Unpredictability of the asymptotic attractors in phase-coupled oscillators

Ying-Cheng Lai
Department of Physics and Astronomy and Department of Mathematics, Kansas Institute for Theoretical and Computational Science, The University of Kansas, Lawrence, Kansas 66045
(Received 31 October 1994)

An array of phase-coupled oscillators may exhibit multiple coexisting chaotic and nonchaotic attractors. The system of coupled circle maps is such an example. We demonstrate that it is common for this type of system to exhibit an extreme type of final state sensitivity in both parameter and phase space. Numerical computations reveal that there exist substantial regions of the parameter space where arbitrarily small perturbations in parameters or initial conditions can alter the asymptotic attractor of the system completely. Consequently, asymptotic attractors of the system cannot be predicted reliably for specific parameter values and initial conditions.

PACS number(s): 05.45.+b

I. INTRODUCTION

Phase-coupled oscillators, oscillators interacting with each other through their phases of oscillation, arise in a variety of biological and physical systems [1–10]. The importance of phase interaction for biological systems was recognized by Winfree [3]. Phase dynamics in cardiac pacemaker cells were subsequently studied by Jalife and Antzelevitch [4]. Systems of coupled oscillators were used by Kopell [5] to understand biological phenomena such as locomotion by fish. A simple class of phase-coupled oscillators, the integrate-and-fire model, was used by Mirollo, Strogatz, and Brailove to study the synchronization phenomenon of certain biological species [6]. Studies also revealed that chaos can occur in these coupled biological oscillators. For instance, chaos was found by Tsuda, Koener, and Shimizu in a phase-coupled physiological neural network model [7]. Glass and Zeng studied a system of two phase-coupled simple cardiac cells and found a variety of complex dynamical phenomena including chaos [8]. In physical situations, systems of phase-coupled oscillators were used by Wiesenfeld and Hadley [9] to model series arrays of Josephson junctions [10]. A detailed numerical study of the dynamics of a family of globally phase-coupled maps was carried out by Kaneko [11].

This paper concerns a study of a family of phase-coupled maps—globally coupled circle maps. The system is written as follows:

$$x_{n+1}(i)=x_n(i)+a\sin[x_n(i)]+\omega+\frac{\delta}{N}\sum_{j=1}^{N}\sin[x_n(j)] \bmod(2\pi),\quad i=1,2,\ldots,N\,, \qquad (1)$$

where n denotes the discrete time, i is an index denoting the discrete spatial sites, and N is the number of coupled maps. In Eq. (1), a and ω are parameters of the circle map, δ is a parameter specifying the coupling strength among oscillators, and $x_n(i)$ is the state variable representing the phase of the oscillator at position i and time n. Notice that the global coupling term is characterized by summation of terms sin(phase), which is common in situations of phase interaction [11]. This is different from the more extensive studied coupled map lattices (e.g., coupled logistic map lattice [12] and coupled Hénon map lattice [13,14]) in the literature, where the coupling terms are usually represented by a simple algebraic sum of some or all state variables. When the coupling vanishes, Eq. (1) reduces to that of N independent circle maps, and it is known that the circle map may exhibit complicated dynamical phenomena such as quasiperiodic and chaotic motions. The "Arnold tongue" phenomenon was discovered in the parameter space of the circle map [15]. The system of coupled circle maps, Eq. (1), was proposed by Wiesenfeld and Hadley [9] as a simple model for understanding the phenomenon of attractor crowding in an array of coupled Josephson junctions which is, more precisely, described by a set of coupled ordinary differential equations [10]. Kaneko subsequently performed detailed studies of a similar class of systems and found the existence of different dynamical phases including coherent, ordered, partially ordered, and turbulent phases [11]. These pioneering studies have revealed that Eq. (1) can exhibit very rich dynamical phenomena.

