Controlling on-off intermittent dynamics

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On-off intermittent chaotic behavior occurs in physical systems with symmetry. The phenomenon refers to the situation where one or more physical variables exhibit two distinct states in their time evolution. One is the ''off'' state where the physical variables remain constant, and the other is the ''on'' state where the variables temporarily burst out of the ''off'' state. We demonstrate that by using arbitrarily small feedback control to an accessible parameter or state of the system, the ''on'' state can be eliminated completely. This could be practically advantageous where the desirable operational state of the system is the ''off'' state. Relevant issues such as the influence of noise and the time required to achieve the control are addressed. It is found that the average transient time preceding the control obeys a scaling law that is *qualitatively different* from the algebraic scaling law which occurs when one controls chaos by stabilizing unstable periodic orbits embedded in a chaotic attractor. A theoretical argument is provided for the observed scaling law. [S1063-651X(96)05208-7]

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I. INTRODUCTION

Recently, the phenomenon of on-off intermittency has been discovered in nonlinear dynamical systems [1,2]. In such a case, one or more dynamical variables of the system exhibit two distinct states as the system evolves in time. One state is the ''off'' state where the dynamical variables remain approximately constant in various time intervals. The length of these time intervals can be either short or long. There can also be occasional bursts of the dynamical variables away from their constant values in the ''off'' state. These bursts are referred to as the ''on'' state which occurs intermittently as time progresses. Mechanisms for generating the behavior of on-off intermittency have been investigated [1–5]. A condition for on-off intermittency to occur is that the system should be driven either randomly or chaotically [1–3].

To give a concrete example of on-off intermittency, we consider the following two-dimensional map in (x,y):

$$x_{n+1}=f(x_n),$$
$$y_{n+1}=ax_n(y_n-b)(1+b-y_n)+b, \tag{1}$$

where $f(x)$ is a random process or a deterministic chaotic process, and a and b are parameters. The map Eq. (1) reduces to the one studied by Heagy, Platt, and Hammel [3] when $b=0$. If x_n is uniformly distributed in the unit interval [0,1], the y dynamics exhibits on-off intermittent behavior when $a>a_c=e=2.718\,28\ldots$ [3], where a_c denotes the critical parameter value for the onset of on-off intermittency. Random variables with uniform probability distribution in the unit interval can be generated directly by a random number generator, or they can be generated by the tent map or the $2x$ mod(1) map. Figure 1 shows a time series y_n originated from an arbitrary initial condition $y_0\in(b,1+b)$ for $a=2.8>a_c$ and $b=0.5$, where $f(x)$ is the $2x$ mod(1) map. Clearly, for most of the time y_n stays near $y=b$ (the ''off'' state) [6]. But there are occasional bursts of y_n from $y=b$ (the ''on'' state). Heagy, Platt, and Hammel showed that time intervals where the trajectory stays near the ''off'' state T, or the length of the laminar phase, defined as the time between two successive bursts, obeys certain probability distribution $P(T)$: For $a>a_c$, $P(T)\sim T^{-\gamma}$, where $\gamma>0$ is a scaling exponent. One interesting result is that for parameter values slightly above the transition point, i.e., $a\gtrsim a_c$, the scaling exponent assumes a universal value of $\gamma=\frac{3}{2}$ [3].

On-off intermittency can also occur in more realistic physical models. Ott and Sommerer considered a mechanical system with symmetry where particles move in a potential field and are subject to forcing and friction [4]. There is an invariant subspace in the system due to symmetry. In the invariant subspace there is a chaotic attractor. Depending on whether the chaotic attractor attracts initial conditions in the vicinity of the invariant subspace, distinct dynamical behaviors can occur. In particular, if almost all initial conditions

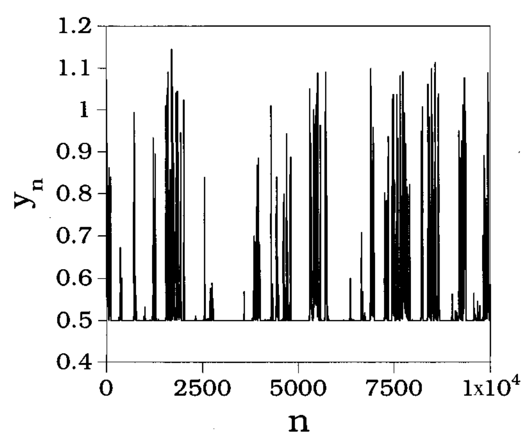

FIG. 1. An on-off intermittent time series generated by Eq. (1) from an arbitrarily initial condition. The parameters are $a=2.8$ and $b=0.5$.

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are repelled away from the invariant subspace (in which case we say the invariant subspace is transversely unstable), and if there are no other attractors in the phase space, dynamical variables not restricted to the invariant subspace can exhibit on-off intermittency. Depending on the number of invariant subspaces, similar mechanical systems can exhibit multiple-state on-off intermittency where there are more than one ''off'' states [5]. It was also demonstrated [4,5] that in systems with symmetry, on-off intermittency is in fact closely related to the phenomena of riddled basins and intermingled basins [7] which occur when the chaotic attractor in the invariant subspace attracts nearby initial conditions on average. Notice that, in terms of symmetry, Eq. (1) has a one-dimensional invariant subspace $y=b$, since a trajectory with $y_0=b$ will have $y_n=b$ for all subsequent iterations. Since dynamical systems with symmetry are fairly common, we expect on-off intermittency to occur commonly, too.

In this paper we investigate controlling chaotic dynamical systems that exhibit on-off intermittency. Specifically, we assume that the desirable operational state of the system is the ''off'' state and the ''on'' state is undesirable. Thus we wish to avoid temporal bursts of dynamical variables from the ''off'' state and wish to keep these variables in the vicinity of the ''off'' state. We are interested in using only arbitrarily small perturbations to the system, as we do not wish to change the system appreciably. We thus address the following question: Given a dynamical system that exhibits on-off intermittency, can one apply small feedback control to the system so as to force the system to operate in only the desirable ''off'' state?

Control of chaos by using only small perturbations to the system was proposed by Ott, Grebogi, and Yorke (OGY) in 1990 [8]. The idea is to stabilize unstable periodic orbits that occur naturally due to chaotic dynamics of the system. In this regard, one chooses an unstable periodic orbit embedded in the chaotic attractor, the one which yields the best system performance according to some criterion. One then defines a small region around the desirable periodic orbit. For a chaotic attractor, a trajectory originated from a random initial condition will come arbitrarily close to the target unstable periodic orbit at some later time. When this occurs, small judiciously chosen temporal parameter perturbations are applied to force the trajectory to stay in the vicinity of the unstable periodic orbit, because, without control, the trajectory will subsequently leave the periodic orbit. This method is extremely flexible because it allows for the stabilization of different periodic orbits, depending on one's needs, for the same set of nominal values of the parameter. This idea has since stimulated further theoretical investigation [9] and has been successfully applied to various physical [10], chemical [11], and biological [12] systems.

