Symmetry-breaking bifurcation with on-off intermittency in chaotic dynamical systems

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When a dynamical system possesses certain symmetry, there can be an invariant subspace in the phase space. In the invariant subspace there can be a chaotic attractor. As a parameter changes through a critical value, the chaotic attractor can lose stability with respect to perturbations transverse to the invariant subspace. We show that the loss of the transverse stability can lead to a symmetry-breaking bifurcation characterized by lack of the system symmetry in the asymptotic attractor. An accompanying physical phenomenon is an extreme type of temporally intermittent bursting behavior. The mechanism for this type of symmetry-breaking bifurcation is elucidated. [S1063-651X(96)51505-9]

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Symmetry is quite common in nonlinear dynamical systems [1]. An interesting question is whether the system symmetry could be seen in the asymptotic attractor of the system. When the attractor does not possess the system symmetry, we say that symmetry is broken for the asymptotic attractor. In general, symmetry exists in the attractor for some parameter regimes. Disappearance of the symmetry occurs when a system parameter passes through a critical value. This is referred to as the symmetry-breaking bifurcation. As the parameter changes further, the attractor can gain partial or full symmetry of the system through the so-called symmetry-increasing bifurcations [2,3]. As a simple example, consider the one-dimensional odd-logistic map: $x \rightarrow ax - x^3$ [2]. This map is invariant under the symmetric operation: $x \rightarrow -x$. When $a<1$, the fixed point $x=0$ is stable, which satisfies the symmetry trivially. For $a>1$, the attractor has either $x>0$ or $x<0$. Thus, a symmetry-breaking bifurcation occurs at $a_c=1$. The attractor recovers the symmetry when $a>a_s=3\sqrt{3}/2$, at which the $x>0$ attractor merges with the $x<0$ attractor and, hence, a_s is the symmetry-increasing bifurcation point [2]. The phenomenon of symmetry-increasing is also believed to be relevant to physical phenomena such as the time-averaged patterns seen in spatiotemporal dynamical systems [4].

In this paper, we describe a type of symmetry-breaking bifurcation that occurs in chaotic systems with symmetric low-dimensional invariant subspace. Denote the invariant subspace by **S**. Since **S** is invariant, initial conditions in **S** result in trajectories which remain in **S** forever. We restrict our investigation to the situation where there is a chaotic attractor in **S**. In this case, whether the chaotic attractor attracts or repels initial conditions in the vicinity of **S** is determined by the sign of the largest transverse Lyapunov exponent $\lambda_\perp$ computed for trajectories in **S** with respect to perturbations in the subspace **T** which is *transverse* to **S**. When $\lambda_\perp$ is negative, **S** attracts trajectories transversely in the phase space and, the chaotic attractor in **S** is also an attractor of the whole phase space. When $\lambda_\perp$ is positive, trajectories in the vicinity of **S** are repelled away from it, and, consequently, the chaotic attractor is transversely unstable and it is hence not an attractor of the whole phase space. The bifurcation from the former behavior to the latter behavior has been investigated [5,6] and it is called the ''blowout'' bifurcation [5]. It is also known that when $\lambda_\perp$ is slightly positive, some dynamical variables of the system can exhibit an extreme type of temporarily intermittent bursting behavior: on-off intermittency [7]. The purpose of this paper is to show that for the class of chaotic systems with low-dimensional invariant subspace, symmetry-breaking bifurcation can occur when $\lambda_\perp$ becomes positive from being negative. As a physical consequence, immediately after the symmetry is broken, dynamical variables that violate the system symmetry exhibit on-off intermittency.

