Stabilizing chaotic-scattering trajectories using control

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The method of stabilizing unstable periodic orbits in chaotic dynamical systems by Ott, Grebogi, and Yorke (OGY) is applied to control chaotic scattering in Hamiltonian systems. In particular, we consider the case of *nonhyperbolic* chaotic scattering, where there exist Kolmogorov-Arnold-Moser (KAM) surfaces in the scattering region. It is found that for short unstable periodic orbits not close to the KAM surfaces, both the probability that a particle can be controlled and the average time to achieve control are determined by the initial exponential decay rate of particles in the hyperbolic component. For periodic orbits near the KAM surfaces, due to the stickiness effect of the KAM surfaces on particle trajectories, the average time to achieve control can greatly exceed that determined by the hyperbolic component. The applicability of the OGY method to stabilize intermediate complexes of classical scattering systems is suggested.

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I. INTRODUCTION

Chaotic scattering is an important physical process [1] in which some output variables characterizing the particle trajectory after the scattering display a sensitive dependence on some input variables characterizing the particle trajectory before the scattering. In other words, arbitrarily small changes in the input variables can cause large changes in the output variables. It is now understood that chaotic scattering is due to nonattracting chaotic invariant sets in the phase space. Those sets embed within themselves an infinite number of unstable periodic orbits [1].

Figure 1 shows schematically three potential hills [2] denoted by A, B, and C that can give rise to chaotic scattering. The potential is appreciable and positive in the scattering region but is negligible at large distances. A particle incident from $x=-\infty$ can be trapped in the scattering region for a finite amount of time before leaving the region. By appropriately choosing the potential form and arranging the configuration of the potential hills, it is possible to form an infinite number of unstable periodic orbits in the scattering region [2]. In the phase space, the closure of these orbits is a nonattracting chaotic invariant set responsible for the chaotic scattering. In this case, a line segment in phase space crossing the stable manifold of the chaotic invariant set contains a Cantor set of intersecting points. Particles initiating from the Cantor set will spend an infinite amount of time in the scattering region wandering on the chaotic invariant set. Nonetheless, this Cantor set has Lebesgue measure zero, indicating that almost all initial conditions for the scattering particle are not on the Cantor set but in the neighborhood of it. Consequently, typical particles incident on the scattering region will be trapped for only a *finite amount of time.* There are practical situations (e.g., chemical reactions) in which it is desirable to keep a particle in the scattering region for an arbitrary long time. We thus address the following question: Can one keep a particle in the scattering region for as long as one wishes by arbitrarily small variations of an available system parameter? A solution to this problem is to stabilize the scattering trajectory in the neighborhood of some unstable periodic orbit in the scattering region.

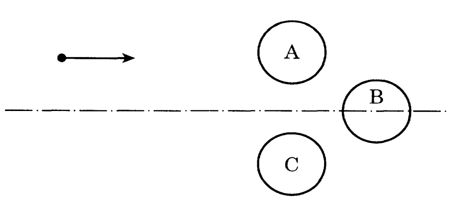

FIG. 1. A schematic illustration of a potential system that can exhibit chaotic scattering.

Control of chaos using unstable periodic orbits embedded in a chaotic attractor was proposed by Ott, Grebogi, and Yorke (OGY) [3]. The basic idea of the OGY method is as follows. First one chooses an unstable periodic orbit embedded in the chaotic set, the one which yields the best system performance according to some criteria. Second, one defines a small region around the desired periodic orbit. Due to ergodicity, a chaotic tra-

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jectory will eventually fall into this small region. When this occurs, small, judiciously chosen temporal parameter perturbations are applied to force the trajectory to approach the unstable periodic orbit. This method is extremely flexible because it allows for the stabilization of different periodic orbits, depending on one's needs, for the same set of nominal values of the parameter. Note that for chaotic attractors, the probability that a chaotic trajectory enters the neighborhood of the desired unstable periodic orbit is one [3]. For the case of transient chaos [4] (e.g., chaotic scattering), the chaotic set is nonattracting and, hence, almost all initial conditions escape the chaotic set except a set of measure zero. Therefore, to stabilize unstable periodic orbits embedded in this set, one must launch an *ensemble* of initial conditions towards the chaotic set. There is a finite probability that some of these initial conditions enter the neighborhood of the desired unstable periodic orbit and can then be stabilized. In this spirit, one controls transient chaos [5] and hence, controls chaotic scattering.