In this paper, we study Eq. (1) from the viewpoint of predicting the asymptotic attractors at specific parameter values and initial conditions using computers. Our investigation was motivated by a recent finding of an extreme type of sensitive dependence of asymptotic attractors on both parameters and initial conditions in the system of coupled Hénon maps [14]. In certain parameter regimes of the system where there are multiple coexisting asymptotic attractors of different natures (e.g., chaotic, quasiperiodic, or periodic), which attractor the system asymptotes to depends extremely sensitively on the choice of both parameters and initial conditions. Perturbations in parameter and/or initial condition, no matter how small, have a finite probability of completely altering the system's asymptotic attractor. Similar behavior was also observed qualitatively on a system of diffusively coupled ordinary differential equations (the Duffing oscillator)

[14], whose dynamics, however, are more similar to that of a lattice of globally coupled [16] two-dimensional maps when the Poincaré surface of section is examined [17]. In view of the high relevance of phase-coupled oscillator systems in biology and physics, it is important to examine whether phase-coupled oscillator systems such as Eq. (1) would exhibit similar extreme sensitive dependence in parameter and phase space. The main result of this paper is that there exist substantial regions of finite area in parameter space in which there are multiple distinct attractors (chaotic or nonchaotic), and the extreme sensitive dependence so described above occurs. This indicates that extreme sensitive dependence of asymptotic attractors occurs not only in systems where the coupling is described by algebraic sum of state variables [14], but also occurs in systems where individual elements interact with each other through their phases, mathematically represented by the coupling term in Eq. (1). The results of this paper, together with previous ones [14], suggest that extreme sensitivity of the final state may be a common dynamical phenomenon in systems consisting of spatially coupled nonlinear oscillators.

This paper is organized as follows. In Sec. II, we demonstrate that Eq. (1) exhibits extreme sensitive dependence of asymptotic attractors on parameters. In Sec. III, phase-space final state sensitivity is studied. Discussions are presented in Sec. IV.

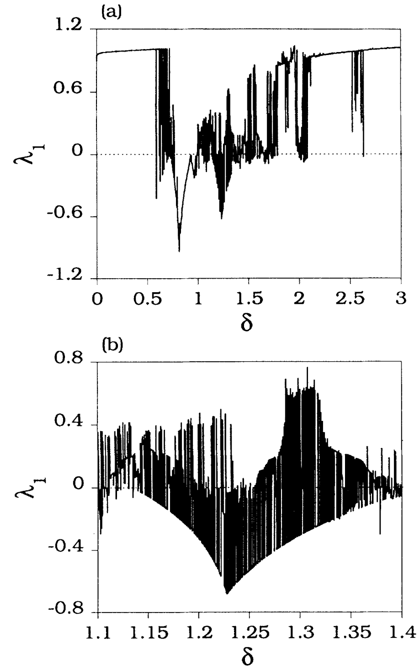

FIG. 1. For the system of globally coupled circle maps Eq. (1), the maximum Lyapunov exponent vs the coupling strength δ for $N=20$, $a=4$, and $\omega=2$. (a) $0 \le \delta \le 3$, (b) $1.1 \le \delta \le 1.4$.

II. EXTREME PARAMETER SPACE SENSITIVITY

The general diagnostic tool to determine the nature of asymptotic attractors is the maximum Lyapunov exponent. For a system of N coupled circle maps, there are N Lyapunov exponents. Let λ_1 be the maximum of these N exponents. For a given parameter and an initial condition, if $\lambda_1 > 0$ (<0), the asymptotic attractor is chaotic (periodic), while $\lambda_1 = 0$ indicates quasiperiodic attractor. In numerical experiments, we compute $\lambda_1(n)$ until $|\lambda_1(n+1) - \lambda_1(n)| < 10^{-8}$, where n is the discrete time, or when the total number of iterations reaches 50 000. The length of the initial transient is chosen to be 10 000. Such a computational setting is in general sufficient to detect different asymptotic attractors, i.e., to distinguish chaotic from quasiperiodic and periodic attractors.

We first consider the case where $a=4$ and $\omega=2$, a parameter setting for which the single circle map has a chaotic attractor [18]. The maximum Lyapunov exponent λ_1 is computed as the coupling δ is increased from zero. Figure 1(a) plots, for $N=20$, λ_1 versus δ, where 1000 values of δ are chosen uniformly in a range $0 \le \delta \le 3$. The set of initial conditions is chosen randomly and then fixed as δ is varied. In general, when δ is small [<0.5 in Fig. 1(a)], the dynamics of maps at different spatial sites are chaotic ($\lambda_1 > 0$) and almost independent of each other. Interesting dynamics occur when the coupling is larger than 0.5 in Fig. 1(a). There are regimes of coupling values in which λ_1 fluctuates wildly with magnitude much greater than that of numerical fluctuations which are almost invisible in Fig. 1(a) (note the smoothness of the curve near $\delta=0$). These wild fluctuations of λ_1 persists as smaller scales of δ variations are examined, as shown in Fig. 1(b), where a blowup of part of Fig. 1(a) for $1.1 \le \delta \le 1.4$ is plotted. Figures 1(a) and 1(b) thus suggest that in certain regimes of δ, the attractor to which the system asymptotes depends extremely sensitively on δ.