We consider the following general class of N-dimensional dynamical systems:

$$\begin{aligned} \mathbf{x}_{n+1} &= \mathbf{f}(\mathbf{x}_n), \\ \mathbf{y}_{n+1} &= F(\mathbf{x}_n, p)\mathbf{G}(\mathbf{y}_n - \mathbf{b}) + \mathbf{b}, \end{aligned} \tag{1}$$

where $\mathbf{x} \in R^{N_\parallel}$ $(N_\parallel \geqslant 1)$, $\mathbf{y} \in R^{N_\perp} (N_\perp \geqslant 1)$, and $N_\parallel + N_\perp = N$. The vector function $\mathbf{G}(\mathbf{y})$ possesses certain symmetry, e.g., $\mathbf{G}(-\mathbf{y}) = -\mathbf{G}(\mathbf{y})$. We assume that both the **x** and **y** dynamics are bounded. The symmetric invariant subspace is $\mathbf{y}=\mathbf{b}$ (constant). The vector function $\mathbf{f}(\mathbf{x})$ is a map that has a chaotic attractor. The scalar function $F(\mathbf{x},p)$ can be regarded as a ''driving'' from the **x** dynamics in the invariant subspace to the symmetric **y**-subsystem, and p is the bifurcation parameter. The largest transverse Lyapunov exponent $\lambda_\perp$ can be computed by monitoring the growth rate of an infinitesimal vector in the symmetric subspace $R^{N_\perp}$: $\lambda_\perp = \lim_{L\rightarrow\infty}(1/L)\Sigma_{n=1}^{L} \ln|F(\mathbf{x}_n, p)\mathbf{DG}(\mathbf{y}_n)|_{\mathbf{y}_n=\mathbf{b}} \cdot \mathbf{u}|$, where **u** is a randomly chosen vector in $R^{N_\perp}$. The largest Lyapunov exponent λ_y of the **y** subsystem follows from the same definition except that $\mathbf{y}_n$ is not set to be **b** when the Jacobian matrices $\mathbf{DG}(\mathbf{y}_n)$ are evaluated. Assume that as the parameter p passes through a critical value p_c, $\lambda_\perp$ crosses zero from the negative side. Our main goal is to understand how symmetry breaking occurs when $\lambda_\perp$ crosses zero. To be concrete, we consider the following version of Eq. (1):

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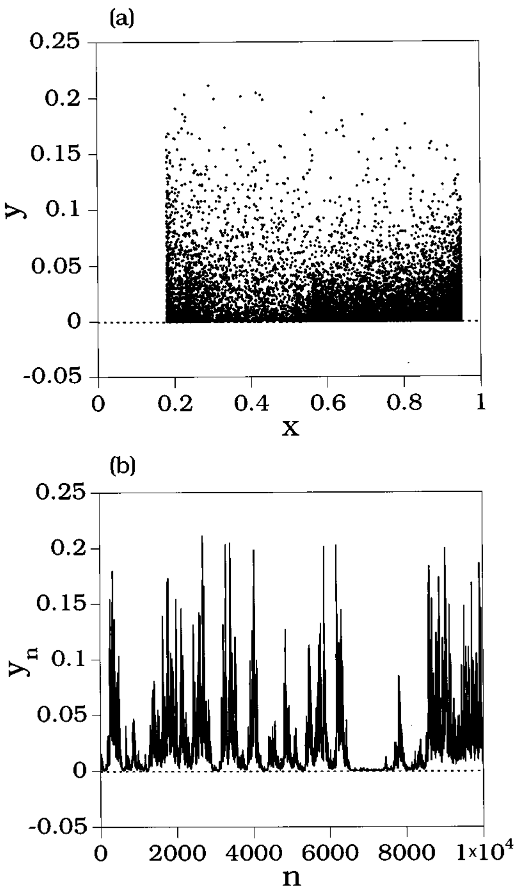

FIG. 1. (a) The chaotic attractor without symmetry of the system for Eq. (2) at $r=3.8$ and $p=1.74$, after 10^6 preiterations. (b) The time series y_n exhibiting on-off intermittency but without the y symmetry in Eq. (2).

