)dy = 1$, the integral in Eq. (8) is negative so that $\Lambda_y < \Lambda_\perp$. Thus, if $\Lambda_\perp \gtrsim 0$, Λ_y can remain negative for $p \gtrsim p_c$. We emphasize that Eq. (8) only holds for $p \gtrsim p_c$ because, in this case, the transverse Lyapunov exponent $\Lambda_\perp$ is only slightly positive and the y dynamics exhibits on-off intermittency. For $p \gtrsim p_c$, it is known that the on-off intermittent behavior in y exhibits universal scaling behaviors regardless of the details of the x dynamics in the invariant subspace [10]. This implies that the natural invariant density in y is roughly independent of $\rho(x)$ for $p \gtrsim p_c$, and so we have, in this case, $\rho(x,y) \approx \rho(x)\rho(y)$.

We now discuss the scaling laws. Immediately before the blowout bifurcation, the asymptotic value of $\Lambda_\perp$ is slightly negative. The histogram of values of $\Lambda_\perp$ computed at finite times has a spread into the positive side. A typical trajectory can experience stretches of time being actually repelled away from the invariant subspace, although the trajectory eventually asymptotes to it. Thus the trajectory exhibits transient on-off intermittent behavior before finally approaching the invariant subspace. For a given parameter value $p \lesssim p_c$, the average transient lifetime $\tau_b(p)$ depends on the parameter difference $(p_c - p)$: $\tau_b(p)$ decreases as p decreases from p_c [note that $\tau_b(p_c) = \infty$]. We ask: What is the scaling relation between $\tau_b(p)$ and $(p_c - p)$? Figure 1 shows, for the model system, Eq. (4), at $r = 3.8$, the plot of $\log_{10} \tau_b(p)$ versus $\log_{10}(p_c - p)$ for $-2.5 < \log_{10}(p_c - p) < 0$. In Fig. 1, for each value of p, 1000 trajectories are used to compute the average transient lifetime $\tau_b(p)$. A trajectory is regarded as having reached $y = 0$ if it stays within 10^{-100} of $y = 0$ for 10 000 iterations. The data can be fitted by a straight line with a slope approximately -1. Thus we have evidence of the following algebraic scaling relation between $\tau_b(p)$ and $(p_c - p)$:

$$\tau_b(p) \sim (p_c - p)^{-1}. \tag{9}$$

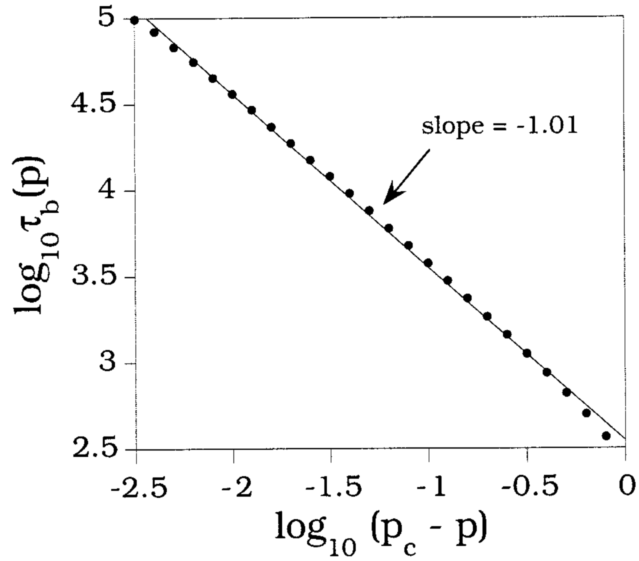

FIG. 1. For Eq. (4) at the parameter setting $r = 3.8$ and $p \lesssim p_c$ (before the symmetry-breaking bifurcation), the average lifetime of the transient on-off intermittent behavior $\tau_b(p)$ versus $(p_c - p)$ on a logarithmic scale. We see that $\tau_b(p) \sim (p_c - p)^{-1}$. Because the behavior of the transverse Lyapunov exponent in the vicinity of the blowout bifurcation is typically linear, we expect this algebraic scaling relation to be general.

This scaling can be understood by noting that $\tau_b(p)\Lambda_\perp(p) \approx \ln(1/\epsilon_0)$, where ϵ_0 is a small initial distance of the trajectory from $y = 0$, and so we have $\tau_b(p) \sim 1/\Lambda_\perp$. For p near p_c, we have $\Lambda_\perp(p) \sim (p - p_c)$. The scaling law (9) thus follows. Since the behavior of $\Lambda_\perp$ near the blowout bifurcation point is typically linear [5,7], we expect the scaling relation, Eq. (9), to be general.