Sensitive dependence of asymptotic attractors on parameters can be quantified by the so-called uncertainty exponent α which was introduced by Grebogi *et al.* to characterize fractal basin boundaries arising in dynamical systems that possess multiple attractors [19]. The exponent α is defined as follows. Randomly choose a parameter value δ in a parameter interval that contains parameter subintervals in which λ_1 fluctuates between distinct values. Define $\delta' = \delta + \epsilon$, where ϵ is a small perturbation. Determine whether the asymptotic dynamics of the system using these two parameters are qualitatively different (e.g., chaotic versus periodic). Parameters leading to distinct asymptotic attractors upon small perturbation are called uncertain parameter values. For given perturbation ϵ, a fraction of uncertain parameter values $f(\epsilon)$ can be computed by randomly choosing many δ values and determining if δ is uncertain. For fractal sets, $f(\epsilon)$ typically scales with ϵ as $f(\epsilon) \sim \epsilon^{\alpha}$, where α is the uncertainty exponent [19]. Since ϵ represents the precision with which the parameter δ is specified in numerical computations, α determines the probability that the computed asymptotic behavior inaccurately reflects the true dynamics of the system. If $\alpha > 1$, reducing ϵ can improve the probability of correct computation of the final state. If $\alpha = 1$, improvement in ϵ results in an equal improvement in the probability of correct computation of the final state. If $\alpha < 1$, reduction of ϵ will result in only a small reduction of $f(\epsilon)$. In particular, in the extreme case where $\alpha \approx 0$, improvement in the precision ϵ with which δ is specified even over many orders of magnitude may result in only an incremental improvement in ability to predict the asymptotic state correctly. The uncertainty exponent is equivalent to another exponent introduced by Farmer [20] to characterize sensitive parameter dependence. For the globally coupled Hénon map lattice, it was found that $\alpha \approx 0$ [14], which indicates an extreme sensitive dependence of asymptotic attractors on parameters.

To determine the degree of sensitivity to parameters in Eq. (1), we have computed the uncertainty exponent. Figure 2 plots, for $N=20$, $f(\epsilon)$ versus ϵ in the base-10 logarithmic scale, where the parameter δ was randomly chosen in [0,3] and $f(\epsilon)$ was computed by accumulating the number of uncertain parameter values to 200. Here, a δ value is uncertain if δ yields $\lambda_1 > 0$ (chaotic attractor) and δ' gives $\lambda_1 \leq 0$ (nonchaotic attractor), or vice versa. The slope of the plot in Fig. 2, which is approximately the uncertainty exponent α, is estimated to be $0.000\,53 \pm 0.003\,38$ at a 95% confidence level. Thus, α cannot be distinguished from 0 in this case. Computations with several other values of N yield similar results.

A near-zero uncertainty exponent has a significant consequence regarding our ability to predict, even qualitatively, the type of asymptotic attractors for specific parameter values. Assume that α takes its upper bound value of approximately 0.004 in Fig. 2. Assume that the value of δ can be specified to within 10^{-16}; then there is a probability of $f(\epsilon) \sim 10^{0.004(-16)} \approx 0.86$ that the final asymptotic state computed using δ is incorrect. Improving the precision with which δ is specified offers little improvement in the probability of computing the final state of the system correctly. For example, suppose computer precision is improved by 16 decades to 10^{-32}. Then the probability of incorrectly computing the asymptotic state is still $\sim 10^{0.004(-32)} \approx 0.74$. This means that vast improvement in the computer precision yields almost no improvement in our ability to predict the asymptotic attractor. Such a near-zero uncertainty exponent thus indicates that the system of coupled circle maps Eq. (1) exhibits an extreme sensitive dependence of asymptotic attractors on parameters.