Our method to confine a trajectory in the ''off'' state is based on OGY's idea of controlling chaos. The strategy is to wait for a trajectory to come sufficiently close to the ''off'' state and then to apply external perturbation to an accessible parameter or state of the system. The magnitude of the perturbation is proportional to the distance of the trajectory to the ''off'' state and can thus be made arbitrarily small. We will discuss algorithms for both discrete maps and flows. A relevant issue is the average transient time that a typical trajectory wanders before falling into a small neighborhood of the ''off'' state (the controlling neighborhood) and being controlled. We call this time the ''waiting time.'' The smaller the size of the controlling neighborhood, the longer the waiting time will be. We find that the waiting time obeys a scaling law that is *qualitatively different* from that which occurs in situations where one applies the OGY method to stabilize unstable periodic orbits embedded in a chaotic attractor. Due to the dynamical properties of on-off intermittency, the required waiting time is actually much less than that required in the case of stabilizing unstable periodic orbits.

This paper is organized as follows. In Sec. II we review the mechanism for generating on-off intermittency in chaotic dynamical systems and describe the control method for both maps and flows. In Sec. III we test the control algorithm for cases without and with external noise. In Sec. IV we give an argument for the scaling of the waiting time. We also provide numerical data from both maps and flows that support the scaling. In Sec. V we present discussions.

II. MECHANISM FOR ON-OFF INTERMITTENCY AND METHOD OF CONTROL

We first consider chaotic systems described by discrete maps,

$$\mathbf{z}_{n+1}=\mathbf{F}(\mathbf{z}_n;p), \tag{2}$$

where $\mathbf{z}_n\in\mathbb{R}^N$ and p is an accessible parameter of the system. One general condition for on-off intermittency to occur is that the phase space contains an invariant subspace in which the dynamics is either chaotic or is generated by some stochastic process. In either case, the dynamical variables in the invariant subspace are random in their time evolution. These random variables serve as the ''driving signals'' to the dynamics in the subspace that is perpendicular to the invariant subspace. To be specific, let $\mathbf{x}_n\in\mathbb{R}^{N_\parallel}$ be the dynamical variables in the $N_\parallel$-dimensional invariant subspace defined by $\mathbf{y}_n=\mathbf{b}$, where $\mathbf{y}_n\in\mathbb{R}^{N_\perp}$ denotes the dynamical variables of the $N_\perp$-dimensional subspace perpendicular to $\mathbb{R}^{N_\parallel}$, and $N_\parallel+N_\perp=N$. The subspace $\mathbb{R}^{N_\parallel}$ is invariant in the sense that if an initial condition in the full phase space $\mathbb{R}^N$ has $\mathbf{y}_0=\mathbf{b}$, the trajectory resulting from this initial condition has $\mathbf{y}_n=\mathbf{b}$ for subsequent iterations $n>0$. Depending on the parameter of the system, on-off intermittency can occur for the dynamical variables $\mathbf{y}_n$ in the perpendicular subspace $\mathbb{R}^{N_\perp}$:$\mathbf{y}_n=\mathbf{b}$ is the ''off'' state and $\mathbf{y}_n\neq\mathbf{b}$ is the ''on'' state. Taking Eq. (1) as an example, the invariant subspace is the one-dimensional x space defined by $y=b$, and the variable x is random. On-off intermittency occurs in y, which is the variable in the one-dimensional subspace perpendicular to x.

Previous studies have established the dynamical mechanism for on-off intermittency [1–4]. Specifically, if the dynamics in the invariant subspace is weakly unstable with respect to perpendicular perturbations, trajectories can be repelled away from the invariant subspace even though they can stay near the invariant subspace for some period of time. To quantify this situation, we write the dynamics in the perpendicular subspace as

$$\mathbf{y}_{n+1}=\mathbf{G}(\mathbf{x}_n,\mathbf{y}_n;p), \tag{3}$$

and we define the perpendicular (or transverse) Lyapunov spectrum as

$$h_\perp^i=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\ln|\mathbf{DG}(\mathbf{x}_n,\mathbf{y}_n,p)|_{\mathbf{y}_n=\mathbf{b}}\cdot\mathbf{u}_i|, \tag{4}$$

where $\mathbf{DG}(\mathbf{x}_n,\mathbf{y}_n;p)|_{\mathbf{y}_n=\mathbf{b}}$ is the Jacobian matrix of the map $\mathbf{G}$ evaluated at $\mathbf{y}_n=\mathbf{b}$, and $\mathbf{u}_i$ is one of the eigenvectors in the eigenspace of $\Pi_{n=1}^{\infty}\mathbf{DG}(\mathbf{x}_n,\mathbf{y}_n;p)|_{\mathbf{y}_n=\mathbf{b}}$. For a randomly chosen unit vector $\mathbf{u}$, Eq. (4) yields the largest perpendicular Lyapunov exponent, which we denote by $h_\perp$. If $h_\perp$ is slightly positive, on average trajectories will be repelled away from the invariant subspace $\mathbf{y}_n=\mathbf{b}$ so that $\mathbf{y}_n\neq\mathbf{b}$ can occur. This corresponds to the ''on'' behavior. But since $h_\perp$ is only slightly positive, in any finite time trajectories can be attracted towards and then stay in the vicinity of the invariant subspace, which leads to the ''off'' behavior. These behaviors can be more precisely quantified by the finite time fluctuations in the perpendicular Lyapunov exponent $h_\perp$ [4,5].