After the symmetry-increasing bifurcation, the asymptotic attractor recovers the system symmetry because trajectories now visit the symmetric components both above and below the invariant subspace. This occurs when trajectories in one attracting component are reinjected into its symmetric component so that the basins of the symmetric components are connected. Immediately after the symmetry-increasing bifurcation, a typical trajectory usually stays in one component for long time before entering the channels through which the basins are connected and being reinjected into the other component. The trajectory can then stay there for some time before coming back to the first component. Thus we expect an intermittent switching behavior to occur [15]. In this case, y_n occurs on both sides of $y = 0$. Whenever $|y_n|$ exceeds $\bar{y} = 0.5$, it jumps from one side of $y = 0$ to the other. Let $\tau_s(p)$ be the average time that a typical trajectory stays in one component. As p increases from p_s, we expect $\tau_s(p)$ to decrease because reinjection of trajectories occurs more often when p is larger. We ask: What is the scaling relation between $\tau_s(p)$ and $(p - p_s)$? Figure 2 shows $\tau_s(p)$ versus $(p - p_s)$ on a logarithmic scale for $10^{-6} < (p - p_s) < 10^{-1}$, where, for each p, 10 000 events of switching are accumulated to compute the average switching time for a trajectory resulting from a random initial condition. The data can be well fitted by a straight line with a slope of about -1. Thus we have the algebraic scaling relation

$$\tau_s(p) \sim (p - p_s)^{-\gamma}, \tag{10}$$

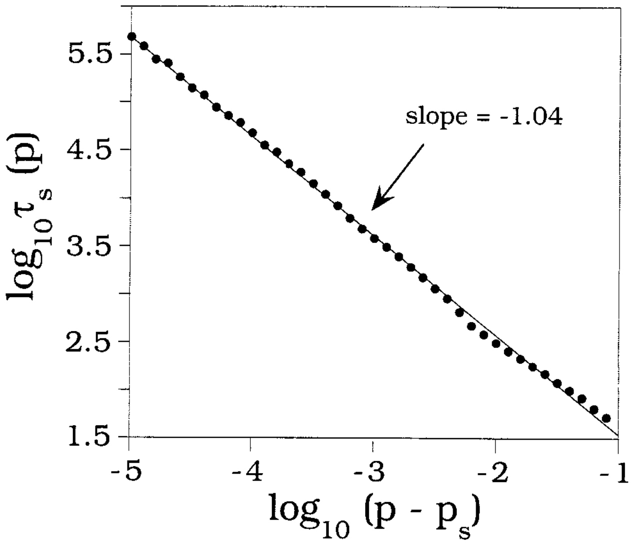

FIG. 2. For Eq. (4) at the parameter setting $r=3.8$ and $p \gtrsim p_s$ (after the symmetry-increasing bifurcation), the average intermittent switching time $\tau_s(p)$ versus $(p-p_s)$ on a logarithmic scale. We have $\tau_s(p) \sim (p-p_s)^{-\gamma}$, where $\gamma=1$ is the scaling exponent. In general, we expect this algebraic scaling relation to hold but the scaling exponent depends on the details of the system.

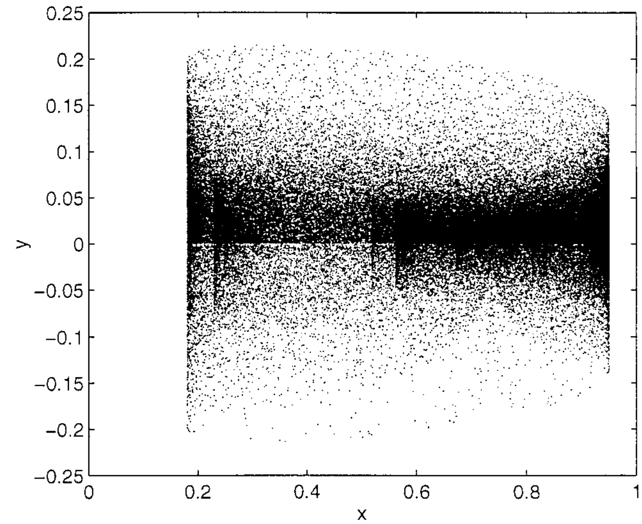

FIG. 3. For $p=1.75$, a trajectory of 10^5 points resulting from a random initial condition in the upper half plane $y>0$ under the influence of noise $10^{-5}\sigma_n$, where σ_n is a Gaussian random variable of zero mean and unit variance. The trajectory spends 79 068 iterations in $y>0$ and is then kicked into the lower half plane $y<0$ by noise. Due to noise, the final attractor possesses the system symmetry.