A near-zero uncertainty exponent is in fact a manifestation of riddled parameter space [14,21]. A parameter space is riddled if near every parameter value that leads to chaos, there are parameter values arbitrarily nearby that lead to nonchaotic attractors. To see the riddling of parameter space in Eq. (1), we have examined, for $N=20$, a two-dimensional parameter region defined by $0 \leq a \leq 6$ and $0 \leq \delta \leq 3$. In this region, a 200×200 uniform grid of parameter pairs is chosen and λ_1 is computed for each parameter pair. Figures 3(a)–3(c) show the parameter

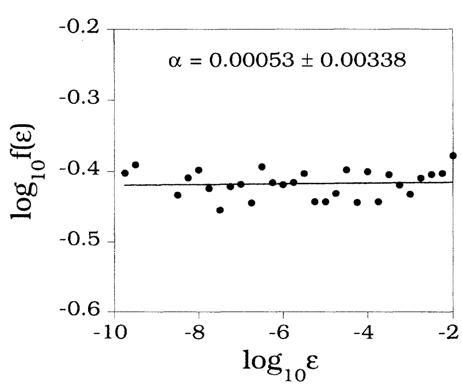

FIG. 2. Plot of $\log_{10} f(\epsilon)$ vs $\log_{10}\epsilon$, where ϵ is a small perturbation in the parameter δ. Other parameter settings are the same as in Fig. 1. The parameter space uncertainty exponent is $\alpha = 0.000\,53 \pm 0.003\,38$, a value that cannot be distinguished from zero.

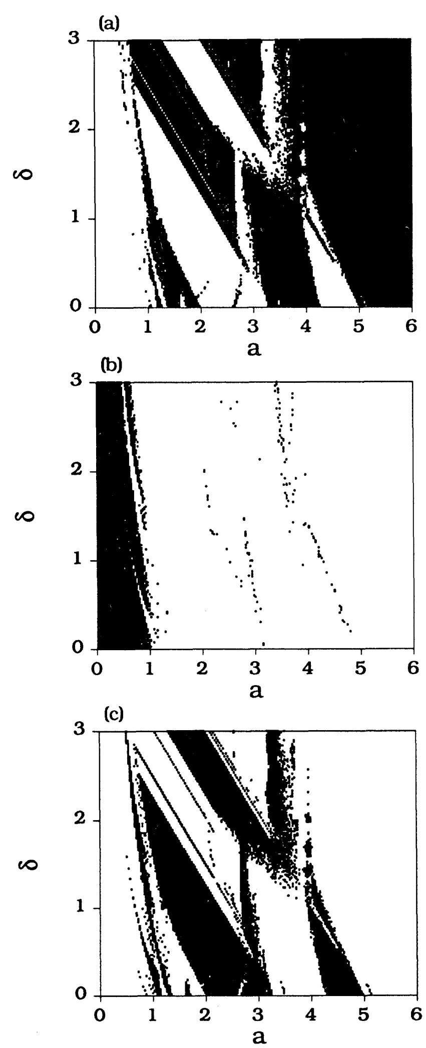

FIG. 3. In the two-dimensional parameter region $0 \leq a \leq 6$ and $0 \leq \delta \leq 3$, parameter pairs that lead to (a) chaotic, (b) quasiperiodic, and (c) periodic attractors. Other parameter settings are the same as in Fig. 1.

pairs that lead to chaotic, quasiperiodic, and periodic attractors, respectively. Chaotic, quasiperiodic, and periodic attractors are determined by the numerical criteria $\lambda_1 > 10^{-5}$, $|\lambda_1| \leq 10^{-5}$, and $\lambda_1 < -10^{-5}$, respectively. In the computation, the set of initial conditions is chosen randomly and then fixed for all 40 000 parameter pairs. Most regions in this two-dimensional parameter space consist of "open sets" of finite area where the asymptotic attractors are chaotic, quasiperiodic, or periodic. In particular, when the coupling $\delta = 0$, the asymptotic attractor is nonchaotic for $a < 1$, and no quasiperiodic attractors exist for $a > 1$. These are known properties of the circle map [15]. Note, however, there are also regions where parameter pairs that lead to chaotic and nonchaotic attractors are mixed (near $a = 3.5$ and $\delta = 1.5$). To better visualize these mixed parameter regions, Figs. 4(a)–4(c) show blowups of Figs. 3(a)–3(c), respectively, in the smaller region $3 \leq a \leq 4$ and $1 \leq \delta \leq 2$. In Fig. 4(a), there are chaotic parameter regions with riddled "holes" corresponding to parameter pairs that lead to nonchaotic attractors. Further blowups of part of Fig. 4(a) exhibit similar riddled structures for chaotic and nonchaotic parameter pairs. Thus the parameter space of the system of coupled circle maps Eq. (1) is riddled [14].