While controlling transient chaos has been successfully demonstrated [5], the previous study concentrated on dissipative transient chaos. Although the strategy of Ref. [5] can be trivially applied to the case of *hyperbolic* chaotic scattering, the case of *nonhyperbolic* chaotic scattering is nontrivial and is worth investigating. Here by "hyperbolic chaotic scattering" we mean that all periodic orbits are unstable and there are no Kolmogorov-Arnold-Moser (KAM) surfaces in the scattering region and, by "nonhyperbolic chaotic scattering" we mean that KAM surfaces coexist with chaotic invariant sets. The purpose of this paper is to study the control of nonhyperbolic chaotic scattering.

It is known [6] that the stickiness effect of the KAM surfaces leads to an algebraic decay of particles from the chaotic region [6] at large times. Our major finding is that for stabilizing short unstable periodic orbits not close to KAM surfaces, the KAM surfaces practically have no effect on the probability that a particle can be controlled and on the average time to achieve the control. The probability that a particle can be controlled is defined as the ratio of the number of particles entering the neighborhood of the target unstable periodic orbit to the total number of particles launched towards the scattering region. The major contribution to the percentage of initial conditions that can be controlled results from the *hyperbolic* component which gives rise to the initial *exponential* decay of particles from the scattering region. The average time to achieve control appears to be bounded by the inverse of the exponential decay rate, as if there were no KAM surfaces. (This bound of the average controlling time has been previously demonstrated in the case of controlling dissipative transient chaos [5].) While for stabilizing unstable periodic orbits near KAM surfaces, the stickiness effect of the KAM surfaces has an influence on scattering trajectories approaching the periodic orbits. In this case, the average time to achieve control can greatly exceed the time determined by the hyperbolic component.

Another issue concerns the possible occurrence of complex-conjugate eigenvalues (on the unit circle) of unstable periodic orbits in Hamiltonian systems. In this situation, we choose a modified algorithm for controlling Hamiltonian chaos [7], which makes use of the stable and unstable directions of periodic orbits.

At this point, it is worth comparing our approach to control chaotic scattering with another one worked out by Lu *et al.* [8]. They investigated the scattering between two atoms interacting via a Morse potential in the presence of a laser field. The scattering is nonhyperbolic because there are KAM surfaces in the phase space. The undriven system is integrable. By switching on the driving force (the electromagnetic field of the laser), trajectories can be trapped by a newly created stable resonance island. In this type of control the stabilized state is *inside* a KAM surface and is, thus, a state which is not accessible by scattering trajectories in the permanently driven system. Note, furthermore, that the perturbation (the driving force) must be *strong* in such cases, and there is, therefore, no correspondence between the controlled and uncontrolled states. This type of approach is different from ours based on the OGY method where a precise tailoring of the controlling process is possible by applying weak perturbations around preselected unstable periodic orbits in the chaotic sea of scattering trajectories.

The organization of the paper is as follows. In Sec. II we review briefly the OGY method and the modified algorithm [7] that is particularly suitable for Hamiltonian systems. In Sec. III we describe a simple scattering model, the Gaspard-Rice scattering map [9] (which will be used as an illustrative example), and demonstrate that our modified control algorithm is quite effective to stabilize unstable periodic orbits. In Sec. IV we discuss the effect of KAM surfaces. In Sec. V, we present the conclusion.

II. THE METHOD OF CONTROL

We consider a chaotic scattering system that can be described by the following two-dimensional symplectic map (since energy is conserved) on the Poincaré surface of section,

$$\mathbf{X}_{n+1}=\mathbf{F}(\mathbf{X}_n,p)\ , \tag{1}$$

where n denotes the discrete time, $\mathbf{X}_n \in \mathbb{R}^2$, $\mathbf{F}$ is a smooth function in its variables and $p \in \mathbb{R}$ is an externally accessible parameter. For a scattering system, p can be, say, the amplitude or the frequency of some external electromagnetic field. Since we do not want to change the dynamics substantially, we restrict parameter perturbations to be small. In other words, we require

$$|p-p_0|<\delta\ , \tag{2}$$

where p_0 is some nominal parameter value and δ is a small number defining the range of parameter variation. To achieve the control, we launch particles originating from an ensemble of initial conditions towards the scattering center. Most of the particles will be scattered away from the system without getting close to the target periodic orbit. Our objective is to program the parameter p in such a way that trajectories which do enter the neighborhood of the target periodic orbit are stabilized, where by "neighborhood" we mean a small region whose size is proportional to δ.