where the scaling exponent γ has the value of -1 for the model system, Eq. (4). The scaling can be derived by considering the probability $P(p|y_n \geqslant \overline{y})$ that a trajectory has $y_n \geqslant 0.5$ at p. For $p \leqslant p_s$, we have $P(p|y_n \geqslant \overline{y})=0$ because trajectories are confined within $\overline{y}$. Let $y_{\max}(p)$ be the maximum value of y_n. For $p \gtrsim p_s$, we have

$$\begin{aligned} y_{\max}(p) &= p x_{\max}/(2\pi) = (p_s + \Delta p) x_{\max}/(2\pi) \\ &= \overline{y} + \Delta p x_{\max}/(2\pi), \end{aligned}$$

where $\Delta p \equiv p - p_s$. We see that $y_{\max}(p) - \overline{y} \sim \Delta p$. Let $\rho(y,p)$ be the natural invariant density of y: we obtain

$$P(p|y_n \geqslant \overline{y}) = \int_{\overline{y}}^{y_{\max}(p)} \rho(y,p)\,dy \sim [y_{\max}(p) - \overline{y}] \sim \Delta p. \tag{11}$$

Since $\tau_s(p) \sim 1/P(p|y_n \geqslant \overline{y})$, Eq. (11) immediately gives the scaling relation, Eq. (10), with the scaling exponent -1. From the above derivation, we see that the scaling exponent is determined by the explicit form of the y dynamics. In particular, it is determined by the relation of $y_{\max}(p)$ as a function of p. Thus the scaling exponent -1 is specific to our model system, Eq. (4). Nevertheless, we expect the scaling relation, Eq. (10), to be more general, with the scaling exponent determined by the behavior of $y_{\max}(p)$ in the vicinity of $\overline{y}$.

III. EFFECT OF NOISE

When noise is present, a trajectory can no longer stay in the symmetry-broken attractor forever. After the blowout bifurcation, although the transverse Lyapunov exponent is positive, there is a set of measure zero points embedded in the chaotic attractor in the invariant subspace, the elements of which are transversely attractive. Thus, when the trajectory comes close to the invariant subspace, noise can ''kick'' the trajectory through the invariant subspace. The trajectory can then stay in the other symmetric component for a finite amount of time and is then ''kicked'' back by noise. As such, an intermittent behavior occurs in which the trajectory switches between the symmetric components, thereby restoring the system symmetry in the resulting attractor. Figure 3 shows a trajectory of 10^5 points on the attractor for $p=1.75$ in Eq. (4) when an additive noise term $\epsilon\sigma_n$ is added to the y equation in Eq. (4), where $\epsilon=10^{-5}$ and σ_n is a random number obeying the Gaussian probability distribution with zero mean and unit variance. Adding noise terms to both the x and y equations in Eq. (4) or changing the noise distribution to, say, uniform distribution does not affect the result. For the initial condition that yields the attractor in Fig. 3, the trajectory first spends 79 068 iterations in the $y>0$ component and then switches to the $y<0$ component and, hence, the attractor now possesses the system symmetry.

The average switching time $\tau(\epsilon)$ between the symmetric components depends on the noise amplitude ϵ. As ϵ increases, we expect $\tau(\epsilon)$ to decrease because it becomes ''easier'' for a trajectory to be kicked through the invariant subspace by larger noise. We find that $\tau(\epsilon)$ scales with ϵ algebraically. Figure 4(a) shows $\log_{10}\tau(\epsilon)$ versus $\log_{10}\epsilon$ for $10^{-6} \leqslant \epsilon \leqslant 10^{-2}$, where the parameter setting is the same as in Fig. 3. To obtain the figure, we randomly choose 1000 initial conditions with $0<x_0<1$ and $0<y_0<0.5$. The staying time of each trajectory in the $y>0$ attractor is then computed. The average switching time is taken to be the average staying time of these 1000 trajectories in $y>0$. The plot can be fit with a straight line, indicating the scaling law

$$\tau(\epsilon) \sim \epsilon^{-\eta}, \tag{12}$$

where $\eta>0$ is the algebraic scaling exponent. In Fig. 4(a), the scaling exponent is $\eta \approx 1.13$. Since the additive noise is