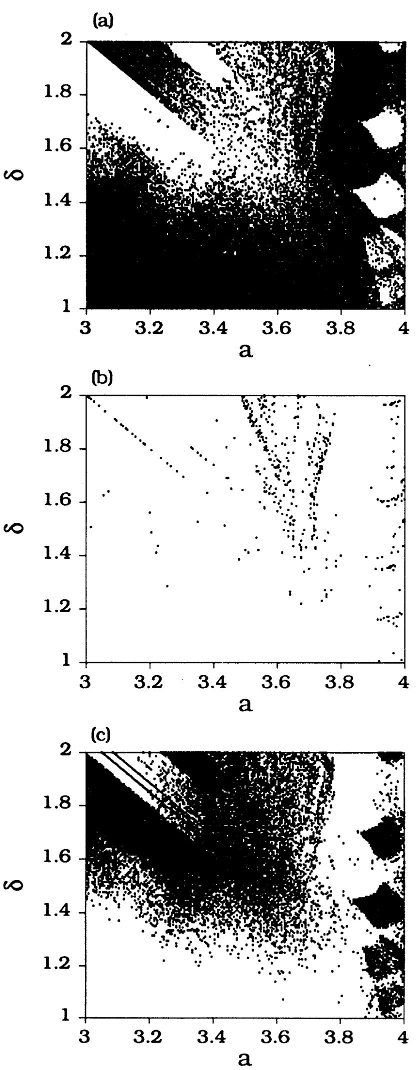

FIG. 4. Blowups of Figs. 3(a)–3(c) for $3 \leq a \leq 4$ and $1 \leq \delta \leq 2$. The parameter space contains riddled regions where for a parameter pair that leads to chaotic attractors, there are parameter pairs arbitrarily nearby that lead to nonchaotic attractors.

III. EXTREME SENSITIVITY IN PHASE SPACE

The extreme sensitive parameter dependence seen above can be related to a similar type of extreme sensitivity of asymptotic attractors on initial conditions in phase space [22,14]. Intuitively, this can be understood by realizing that perturbations in parameter space are equivalent to perturbations in phase space if equations of the system have smooth dependence on both state variables and parameters. Hence, a near-zero uncertainty exponent in the parameter space implies a near-zero uncertainty exponent in phase space. To investigate phase-space sensitivity at fixed parameter values for Eq. (1), we choose a two-dimensional plane among the N phase-space variables and systematically examine the type of attractors resulting from many initial conditions chosen on this plane by computing λ_1 for each initial condition.

Figures 5(a)–5(d) show histograms of λ_1 values resulting from 4096 initial conditions chosen from a uniform 64×64 grid in the region $0 \leq x(8)$, $x(9) \leq 2\pi$, where

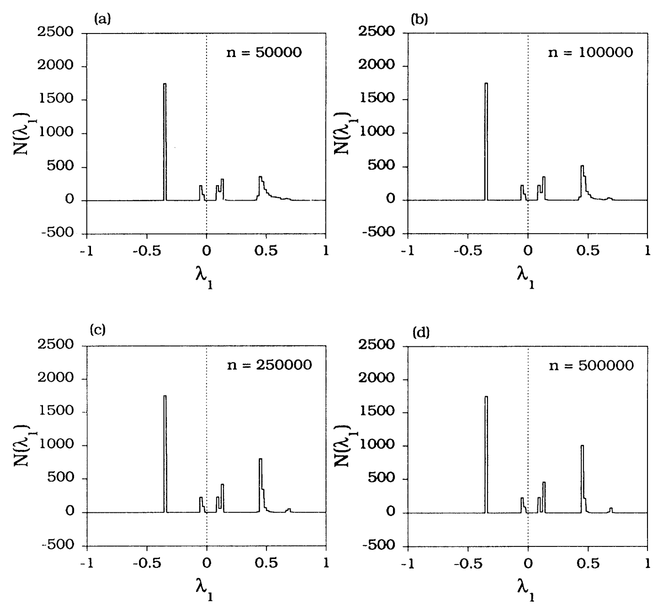

FIG. 5. Histograms of 4096 values of λ_1 computed from a 64×64 grid of initial conditions chosen from $0 \leq x(8) \leq 2\pi$ and $0 \leq x(9) \leq 2\pi$. Parameter settings are $N = 20$, $a = 4$, $\omega = 2$, and $\delta = 1.288$. Histograms are computed at time steps (a) $n = 5 \times 10^4$, (b) $n = 10^5$, (c) $n = 2.5 \times 10^5$, and (d) $n = 5 \times 10^5$. Thee are four distinct chaotic attractors and two distinct periodic attractors.