Based on the dynamical mechanism for on-off intermittency, the design of the control algorithm for confining trajectories in the vicinity of the invariant subspace is quite straightforward. Assume that at some nominal parameter value $p=p_0$, the system exhibits on-off intermittency. Our goal is to apply arbitrarily small perturbations to the parameter p so that a trajectory stays in the ''off'' state for as long as control is present. The strategy is similar to the OGY idea of controlling chaos via stabilization of unstable periodic orbits [8]. Due to the chaotic nature of the $\mathbf{y}$ dynamics in the vicinity of the invariant subspace, a trajectory resulting from a random initial condition will come arbitrarily close to the ''off'' state at some later time. When this occurs, judiciously chosen and time-dependent parameter perturbations δp_n around p_0 are applied to keep the trajectory in the ''off'' state. The magnitude of the perturbation is proportional to the $\mathbf{y}$ distance of the trajectory to the ''off'' state and can therefore be made arbitrarily small. Assuming $\delta\mathbf{y}_n\equiv|\mathbf{y}_n-\mathbf{b}|\to 0$, we expand Eq. (3) around $\mathbf{y}=\mathbf{b}$. This yields

$$\delta\mathbf{y}_{n+1}\approx\frac{\partial\mathbf{G}}{\partial\mathbf{y}_n}\cdot\delta\mathbf{y}_n+\frac{\partial\mathbf{G}}{\partial p}\,\delta p_n, \tag{5}$$

where the matrix $\partial\mathbf{G}/\partial\mathbf{y}_n$ and the vector $\partial\mathbf{G}/\partial p$ are evaluated at $\mathbf{y}_n=\mathbf{b}$ and $p=p_0$. In order to compute the necessary parameter perturbation δp_n to keep $\delta\mathbf{y}_{n+1}\approx 0$ for subsequent iterations, we choose a unit vector $\mathbf{u}$ in the $\mathbf{y}$ subspace to form the dot product $\mathbf{u}\cdot\delta\mathbf{y}_{n+1}$. Letting $\mathbf{u}\cdot\delta\mathbf{y}_{n+1}=0$ yields

$$\delta p_n=-\frac{\mathbf{u}\cdot(\partial\mathbf{G}/\partial\mathbf{y}_n)\cdot\delta\mathbf{y}_n}{\mathbf{u}\cdot(\partial\mathbf{G}/\partial p)}. \tag{6}$$

In principle, we can choose the unit vector $\mathbf{u}$ arbitrarily provided that the denominator in Eq. (6), $\mathbf{u}\cdot(\partial\mathbf{G}/\partial p)$, is not close to zero. In practice, we define a maximum allowed magnitude for the parameter perturbation $\delta p_{\max}$. If the computed $|\delta p_n|$ exceeds $\delta p_{\max}$, we set $\delta p_n=0$. Doing this would cause loss of control occasionally. But we find in our numerical experiments (to be described later) that robust control can still be achieved since setting $\delta p_n=0$ is done only rarely. Note that $\delta p_{\max}$ can be made arbitrarily small.

We now briefly describe the control algorithm for flows,

$$\frac{d\mathbf{z}}{dt}=\mathbf{F}(\mathbf{z};p), \tag{7}$$

where $\mathbf{z}(t)\in\mathbb{R}^N$. In terms of the dynamical variables $\mathbf{x}(t)\in\mathbb{R}^{\parallel}$ and $\mathbf{y}(t)\in\mathbb{R}^{\perp}$ in the invariant and perpendicular subspaces, respectively, the dynamical equation in the perpendicular subspace can be written as

$$\frac{d\mathbf{y}}{dt}=\mathbf{G}(\mathbf{x},\mathbf{y};p), \tag{8}$$

where $\mathbf{x}$ is the random driving signal generated by the dynamics in the invariant subspace. In principle, one can regard the flow as a map constructed on some appropriate Poincaré surface section and design the feedback control from the map. But the control so designed is usually vulnerable to external noise as the time between successive parameter perturbations is the typical time that a trajectory takes to return to the surface of section after passing through it. This time, however, can be long. Therefore we seek to apply control at small time steps Δt. Letting $\Delta t\ll T$, where T is the average time between two successive passes of the trajectory through the surface of section, Eq. (8) can be approximated by

$$\mathbf{y}(t+\Delta t)=\mathbf{y}(t)+\mathbf{G}(\mathbf{x}_n,\mathbf{y};p)\Delta t, \tag{9}$$

which can be regarded as a discrete map defined by iterations of time step Δt. Expanding Eq. (9) in the vicinity of the ''off'' state $\mathbf{y}(t)=\mathbf{b}$, we obtain, for the parameter perturbation to be applied at time t,

$$\delta p(t)=-\frac{\mathbf{u}\cdot\delta\mathbf{y}(t)+\mathbf{u}\cdot(\partial\mathbf{G}/\partial\mathbf{y})|_{\mathbf{y}=\mathbf{b},p=p_0}\cdot\delta\mathbf{y}(t)\Delta t}{\mathbf{u}\cdot(\partial\mathbf{G}/\partial p)|_{\mathbf{y}=\mathbf{b},p=p_0}\Delta t}, \tag{10}$$

where $\delta\mathbf{y}(t)=\mathbf{y}(t)-\mathbf{b}$. Again, we set $\delta p(t)=0$ if $|\delta p(t)|\geqslant\delta p_{\max}$, where $(\delta p_{\max})/p_0\ll 1$.

III. NUMERICAL RESULTS

A. A two-dimensional map

Our first numerical example is the discrete map Eq. (1). For this system, both the invariant and perpendicular subspaces are one dimensional. We test our control algorithm using the case shown in Fig. 1 where the parameter setting is $a=2.8$ and $b=0.5$. Assuming b is the accessible parameter to be perturbed, we set the nominal value of b at $b_0=0.5$ and allow b to vary around b_0. The required parameter perturbation is given by

$$\delta b_n=\frac{ax_n(y_n-b_0)}{ax_n-1}, \tag{11}$$

where we set $\delta b_n=0$ if $\delta b_n\geqslant\delta b_{\max}=10^{-3}$. The size of the small neighborhood for triggering the control is also $\epsilon=10^{-3}$. Parameter control is applied when $(y_n-b_0)\leqslant\epsilon$. Figure 2 shows a controlled time series y_n, where the trajectory starts from an arbitrary initial condition. At time step

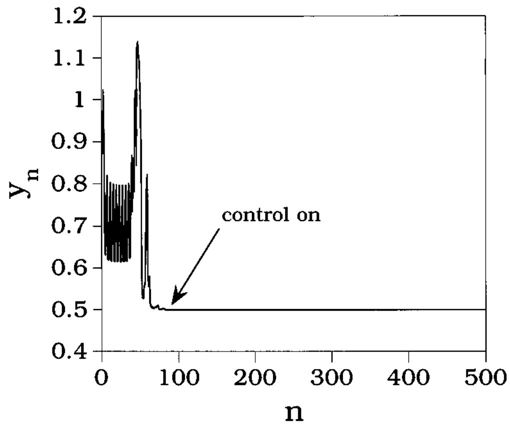

FIG. 2. A controlled trajectory. The control is applied at time $n=86$ when the trajectory comes within 10^{-3} of the desirable "off" state. The maximum allowed parameter perturbation is 10^{-3}.

$n=86$, the trajectory comes within 10^{-3} of the desirable "off" state, and control is activated. The trajectory then stays near the "off" state for as long as the control is present. After a few iterations with control, y_n comes within about 10^{-16} (computer roundoff error) of the "off" state, and the required parameter perturbation also reduces to the same order of magnitude. Notice that the x dynamics remains chaotic, regardless of whether y is controlled or not.