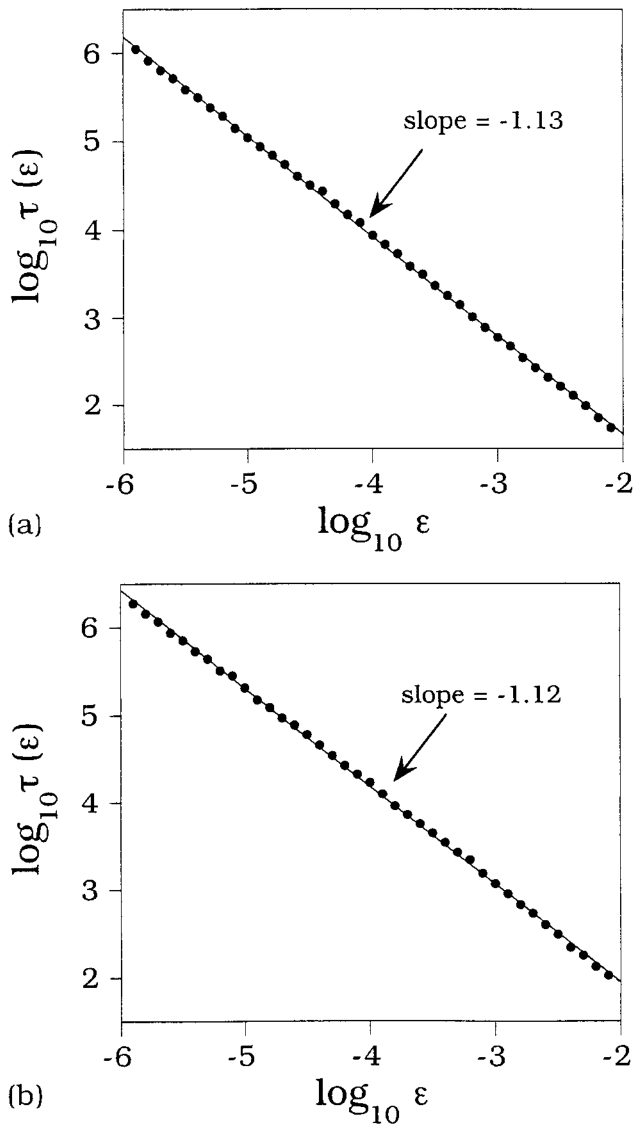

FIG. 4. The average noise-induced switching time $\tau(\epsilon)$ versus the noise amplitude ϵ on a logarithmic scale for $p=1.75$. (a) Gaussian noise and (b) uniform noise $\epsilon\gamma_n$, where γ_n is a random number uniformly distributed in $[-1,1]$. The algebraic scaling law, Eq. (12), holds regardless of the details of the noise distribution.