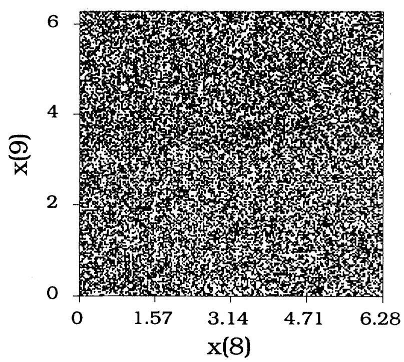

FIG. 6. Basins of the four chaotic attractors in Fig. 5 (black dots) computed using a grid of 200×200 initial conditions on the $x(8)$-$x(9)$ plane. Blank regions are the basin of the two periodic attractors in Fig. 5.

$N=20$, $a=4$, $\omega=2$, and $\delta=1.288$, and the values of $x(j)$ $(j=1,\ldots,N,\ j\neq 8,9)$ are chosen randomly and then fixed. These histograms are obtained at time steps $n=5\times10^4$ (a), 10^5 (b), 2.5×10^5 (c), and 5×10^5 (d), with 10 000 preiterates. There are four peaks with $\lambda_1>0$ and two peak with $\lambda_1<0$, indicating the existence of four distinct chaotic attractors and two distinct periodic attractors. As time progresses, the peaks with $\lambda_1>0$ sharpen, as in Figs. 5(a)–5(d). The total number of initial conditions leading to λ_1 in any of these four positive peaks is unchanged, indicating that these peaks correspond to chaotic attractors rather than superlong chaotic transients [23]. Figure 6 shows the basin of these four chaotic attractors (black dots) in the $x(8)$-$x(9)$ plane, where the blank regions are the basin of the two periodic attractors with $\lambda_1<0$. Clearly, basins of chaotic and periodic attractors appear to be extremely intermingled, similar to the riddled parameter regions shown in Fig. 4(a). For initial conditions that lead to chaotic attractors, there are initial conditions arbitrarily nearby that lead to the periodic attractors. Figure 7 shows $\log_{10}f(\epsilon)$ versus

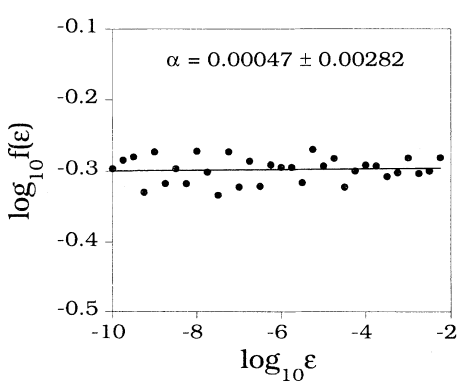

FIG. 7. Plot of $\log_{10}f(\epsilon)$ vs $\log_{10}\epsilon$, where ϵ is the small perturbation chosen from an arbitrary line in the $x(8)$-$x(9)$ plane. Parameter settings are the same as in Fig. 5. The phase-space uncertainty exponent is $\alpha=0.000\,47\pm0.002\,82$, a value that cannot be distinguished from zero.

$\log_{10}\epsilon$, where ϵ is a small perturbation in initial conditions and $f(\epsilon)$ is the fraction of uncertain initial condition at perturbation ϵ. The initial conditions are drawn on an arbitrary line in the $x(8)$-$x(9)$ plane. The phase-space uncertainty exponent is estimated to be $\alpha=0.000\,47\pm0.002\,82$, a value which cannot be distinguished from zero, similar to the uncertainty exponent computed in parameter space. Hence, Eq. (1) also exhibits an extreme sensitive dependence of asymptotic attractors in the phase space.