When noise is present, control can still be achieved. But the controlled trajectory stays in a neighborhood of the "off" state of size that is proportional to the noise amplitude. Figure 3(a) shows $\log_{10}(y_n - b_0)$ versus time n when a term $10^{-5}\sigma_n$, where σ_n is a random number uniformly distributed in [0,1], is added to Eq. (1). After control is turned on, the trajectory stays within about $10^{-4.5}$ of the "off" state. Figure 3(b) shows the required parameter perturbation after control is on. On average, the magnitude of the parameter perturbation is about $10^{-4.7}$, which is comparable to the noise amplitude.

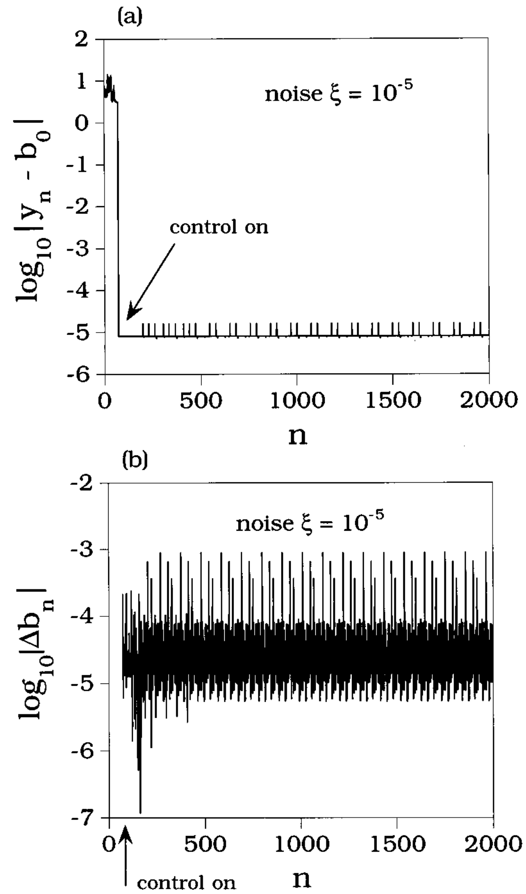

FIG. 3. (a) A controlled trajectory when random noise of amplitude 10^{-5} is added to Eq. (1). Shown is $\log_{10}|y_n - b_0|$ versus time n. (b) The required magnitude of the parameter perturbation. Shown in $\log_{10}|\Delta b_n|$ versus time n.

B. A five-dimensional flow

Our second numerical example is a flow that exhibits on-off intermittency. We consider a mechanical system where particles move under the influence of the following potential in the plane:

$$V(\mathbf{x}) = (1-x^2)^2 + (y^2-a^2)^2(x-d) + b(y^2-a^2)^4, \tag{12}$$

where $\mathbf{x} \equiv (x,y)$, a, d, and $b(>0)$ are parameters. We assume particles are also subjected to friction and periodic forcing of the form $f_0 \sin(\omega t)$ in the x direction. There are now two symmetric lines defined by $y = \pm a$ on which $V(\mathbf{x})$ is independent of the coordinate y and reduces to Duffing's two-well potential in x [13]. The equation of motion is

$$\frac{d^2\mathbf{x}}{dt^2} = -\alpha \frac{d\mathbf{x}}{dt} - \nabla V(\mathbf{x}) + f_0 \sin(\omega t)\mathbf{x}_0, \tag{13}$$

where α is the friction coefficient and $\mathbf{x}_0$ is the unit vector in x. The system can be written as five first-order autonomous differential equations in terms of dynamical variables x, $v_x \equiv dx/dt$, $z = \omega t$, y, and $v_y \equiv dy/dt$,

$$\begin{aligned} \frac{dx}{dt} &= v_x, \\ \frac{dv_x}{dt} &= -\alpha v_x + 4x(1-x^2) - (y^2-a^2)^2 + f_0 \sin z, \\ \frac{dz}{dt} &= \omega, \\ \frac{dy}{dt} &= v_y, \\ \frac{dv_y}{dt} &= -\alpha v_y - 4y(y^2-a^2)(x-d) - 8by(y^2-a^2)^3. \end{aligned} \tag{14}$$

Note that on the two symmetric lines $y = \pm a$, if $v_y = 0$, the equations of motion reduce to

$$\frac{dx}{dt}=v_x,$$

$$\frac{dv_x}{dt}=-\alpha v_x+4x(1-x^2)+f_0\sin z, \tag{15}$$

$$\frac{dz}{dt}=\omega,$$

which is the set of equations describing a forced-damped Duffing oscillator [13] in which chaos occurs commonly. Since Eq. (15) is independent of y and v_y, a trajectory with initial condition in the subspaces $y=\pm a$ and $v_y=0$ will remain in the subspaces forever. The conditions $y=\pm a$ and $v_y=0$ thus define two three-dimensional invariant subspaces, where the two Duffing chaotic attractors are located, in the five-dimensional phase space. The system exhibits on-off intermittency in a wide range of parameter values [5]. Figure 4(a) shows an on-off intermittent time series of $y(t)$ where the parameter setting is $a=0.8$, $b=0.008$, $d=-1.8$, $f_0=2.3$, $\alpha=0.05$, and $\omega=3.5$. Due to the presence of two invariant subspaces at $y=\pm a$ and $v_y=0$, there are two ''off'' states. This can be understood as follows: The perpendicular Lyapunov exponents with respect to both invariant subspaces are slightly positive ($h_\perp\approx 0.0006$). Thus a typical trajectory spends a long time near one invariant subspace, is repelled away from this subspace, then is possibly attracted to the other invariant subspace or the same subspace, temporarily spending a long stretch of time there, is repelled away again, etc. Such a behavior is called two-state on-off intermittency [5]. The two ''off'' states correspond to two wells in the potential $V(x,y)$ in the y direction, as shown by the potential profile at $x=1$ in Fig. 4(b). Our goal is thus to apply small control so that a trajectory from a random initial condition stays in the vicinity of one of the potential wells in y, assuming that this potential well corresponds to the desirable operational state of the system.