$$x_{n+1}=rx_n(1-x_n),$$
$$y_{n+1}=\frac{1}{2\pi}\,px_n\sin[2\pi(y_n-b)]+b, \quad (2)$$

where p and b are parameters, both the invariant subspace x and the symmetric subspace y are one dimensional, x is restricted to the unit interval [0,1], and r is the parameter in the logistic map. The y equation is invariant under the mirror symmetric operation: $(y-b)\rightarrow-(y-b)$. Since x is bounded, y is also bounded. We choose r such that the logistic map generates a chaotic attractor in the x subspace. The transverse Lyapunov exponent is $\lambda_\perp=\int_0^1 \ln|px|\rho(x)dx$, where $\rho(x)$ is the invariant density of x for the logistic map. Thus, we have $p_c=\exp[-\int_0^1 \ln|x|\rho(x)dx]$, where $\lambda_\perp\geqslant 0$ for $p\geqslant p_c$ and $\lambda_\perp<0$ for $p<p_c$.

To illustrate our findings, we use the following parameter setting for numerical experiments: $r=3.8$ and $b=0$. We find $p_c\approx 1.725$. For $p<p_c$, the asymptotic attractor is $y=0$ which is trivially invariant under $y\rightarrow -y$. For $p\gtrsim p_c$, numerical computation reveals that the resultant attractor no longer possesses the mirror symmetry about $y=0$: for an initial condition with $y_0>0$ (<0), the resulting trajectory has $y_n>0$ (<0) for subsequent iterations. Figure 1(a) shows such a trajectory in the phase space for $p=1.74>p_c$ resulting from an arbitrary initial condition $y_0\in[0,0.5]$. The y value of the trajectory is confined in [0,0.5]. Thus, the mirror symmetry no longer exists in the attractor. Similarly, if we choose $-0.5<y_0<0$, the resulting trajectory will be confined in [−0.5,0]. The key observation is that for the asymptotic attractor in the phase space, the mirror symmetry in the y equation is broken immediately after the transverse Lyapunov exponent $\lambda_\perp$ becomes positive.

A feature associated with the symmetry-breaking is the occurrence of on-off intermittency [7] in y when p is slightly above p_c ($\lambda_\perp\gtrsim 0$). This is shown in Fig. 1(b), where y_n versus the time n is plotted for $p=1.74$. We see that there are time intervals when y_n stays near $y=0$ (the ''off'' state), but there are also intermittent bursts of y_n (the ''on'' state) away from the ''off'' state. This is due to the fact that $\lambda_\perp$ is only slightly positive immediately after the symmetry-breaking bifurcation. Imagine we choose an ensemble of initial conditions in x, compute $\lambda_\perp$ for each initial condition at a finite time, and then construct a histogram of these exponents. Since the asymptotic $\lambda_\perp$ is only slightly positive, there is a spread of the histogram into the negative side, indicating that a trajectory can spend long stretches of time near $y=0$ in finite times. But since $\lambda_\perp$ is positive, occasionally the trajectory can be repelled away from $y=0$. Thus on-off intermittency occurs [8].

The mechanism for the symmetry-breaking bifurcation can be understood as follows. Consider the situation where $\lambda_\perp\gtrsim 0$. Take an initial condition with $y_0>0$. At some later time n, the trajectory will come close to the x axis, i.e., $y_n\gtrsim 0$. Thus we have $y_{n+1}=(1/2\pi)px_n\sin(2\pi y_n)\approx px_n y_n$. Letting $Y_n\equiv-\ln|y_n|\geqslant 0$, we obtain $Y_{n+1}=-\alpha_n+Y_n$, where $\alpha_n\equiv\ln|px_n|$. This is a random walk in Y_n, since x_n is a chaotic variable with some invariant density $\rho(x)$. Taking the time average of Y, we obtain $\overline{Y_{n+1}}=-\overline{\alpha_n}+\overline{Y_n}$. By the ergodic theorem we have $\overline{\alpha_n}=\int\ln|px_n|\rho(x)dx=\lambda_\perp>0$ and,