IV. DISCUSSIONS

Phase-coupled oscillators are spatiotemporal dynamical systems that arise commonly in biological and physical situations where individual oscillators "communicate" with each other through their phases. These systems have been studied extensively in the literature, and it is known that they can exhibit complicated dynamics including quasiperiodic and chaotic motions [1–10] which are common for dissipative dynamical systems. Phase dynamics, such as those described by the circle-type maps, may also exhibit distinct phenomena, such as the "Arnold" tongue [15] and "attractor crowding" [9], which have not been seen in other dissipative systems such as the logistic map and coupled logistic map lattices [12]. The results presented in this paper for the model of a well-studied phase-coupled oscillator [9,11], the system of globally coupled circle maps, indicate that phase-coupled oscillators may exhibit an extreme type of sensitive dependence of asymptotic attractors on initial conditions and parameters. Such a dependence is characterized by near-zero uncertainty exponents in both parameter space and phase space, and it is similar to that found in systems of globally coupled Hénon maps [14]. A consequence of such an extreme sensitive dependency is that even the type of asymptotic attractors (chaotic or nonchaotic) of the system cannot be predicted reliably by using computers with finite precision arithmetic. Statistical properties of the asymptotic attractor, such as the Lyapunov exponents and fractal dimensions, are consequently unpredictable in parameter regimes where this type of dependence occurs.

The precise origin of the extreme unpredictability of the asymptotic attractors for phase-coupled oscillators described in this paper is still not clear at present. A possible mechanism may be due to the occurrence of extremely long chaotic transients in Eq. (1). Such transients were discovered in coupled map lattices [23]. Sagdeev, Usikov, and Zaslavasky [24] also discovered these types of long transients in dissipative dynamical systems that are slightly perturbed away from some Hamiltonian systems. In these systems, trajectories usually behave chaotically for an extremely long time before settling into a regular (nonchaotic) attractor. The long chaotic transients were also called "quasiperiodic" attractors [24,25]. When these transients occur, the effect of numerical roundoff error may become severe over a long period of time. The time within which a trajectory settles into the final nonchaotic attractor depends sensitively on the choice of the initial condition from which the trajectory is originated. While this may explain the extreme sensitive dependence of asymptotic attractors in Eq. (1), it remains unclear whether there are superlong chaotic transients [23] for Eq. (1). Histograms of the largest Lyapunov exponents computed from many initial conditions over 500 000 iterations show no sign of the decay of the peak located at $\lambda_1 > 0$, which would occur if the chaotic sets at $\lambda_1 > 0$ were transients. In fact, the peaks become more localized in the λ_1 axis as time progresses, as shown in Fig. 5. Therefore, if these chaotic sets were transients, their lifetime must be much greater than 500 000 iterations. This is, nonetheless, difficult to verify due to the intensive numerical computations involved.

Another mechanism could be similar to that responsible for the occurrence of riddled basins in certain low-dimensional chaotic systems [21]. For such systems, extreme sensitive dependence of asymptotic attractors on initial conditions characterized by the near-zero uncertainty exponent occurs. Conditions under which riddled basins occur include the existence of a symmetric invariant manifold on which there is a chaotic attractor, the existence of another attractor in the phase space, and the existence of a natural measure zero set on the invariant manifold for which infinitesimal perturbations from the invariant manifold grow exponentially (points in this measure zero set are said to have positive transverse Lyapunov exponents) [21]. As a consequence, for any initial condition in the basin of the chaotic attractor on the invariant manifold, there are points arbitrarily close to it that are in the basin of the other attractor in the phase space. Riddled basins in low-dimensional systems have been fairly well understood, both qualitatively and quantitatively [21]. The system examined in this paper is high dimensional, yet it possesses a high degree of symmetry. Thus, a mechanism similar to that in low-dimensional systems with riddled basins may be responsible for the extreme sensitive dependence described in this paper. However, at present it remains difficult to search for conditions discussed in Ref. [21] under which riddled basins occur, particularly the existence of a symmetric manifold, for the phase-coupled oscillators Eq. (1) due to its high dimensionality.

ACKNOWLEDGMENTS

We thank the Army High Performance Computing Center at the University of Minnesota for providing computer CPU time on the Connection Machine CM5. This work was partially supported by the Kansas Institute for Theoretical and Computational Science at the University of Kansas through the NSF/K*STAR EPSCoR Program.

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