We use Eq. (10) to compute the required parameter perturbation. Note that for the system Eq. (14), the invariant subspace is three dimensional and the perpendicular subspace is two dimensional. We choose a to be the parameter to be perturbed. We thus set $a_0=0.8$ and $\delta a_{\max}=10^{-3}$. The rest of the parameters are the same as in Fig. 4(a). Letting y^{off} $(=\pm a_0)$ and $v_y^{\mathrm{off}}=0$ denote the desirable ''off'' state, we have, for the function $\mathbf{G}(\mathbf{x}(t),\mathbf{y}(t),a)$ and the vector $\partial\mathbf{G}/\partial a$ in Eq. (10), the following:

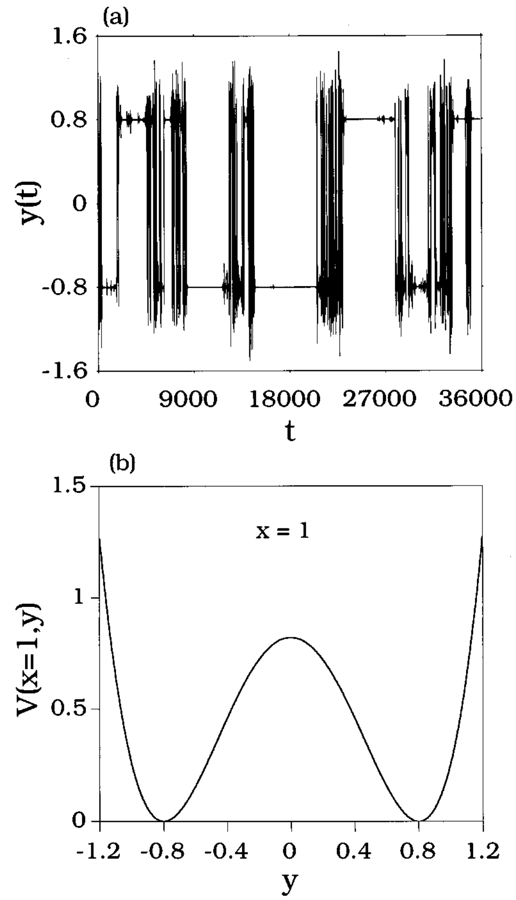

FIG. 4. (a) A two-state on-off intermittent time series generated by the five-dimensional flow Eq. (14) from a random initial condition. In this case there are two ''off'' states. The parameter setting is $a=0.8$, $b=0.008$, $d=-1.8$, $f_0=2.3$, $\alpha=0.05$, and $\omega=3.5$. (b) The potential profile at $x=1$ indicating two wells at $y=\pm 0.8$, respectively.

$$\mathbf{G}^\dagger(\mathbf{x}(t),\mathbf{y}(t);a) =\{v_y(t),-\alpha v_y(t)-4y(t)[y^2(t)-a^2][x(t)-d] -8by(t)[y(t)^2-a^2]^3\},$$

and $(\partial\mathbf{G}/\partial a)^\dagger=\{0,8y(t)a[x(t)-d]\}$. By choosing $\mathbf{u}=(1.1)/\sqrt{2}$, we obtain the following expression for the parameter perturbation:

$$\delta a(t)=-\frac{\{1-8(y^{\mathrm{off}})^2[x(t)-d]\Delta t\}\delta y(t)+[1+(1-\alpha)\Delta t]\delta v_y(t)}{8y^{\mathrm{off}}a_0[x(t)-d]\Delta t}, \tag{16}$$

where $\delta y(t)=y(t)-y^{\mathrm{off}}$, $\delta v_y(t)=v_y(t)-v_y^{\mathrm{off}}$, and $\Delta t\ll 2\pi/\omega$ is the small time interval for $\delta a(t)$ to be applied. If $\delta a(t)\geqslant\delta a_{\max}$, we set $\delta a(t)=0$.

Figure 5(a) shows a case of stabilizing the ''off'' state $y^{\mathrm{off}}=-a_0=-0.8$ and $v^{\mathrm{off}}=0$. The trajectory starts from an arbitrary initial condition. The parameter control is turned on when $|y(t)-y^{\mathrm{off}}|\leqslant 10^{-3}$ and $|v_y(t)-v_y^{\mathrm{off}}|\leqslant 10^{-3}$. The time interval to apply the control is set to be $\Delta t=T/256$, where $T=2\pi/\omega$ is the period of the external forcing. After the control is on, $y(t)$ is stabilized in the vicinity of $-a_0$ for as long as the small control is present [Fig. 5(a)], and simultaneously, $v_y(t)$ is stabilized around $v^{\mathrm{off}}=0$ (not shown).

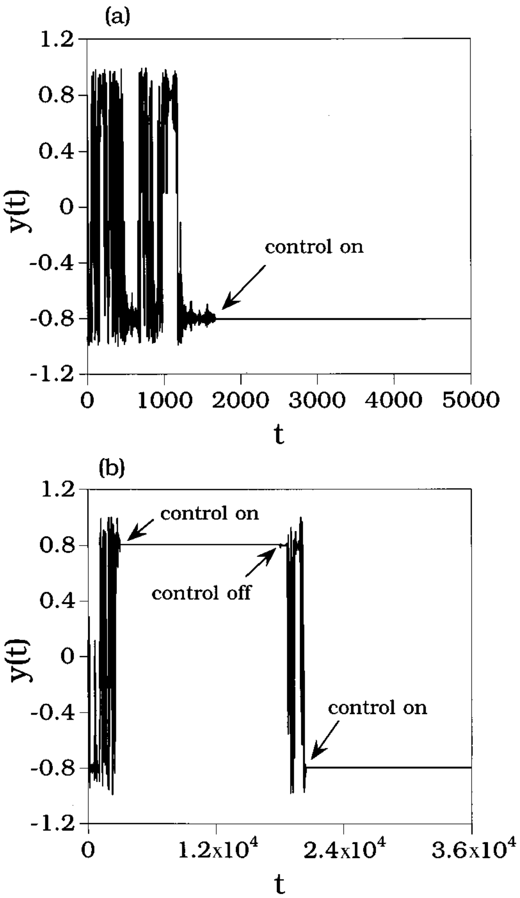

FIG. 5. For the five-dimensional flow Eq. (14), (a) a controlled trajectory stabilized around the desirable ''off'' state $y(t) = -0.8$. (b) A situation where the trajectory is stabilized around $y(t) = 0.8$ first and is then stabilized around the second ''off'' state $y(t) = -0.8$. This demonstrates the flexibility of the control algorithm to select different desirable ''off'' state at different time.

Similar to the spirit of OGY control, in the presence of multiple ''off'' states, our control algorithm is quite flexible to stabilize different ''off'' state that might correspond to a desirable operational state of the system at different time. Figure 5(b) shows a situation where we stabilize the ''off'' state $y^{\mathrm{off}} = a_0$, $v_y^{\mathrm{off}} = 0$ first and then switch to stabilize the ''off'' state $y^{\mathrm{off}} = -a_0$, $v_y^{\mathrm{off}} = 0$. To switch the control from the first ''off'' state to the second ''off'' state after the trajectory is stabilized around the first ''off'' state, we simply turn off the control to let the system evolve at the nominal parameter values. At some later time the trajectory will come arbitrarily close to the second ''off'' state. A new set of parameter perturbations computed with respect to the second ''off'' state is then activated to stabilize the trajectory around this new ''off'' state.