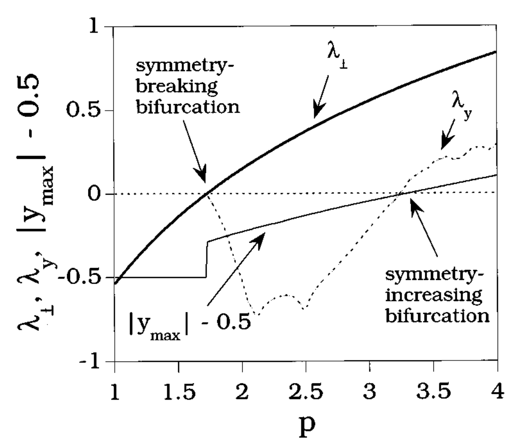

FIG. 2. For Eq. (2), the transverse Lyapunov exponent $\lambda_\perp$ and the average ($|y_{\max}|-0.5$) (computed using 1000 trajectories) versus the parameter p ($r=3.8$). Symmetry-breaking bifurcation occurs when $\lambda_\perp$ becomes positive but ($|y_{\max}|-0.5$) remains negative. Symmetry-increasing bifurcation occurs when ($|y_{\max}|-0.5$) becomes positive. Also shown is the y Lyapunov exponent λ_y versus p (the dotted line). Except in the vicinity of p_s, λ_y remains negative in the parameter range where there is a symmetry breaking.

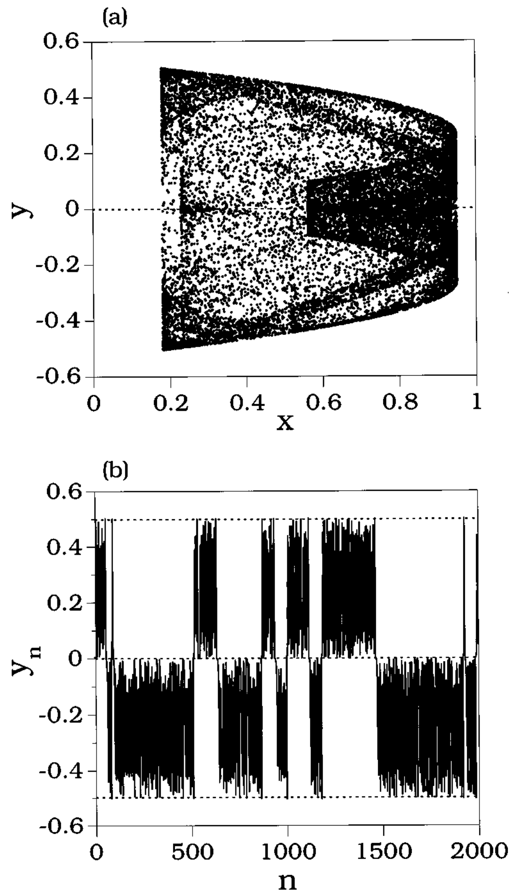

FIG. 3. (a) The chaotic attractor at $p=3.3$, after the symmetry-increasing bifurcation. The attractor is symmetric with respect to the x axis. (b) The time series y_n that exhibits intermittent switchings on the positive and negative y sides. In both (a) and (b), the number of preiterations is 10^6.