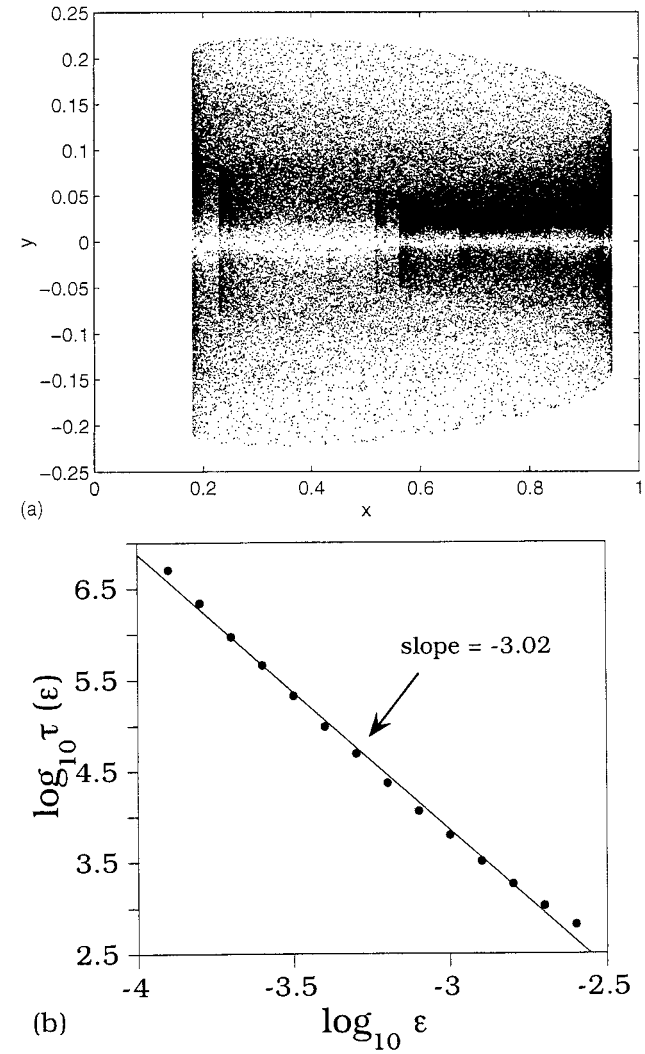

FIG. 5. For $p=1.78$: (a) A trajectory of 10^5 points resulting from a random initial condition in the upper half plane $y>0$ under the influence of Gaussian noise of amplitude $10^{-3.5}$. The trajectory spends 74 190 iterations in $y>0$ and is then kicked into the lower half plane $y<0$ by noise. (b) The average switching time $\tau(\epsilon)$ versus the noise amplitude ϵ on a logarithmic scale.

Gaussian in the computation of Fig. 4(a), it is unclear whether the switching is due to the measure-zero set of transversely attractive points in the invariant subspace or due to some particularly large kick produced by the Gaussian distribution. To resolve this issue, we have redone Fig. 4(a) using noise term $\epsilon\gamma_n$, where γ_n is a random number uniformly distributed in $[-1,1]$. The result is shown in Fig. 4(b), where the algebraic scaling law, Eq. (12), still holds with an almost identical scaling exponent. Thus it is safe to conclude that the algebraic scaling law, Eq. (12), is due to the dynamics of the system, regardless of the form of the noise. This scaling can be understood by noting that for noise amplitude ϵ, when a trajectory falls within distance ϵ of $y=0$, it can be kicked through $y=0$. Thus the average switching time due to noise is roughly the average transient time before a typical trajectory falls within ϵ of $y=0$. Previous work has argued that this transient time scales algebraically with ϵ [16].

As the parameter p increases further from the blowout bifurcation point so that the transverse Lyapunov exponent becomes larger, it becomes more difficult for the trajectory to come close to the invariant subspace $y=0$, thereby weakening the influence of noise on the symmetry-broken attractor. Figure 5(a) shows a trajectory of 10^5 points originated from a random initial condition in the upper half plane $y>0$ for $p=1.78$ ($\Lambda_\perp\approx 0.031$) with additive Gaussian noise of amplitude $\epsilon=10^{-3.5}$. The trajectory stays in $y>0$ for 74 190 iterations. It is then kicked by noise into the lower half plane $y<0$ and stays there for the remaining 25 810 iterations of the 10^5 iterations shown. There appears to be a gap region near $y=0$ in which it is difficult for trajectories to fall. This leads to a longer switching time at the same noise level. Comparing with Fig. 3 where $p=1.75$, we see that in order to have a switching time about 70 000 iterations, the noise amplitude needs to increase by a factor of $10^{1.5}$ when $p=1.78$. Figure 5(b) shows $\log_{10}\tau(\epsilon)$ versus $\log_{10}\epsilon$ for $p=1.78$, where 1000 random trajectories are used to compute the average noise-induced switching time $\tau(\epsilon)$. It can be seen that the algebraic scaling law, Eq. (12), still holds approximately, but at the same noise level, $\tau(\epsilon)$ is roughly two orders of magnitude larger than that of Fig. 4(a). Note that the range of ϵ in Fig. 5(b) is rather small. The reason is that the average switching time has already reached approximately 3×10^6 iterations when $\epsilon\gtrsim10^{-4}$. For a smaller noise amplitude, the switching time for certain initial conditions becomes prohibitively long for the *average* switching time to be computed. Despite this numerical difficulty, the algebraic scaling law, Eq. (12), appears to hold for Fig. 5(b).

IV. DISCUSSIONS

In conclusion, we have presented the scaling laws for symmetry-breaking bifurcation in chaotic dynamical systems with an invariant subspace. Although we illustrate our results mainly by using the model of Eq. (4), the heuristic arguments for the scaling laws do not depend on specific feature of the model. Similar bifurcations have been observed for a large variety of chaotic dynamics in the invariant subspace, for flows, and for coupled map lattices.