When noise is present, our control method still works, but the closeness of the controlled trajectory to the desirable ''off'' state and the magnitude of the required parameter perturbations are now proportional to the noise amplitude. Figure 6 shows a time series $y(t)$ for a case of stabilizing the ''off'' state $y^{\mathrm{off}} = a_0$ and $v^{\mathrm{off}} = 0$ where a noisy term $10^{-2}\sigma(t)$ [$\sigma(t)$ is a random variable uniformly distributed in [0,1]] is added to Eq. (14). It can be seen that while a trajectory can still be stabilized around the desirable ''off'' state after control is on, occasionally the trajectory deviates from the ''off'' state to within a range about 10^{-2} around the ''off'' state.

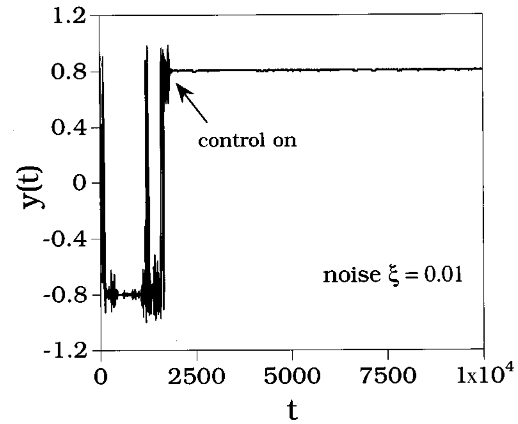

FIG. 6. For the five-dimensional flow Eq. (14), a controlled trajectory stabilized around $y(t) = 0.8$ where external noise of amplitude 10^{-2} is present.

IV. SCALING OF THE AVERAGE TRANSIENT TIME PRECEDING CONTROL

One important issue for achieving stabilization of the desirable ''off'' state is the average transient time, or the waiting time, preceding turning on of the control. In general, this waiting time depends on the size of the controlling neighborhood. The smaller the size of the neighborhood is, the longer the waiting time will be. When one applies the OGY idea to stabilize unstable periodic orbits embedded in a chaotic attractor [8], or to stabilize a chaotic orbit to synchronize two identical chaotic systems [14], one usually finds that the average waiting time $\tau(\epsilon)$ scales with the size of the controlling neighborhood algebraically,

$$\tau(\epsilon) \sim \epsilon^{-\beta}, \tag{17}$$

where $\beta > 0$ is a scaling exponent that can be related to the Lyapunov exponent of the unstable periodic orbit or the chaotic orbit [8,14]. We find that in our cases of controlling on-off intermittency, the average waiting time does not obey the algebraic scaling law. This is shown in Fig. 7 for the case of stabilizing the ''off'' state $y = b_0$ in map Eq. (1), where $\tau(\epsilon)$ versus ϵ is plotted on a logarithmic scale. If the algebraic scaling law holds, such a plot could be fitted by a straight line. As ϵ gets smaller, the average waiting time increases slowly. We find that the average waiting time obeys the following scaling law:

$$\tau(\epsilon) \sim |\ln(\epsilon)|^{\beta}, \tag{18}$$

where β is a scaling exponent. This is shown in Fig. 8(a) for the same parameter setting as in Fig. 7, where $\ln[\tau(\epsilon)]$ versus $\ln|\ln(\epsilon)|$ is plotted. For each value of ϵ, 10^5 random initial conditions were chosen to compute the average time for trajectories to first fall into the ϵ neighborhood of the ''off'' state. The plot can be fitted by a straight line with a slope 1.63 ± 0.02.

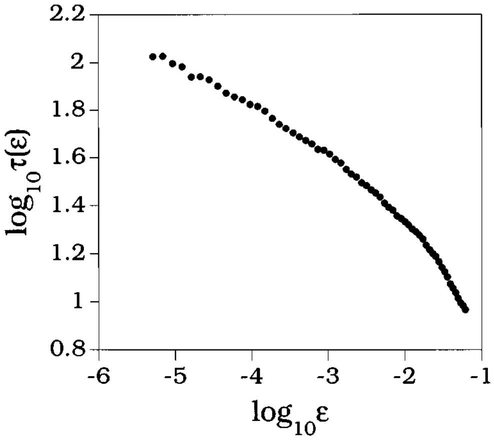

FIG. 7. For the map Eq. (1), the average waiting time $\tau(\epsilon)$ preceding the control versus ϵ on a logarithmic scale. Apparently the scaling between $\tau(\epsilon)$ and ϵ is not algebraic. In the computation, 10^5 trajectories were chosen to compute $\tau(\epsilon)$ for each value of ϵ.

We now give a heuristic argument for the scaling relation Eq. (18). Take a trajectory that starts from a random initial condition. In order for the trajectory to fall within ϵ of the invariant subspace $y=b_0$ (the asymptotic ''off'' state), on average the trajectory must experience attraction towards $y=b_0$ in time $\tau(\epsilon)$. It is thus insightful to study the statistics of the time intervals during which trajectories experience contraction on average. For simplicity we consider the dynamics in the vicinity of $y=b_0$. For $|y_n-b_0|$ small we have $\Delta y_{n+1}\approx a x_n \Delta y_n$ from Eq. (1), where $\Delta y_n = y_n - b_0$. We obtain $\Delta y_m \approx (a^m \Pi_{i=0}^{m-1} x_i)\Delta y_0$. Thus we are led to consider the sequence in x: $\{x_0, x_1, \ldots, x_{m-1}\}$ which satisfies