hence, $\overline{Y_{n+1}}<\overline{Y_n}$, indicating that $\overline{y_{n+1}}>\overline{y_n}$. Since, (1) the change in y_n is finite in one iteration (x_n is bounded); and (2) on average y_n increases for small y_n, we conclude that y_n cannot reach zero asymptotically. Thus, the trajectory cannot attain the system symmetry trivially by having $y_n=0$ (note that if $y_n=0$, then $y_{n+1}=0$ for subsequent iterations). But having $\lambda_\perp>0$ does not guarantee symmetry breaking. For Eq. (2), if y_n exceeds 0.5, y_{n+1} immediately becomes negative, indicating that trajectories on the positive-y chaotic component can be reinjected into the basin of the coexisting negative-y chaotic component. Since the positive-y and negative-y chaotic components are completely symmetric with respect to each other, in this case the system symmetry is not broken for the attractor. In general, symmetry breaking occurs if trajectories on one symmetric chaotic component cannot be reinjected into other coexisting symmetric components. For Eq. (2), we find that reinjection of trajectories between the symmetric components does not occur if the y Lyapunov exponent λ_y remains negative even if $\lambda_\perp$ is positive. If $\lambda_\perp \gtrsim 0$, λ_y can remain negative in the vicinity of p_c since $\lambda_y=\lambda_\perp+\int \ln|\cos(2\pi y_n)|\rho_y(y)dy$ with the integral in y being negative, where $\rho_y(y)$ is the probability distribution of y after the symmetry-breaking bifurcation. We note that before the bifurcation, we have $\lambda_y=\lambda_\perp$ because $y_n=0$ asymptotically and, therefore, $\int \ln|\cos(2\pi y_n)|\rho_y(y)dy=0$. Figure 2 shows $\lambda_\perp$ and the average value of the quantity $(|y_{\max}|-0.5)$ (averaged over 1000 trajectories) versus p for $1 \geqslant p \geqslant 4$. Symmetry breaking occurs for $p_c<p<p_s$, where $p_s \approx 3.306$ is a symmetry-increasing bifurcation point at which $|y_{\max}|$ exceeds 0.5. Also shown in Fig. 2 is λ_y versus p (the dotted line). We see that λ_y remains negative for $p_c<p<p_s$ except when p is very close to p_s (λ_y becomes positive at $p\approx 3.245$). Figures 3(a) and 3(b) show the attractor with the system symmetry recovered and the time series y_n, respectively, for $p=3.33>p_s$. We see that y_n occurs on both sides of the symmetric axis $y=0$. Whenever $|y_n|$ exceeds 0.5, it jumps from one side of $y=0$ to the other [Fig. 3(b)].

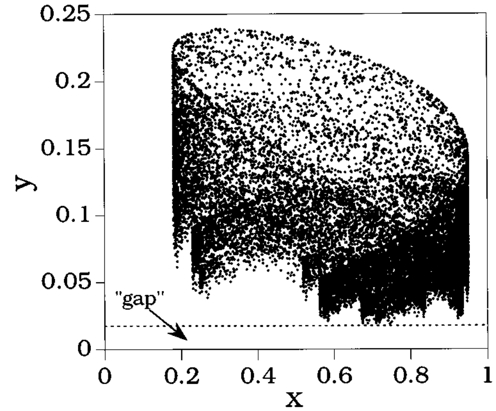

FIG. 4. The symmetry-broken chaotic attractor with a gap from $y=0$ at $p=1.85$ ($\lambda_\perp \approx 0.07$ and $\lambda_y \approx -0.182$). The number of preiterations is 10^6.

An interesting phenomenon is the occurrence of an apparent ''gap'' between the attractor and the x axis in the symmetry-broken attractor. Such a gap is observed, and it is particularly obvious when $\lambda_\perp$ is positive but not close to zero, as shown in Fig. 4, where $p=1.85>p_c$ at which the values of $\lambda_\perp$ and λ_y are: $\lambda_\perp \approx 0.07$, $\lambda_y \approx -0.182$. When such a gap exists, it is extremely difficult for trajectories to get close to $y=0$. We find that whether such a gap occurs is determined by the characteristics of the chaotic driving in the x invariant subspace. This can be heuristically understood as follows. For y_n small we have $y_{n+1}\approx p x_n y_n$, so $y_n \approx (p^n \prod_{i=0}^{n-1} x_i) y_0$ if $y_n<1$. Thus we are led to consider the sequence in x: $x_0, x_1, x_2, \ldots, x_M$ which satisfies $p^M \prod_{i=0}^{M-1} x_i <1$ and $p^{M+1}\prod_{i=0}^{M} x_i \geqslant 1$. We ask, what is the probability distribution $P(M)$ for the length of the sequence M? If the probability for large M is not negligible, we expect that y_n can be arbitrarily close to $y=0$ and consequently no apparent gap would occur. If, on the other hand, the probability for having large M is practically zero, we would expect a gap. We find [9] that for the logistic driving, $P(M)\sim \exp(-KM)$ for large M, where K is a positive constant. Thus, the probability for having large M is prohibitively small, thereby causing the apparent gap in Fig. 4.