In general, chaotic attractors with broken symmetry as a result of the blowout bifurcation can have distinct dynamical characteristics. Immediately after the bifurcation, attractors appear to be ''stuck on'' to the invariant subspace. In this case, a typical trajectory on the attractor can get arbitrarily close to the invariant subspace. As the parameter varies further away from the bifurcation, it is possible for the symmetry-broken attractor to be ''lifted off'' from the invariant subspace. These ''stuck-on'' and ''lifted-off'' attractors were first discovered by Ashwin [11]. Specifically, in Ref. [11], Ashwin studied the two-dimensional map

$$x_{n+1}=g(x_n)+\epsilon x_n y_n,$$

$$y_{n+1}=\lambda y_n e^{-x_n^2-y_n^2}+\tfrac{1}{2}y_n(1-e^{-y_n^2}), \qquad (13)$$

where $g(x_n)$ is a chaotic map, and ϵ and λ are parameters. The invariant subspace is given by $y=0$ in which there is a chaotic attractor. Ashwin studied the case where $g(x)$ is the cubic map $g(x)=\frac{3}{2}\sqrt{3}x(x^2-1)$. This map has a unique chaotic attractor $A_0=[-1,1]$ with basin $x\in(-1.1768,1.1768)$ [17]. For $\epsilon=0$, a blowout bifurcation occurs at $\lambda=1.430$ after which the chaotic attractor A_0 becomes a chaotic saddle in the two-dimensional phase space [6,11,13]. At the blowout bifurcation, a symmetry-broken attractor A is born which lies in the two-dimensional phase space (x,y). Ashwin proved that for $1.430<\lambda<1.850$, the symmetry-broken attractor A must contain A_0 lying in the invariant subspace. Thus, in this parameter range, a typical trajectory on the attractor can come arbitrarily close to the invariant subspace, and consequently, the attractor A must be ''stuck on'' to the invariant subspace. Ashwin also showed that for $\lambda>1.850$, the symmetry-broken attractor A no longer contains A_0 and the attractor A thus ''lifts off'' the invariant subspace. Similar behaviors were also observed in other examples [11].

The phenomenon of symmetry breaking by blowout bifurcation described in this paper is potentially relevant to applications such as synchronization in chaotic systems [18]. The ability for chaotic systems to synchronize with each other provides a possible approach to transmit information via a chaotic carrier [18,19]. It is known that when an appropriately chosen state variable of a chaotic system is used to drive a subsystem, the subsystem synchronizes with its replica if its Lyapunov exponents are negative [18]. For the class of symmetric dynamical system studied in this paper [Eq. (1)], the dynamical variables $\mathbf{x}$ in the invariant subspace can be regarded as the driving to the dynamics in the transverse subspace. Thus synchronism occurs when the largest Lyapunov exponent of the subsystem $\mathbf{y}$ is negative. In this case, the subsystem $\mathbf{y}$ synchronizes with its replica $\mathbf{y}'$ in the sense that $|\mathbf{y}-\mathbf{y}'|\to0$ as $t\to\infty$ if both $\mathbf{y}$ and $\mathbf{y}'$ are driven by the same $\mathbf{x}$. This can indeed be achieved because, as we have seen in our example of Eq. (4), after the symmetry-breaking bifurcation, although the largest transverse Lyapunov exponent $\Lambda_\perp$ becomes positive, the Lyapunov exponents of the $\mathbf{y}$ subsystem can still be negative. In this case, since $\Lambda_\perp>0$, the chaotic attractor in the invariant subspace $\mathbf{y}=\mathbf{0}$ is a repeller in the $\mathbf{y}$ subspace and, hence, the $\mathbf{y}$ dynamics is locally chaotic near $\mathbf{y}=\mathbf{0}$. The $\mathbf{y}$ variables of a typical trajectory would therefore exhibit complicated and nontrivial behavior. But since $\Lambda_{\mathbf{y}}<0$, the $\mathbf{y}$ variables of trajectories starting from different initial conditions will be synchronized asymptotically. In the numerical example presented in this paper, there exist wide parameter regimes ($p>p_c$) for which $\Lambda_y<0$ can be realized. High-dimensional chaotic synchronism is particularly appealing in communication applications for considerations of security [20]. The scenario of symmetry breaking by blowout bifurcation studied in this paper may provide a way to design high-dimensional synchronous chaotic systems [21].

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