$$a^m \prod_{i=0}^{m-1} x_i \equiv (a\overline{x_m})^m \sim \epsilon. \tag{19}$$

where $\overline{x_m^m} \equiv x_0 x_1, \ldots, x_{m-1}$, and $a\overline{x_m}<1$. The integer m is in fact the time interval during which a trajectory is attracted towards the invariant subspace on average. We ask, what is the probability distribution $P(m)$ for the length m of the sequence? To answer this question, we observe that points in the sequence $\{x_0, x_1, \ldots, x_{m-1}\}$ can be divided into two groups: one with $ax_i \geqslant 1$ (or $x_i \geqslant x_c \equiv 1/a$) and one with $ax_i<1$ (or $x_i<x_c$). For the cases studied in this paper, the chaotic or random variable x has a smooth invariant density. As a consequence, the probability that a trajectory in the invariant subspace stays in the contracting region ($ax<1$) for a large number of iterations is not negligible. For instance, if the x dynamics is produced by the tent map, the invariant density $\rho(x)$ is uniform in $x\in[0,1]$. The contracting region is given by $ax<1$ or $x<x_c=1/a$. The probability for x to stay in $x<x_c$ *consecutively* for n iterations is a^{-n}. Consequently, the probability $P(m)$ can attain appreciable values even when m is large. To obtain the scaling for $P(m)$, we note that in the well established laminar phase statistics $P(T)\sim T^{-3/2}$ [3], T is the time that a typical trajectory stays in the ''off'' state, but m in Eq. (19) is the time during which a trajectory experiences attraction towards the invariant subspace. In order for a trajectory to stay in the vicinity of the invariant subspace (the ''off'' state), *on average* the trajectory must be attracted towards the invariant subspace. Therefore, we have $m\sim T$ and we expect $P(m)$ to follow a similar algebraic scaling law. We write $P(m)\sim m^{-\beta}$. From Eq. (19), we have $m\sim \ln\epsilon/\ln(a\overline{x_m})$. For ϵ small (or m large) we approximate $m\sim|\ln\epsilon|$, assuming that $\overline{x_m}$ is roughly independent of m. Since $P(m)$ is the probability that a trajectory enters the ϵ neighborhood of $y=b_0$, we have

$$\tau(\epsilon)\sim 1/P(m)\sim m^{\beta}\sim|\ln(\epsilon)|^{\beta},$$

which is Eq. (18). One implication is that since $m\sim T$, the scaling exponent β in Eq. (18) should be close to the scaling exponent $\frac{3}{2}$ in $P(T)$ when $a\gtrsim a_c$. It should be stressed that the argument leading to Eq. (18) is only heuristic. There are several crude approximations used in the derivation. Nevertheless, the scaling relation Eq. (18) is supported by numerical experiments for both maps and flows, as we will see below.

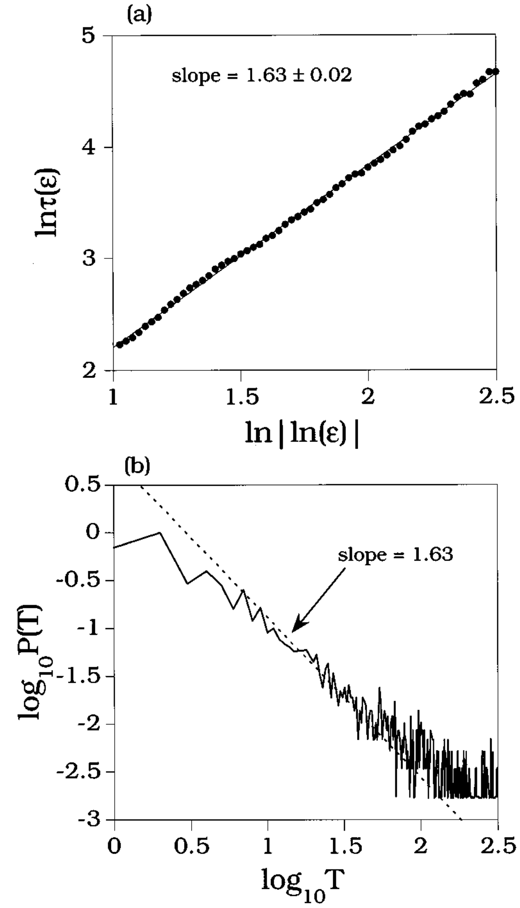

FIG. 8. For the map Eq. (1), (a) the average waiting time $\tau(\epsilon)$ versus $|\ln(\epsilon)|$ on a logarithmic scale. The good fit of the plot to a straight line indicates that the scaling between $\tau(\epsilon)$ and $|\ln(\epsilon)|$ is algebraic with an exponent of approximately 1.63±0.02; (b) a histogram of the length of the laminar phase. The threshold for regarding the trajectory as being in the ''off'' state is 0.01. In total, 10^7 laminar phases are accumulated to produce the histogram. The dotted line with a slope of 1.63 approximates the asymptotic scaling of the probability distribution.

To compare the waiting time scaling exponent to that of $P(T)$, we compute a histogram for the length of the laminar

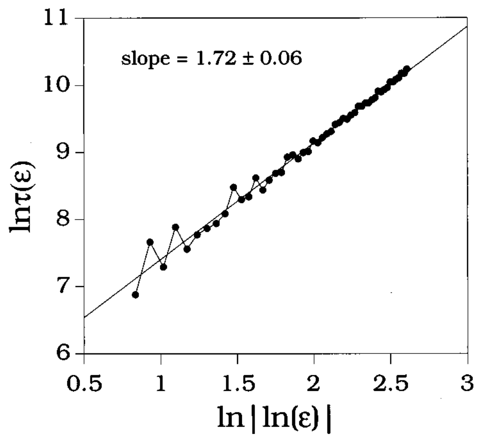

FIG. 9. For the five-dimensional flow Eq. (14), the average waiting time $\tau(\epsilon)$ versus $|\ln(\epsilon)|$ on a logarithmic scale. The fit of the data to a straight line is good, the slope of which is 1.72 ± 0.06. This implies that the algebraic scaling between $\tau(\epsilon)$ and $|\ln(\epsilon)|$ is quite general.

phase for Eq. (1), as shown in Fig. 8(b), where $P(T)$ versus T is plotted on a logarithmic scale. To obtain this picture, we accumulate 10^7 lengths of laminar phase, where if a trajectory comes within 10^{-2} of the ''off'' state, we consider it to be in a laminar phase. Also shown in Fig. 8(b) is a straight dotted line with a slope of 1.63. The asymptotic scaling of $P(T)$ obeys the algebraic scaling law $T^{-3/2}$ approximately, and the scaling exponent for the waiting time in Eq. (18) is close to $\frac{3}{2}$.

The above argument for the average waiting time scaling makes use of the scaling for the probability distribution of the length of the laminar phase: The latter is conjectured to be universal for a large class of systems that exhibit on-off intermittency. Thus we expect the waiting time scaling Eq. (18) to be quite general. The generality is supported by a similar scaling observed for the system of five-dimensional flow Eq. (14), as shown in Fig. 9, where $\tau(\epsilon)$ versus $|\ln(\epsilon)|$ is plotted on a logarithmic scale. Here, due to the intensive computation involved, only 5000 random initial conditions were chosen for each value of ϵ to compute the average waiting time. The scaling exponent is $\gamma=1.72\pm0.06$. The good fit of the data to a straight line indicates that the scaling Eq. (18) is robust for this five-dimensional flow.