The symmetry-breaking bifurcation observed for the map Eq. (2) can also occur in flows. For instance, we have examined the following four-dimensional flow:

$$dx_1/dt = x_2, \quad (3)$$

$$dx_2/dt = -\gamma x_2 + 4x_1(1-x_1^2) + f_0 \sin x_3,$$

$$dx_3/dt = \omega,$$

$$dy/dt = (2\pi)^{-1}(ax_1+b)\sin(2\pi y) - y,$$

where γ, f_0, ω, and p are parameters. The symmetric subspace is y, whose evolving equation is invariant under the symmetric operation $y \to -y$. The variables (x_1, x_2, x_3) constitute the forced Duffing's system [10] and, therefore, chaotic attractor occurs commonly. The transverse Lyapunov exponent can be computed analytically by solving: $d\delta y/dt = (ax_1 + b - 1)\delta y$. We obtain $\lambda_\perp = b - 1 + a\int x_1 \rho(x_1) dx_1 = b - 1$, where the integral is zero because the invariant density $\rho(x_1)$ of $x_1(t)$ is an even function of x_1. Thus, a symmetry-breaking bifurcation occurs at $b_c = 1$. We have observed such a bifurcation with on-off intermittency for a wide range of parameter values in the Duffing's system that yields a chaotic attractor.

In conclusion, we have presented a scenario for symmetry-breaking bifurcation in chaotic dynamical systems with an invariant subspace. We argue that symmetry-breaking bifurcation occurs (1) if the transverse Lyapunov exponent with respect to the symmetric invariant subspace crosses zero from the negative side; and (2) if the repulsion from the invariant subspace is not too strong so that trajectories in one symmetric component cannot be injected into basins of other coexisting symmetric components. When such a symmetry-breaking bifurcation occurs, the dynamical variables that break the symmetry exhibit on-off intermittency. As a parameter varies further, symmetry-increasing bifurcation occurs when trajectories start switching intermittently among the coexisting symmetric chaotic components. Although we illustrate our main result by using the model Eq. (2), the argument for symmetry-breaking bifurcation to occur does not depend on a specific feature of the model. Similar bifurcations have been observed for a large variety of chaotic dynamics in the invariant subspace, and also for flows.

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[8] For characterization of on-off intermittency, see J. F. Heagy, N. Platt, and S. M. Hammel, Phys. Rev. E **49**, 1140 (1994).

[9] The probability distribution $P(M)$ depends on the invariant density $\rho(x)$ of the chaotic process. If $\rho(x)$ appears to contain an infinite number of singularities, such as that produced by the logistic map at $r=3.8$, $P(M)$ becomes exponentially small for large M. If $\rho(x)$ is smooth, $P(M)$ tends to decay slowly as M increases. This can be qualitatively understood by noting that large M requires x to stay in the small x interval for many iterations. This is not likely when a trajectory tends to visit locations of both small and large x values with singularities in an alternating fashion, such as what happens in the logistic map. On the other hand, if $\rho(x)$ is smooth, the probability for a trajectory to have small x values for a successive number of iterations can be quite appreciable, which would cause large value for M. We find that no gap can be seen in the symmetry-broken attractor if $\rho(x)$ is smooth (unpublished results).

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