V. DISCUSSIONS

In this paper we investigate controlling chaotic dynamical systems that exhibit on-off intermittency. We devise an algorithm for stabilizing a trajectory in the vicinity of a desirable state, the ''off'' state, by using arbitrarily small parameter perturbations. It should be noted that the ''off'' states are in general chaotic, because they are usually characterized by chaotic sets embedded in some invariant subspace of the full system. Thus controlling chaos with on-off intermittency can be regarded as a case in the more general study where one seeks to select desirable chaotic states using small feedback control [15]. Numerical examples with both a map and a flow demonstrate that the algorithm works even when there is small-amplitude noise. Our method follows the spirit of the OGY idea of controlling chaos, and it is therefore flexible

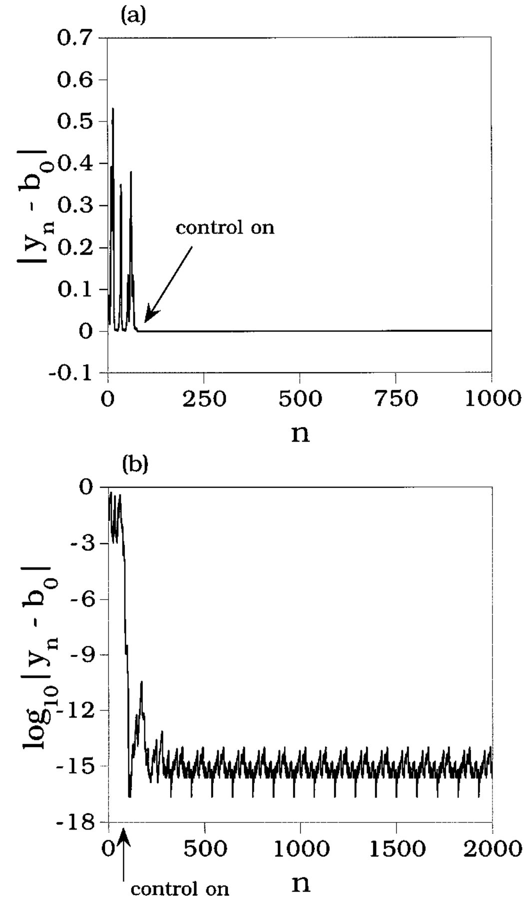

FIG. 10. A controlled trajectory when a simple proportional feedback control scheme Eq. (20) with $\alpha=1.5$ is used for the system Eq. (1) at $a=2.8$. (a) $|y_n-b_0|$ versus n (1000 iterations shown); and (b) $\log_{10}|y_n-b_0|$ versus n (2000 iterations shown). Although only a limited number of iterations are shown in (a) and (b) for the purpose of illustration, robust control can be achieved for as long as the parameter perturbations Eq. (20) are present (verified using 10^8 iterations). The result indicates that on-off intermittent dynamics can be controlled in more realistic situations where the only available information is time series of the on-off dynamical variables.

to stabilize different ''off'' states at different time depending on one's needs, provided that there are multiple ''off'' states in the system. We also study the scaling of the average transient time preceding the control and find that the scaling obeys a qualitatively different law from that in conventional controlling chaos applications where one stabilizes unstable periodic orbits or chaotic orbits embedded in a chaotic attractor. The mechanism for the observed scaling law is elucidated, and numerical confirmation for both map and flow indicates that the scaling law is quite general.

Our control algorithm is based on the knowledge of system equations. It is important to discuss the feasibility of controlling on-off intermittent chaotic systems when detailed system equations are not available, which would occur in practical applications. In our numerical examples, feedback perturbations are applied to a parameter that *directly* characterizes the ''off'' state. It is conceivable that in most situations one can gain a fairly good knowledge about the target ''off'' state by running the system and observing it. This is similar to finding the target unstable periodic orbit to be stabilized in the OGY control strategy. Therefore we expect that it would be possible to find the target ''off'' state and the proper control parameter in more realistic applications. After the ''off'' state is identified, one can then measure the distance of a trajectory from the ''off'' state $\delta\mathbf{y}$. In cases where the on-off intermittent behavior is generated by a random or a chaotic driving signal such as in Eq. (1), the feedback control only depends on $\delta\mathbf{y}$ and the driving signal [ax_n in Eq. (1), see Eq. (11)]. If in experiments one knows the driving signal well (which is possible), the appropriate parameter perturbation can be computed and applied to the system. Thus, for such a case, we expect our control algorithm to work in more realistic situations, but at present there is no assurance of this.

The control algorithm presented in this paper requires the knowledge of both the on-off dynamical variables and the underlying chaotic or random variables. Rarely does one know these underlying variables in practical situations. In general, one only observes the on-off variables. Moreover, it is very difficult to reconstruct the underlying variables from the on-off variables via the delay-coordinate embedding technique [16]. Therefore it is important to test the control algorithm without having a detailed knowledge of the driving variables. We thus propose a straightforward proportional feedback scheme [17] which in principle allows one to control on-off intermittent dynamics in experiments. Taking Eq. (1) as an example and assuming that the only available information is $|y_n - b_0|$, the distance of the on-off variable from the ''off'' state, we use the following parameter perturbation in place of Eq. (11):

$$\delta b_n = \alpha(y_n - b_0), \tag{20}$$

where α is a proportional constant. We then determine, using a trial-error procedure, the range of α values where control can be achieved. It is found that robust control can still be realized when $1.0 \leqslant \alpha \leqslant 2.1$. Figure 10 shows such a case for $\alpha = 1.5$, where y_n versus n and $\log_{10} y_n$ versus n are plotted in (a) and (b), respectively. Clearly, control is successful using the simple direct proportional feedback scheme Eq. (20). For the example of the five-dimensional flow Eq. (14), we use a similar scheme: $\delta a(t) = \alpha\,\delta y(t) + \beta\,\delta v_y(t)$, assuming that only the on-off variables $y(t)$ and $v_y(t)$ can be measured. Again, control is achieved for a wide range of choices of the proportional constants α and β (data not shown). From these examples, we see that on-off intermittent dynamics can be controlled in experimental situations where the only available information is time series of the on-off dynamical variables.

Finally, we discuss implications of the scaling law for the average waiting time. In conventional applications of the OGY control method, the waiting time typically scales with the size of the controlling neighborhood ϵ (or equivalently, the maximum allowed parameter perturbation) algebraically [8]. Thus as ϵ decreases, the required waiting time increases greatly and can become prohibitively long. There is a ''trade-off'' between the waiting time and the maximum allowed perturbation [8]. In our cases of controlling on-off intermittency, the average waiting time scales with ϵ as some power of $|\ln(\epsilon)|$. This indicates that the required waiting time increases only incremently even if ϵ is decreased by many orders of magnitude [Figs. (8a) and (9)]. Therefore it is possible to apply extremely small parameter perturbations to achieve the desirable system performance *in relatively short time* when one controls on-off intermittent dynamics.

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