Specifically, let the unstable orbit of period m to be controlled be $\mathbf{X}_{O_1}(p_0)\to\mathbf{X}_{O_2}(p_0)\to\cdots\to\mathbf{X}_{O_m}(p_0)\to\mathbf{X}_{O_{(m+1)}}(p_0)=\mathbf{X}_{O_1}(p_0)$. The linearized dynamics in the neighborhood of the period-m orbit is

$$\mathbf{X}_{n+1}-\mathbf{X}_{O_{(n+1)}}(p_0)=\mathbf{A}[\mathbf{X}_n-\mathbf{X}_{O_n}(p_0)]+\mathbf{B}(\Delta p)_n \ , \qquad (3)$$

where $p_n=p_0+(\Delta p)_n$, $(\Delta p)_n\le\delta$, $\mathbf{A}$ is the 2×2 Jacobian matrix, and $\mathbf{B}$ is a two-dimensional column vector,

$$\begin{aligned}\mathbf{A}&=\mathbf{D}_X\mathbf{F}(\mathbf{X},p)|_{\mathbf{X}=\mathbf{X}_{O_{(n)}},p=p_0} \ ,\\ \mathbf{B}&=\mathbf{D}_p\mathbf{F}(\mathbf{X},p)|_{\mathbf{X}=\mathbf{X}_{O_{(n)}},p=p_0} \ .\end{aligned} \qquad (4)$$

In Eq. (4), we do not express the Jacobian matrix $\mathbf{A}$ in terms of eigenvalues and eigenvectors because there may exist complex-conjugate eigenvalues on the unit circle at some of the periodic points for Hamiltonian systems [7]. Instead we explore the stable and unstable directions associated with these points. The stable and unstable directions do not necessarily coincide with the eigenvectors at a given periodic point if $m\neq 1$. In the case of complex-conjugate eigenvalues, eigenvectors are not even defined in the real plane. The existence of both stable and unstable directions around each orbit point can be seen as follows. Let us choose a small circle of radius ϵ around some orbit point $\mathbf{X}_{O_n}$. The image of a small circle under $\mathbf{F}^{-1}$ is an ellipse at $\mathbf{X}_{O_{(n-1)}}$. This deformation from a circle to an ellipse means that distances along the major axis of the ellipse at $\mathbf{X}_{O_{(n-1)}}$ contract as a result of the map. Similarly, a small circle at $\mathbf{X}_{O_{(n-1)}}$ maps into an ellipse at $\mathbf{X}_{O_n}$ under $\mathbf{F}$, which means that distances along the inverse image of the major axis of the ellipse at $\mathbf{X}_{O_n}$ expand under $\mathbf{F}$. Therefore, the forward image of the major axis of the ellipse at $\mathbf{X}_{O_{(n-1)}}$ and, the major axis of the ellipse at $\mathbf{X}_{O_n}$ (the image of the small circle at $\mathbf{X}_{O_{(n-1)}}$) approximate the stable and unstable directions at $\mathbf{X}_{O_n}$, respectively. See Refs. [7,10] for a systematic algorithm to compute stable and unstable directions for general two-dimensional maps.

Let $\mathbf{e}_{s(n)}$ and $\mathbf{e}_{u(n)}$ be the stable and unstable directions at $\mathbf{X}_{O_n}$ and, $\mathbf{f}_{s(n)}$ and $\mathbf{f}_{u(n)}$ be two vectors that satisfy $\mathbf{f}_{u(n)}\cdot\mathbf{e}_{u(n)}=\mathbf{f}_{s(n)}\cdot\mathbf{e}_{s(n)}=1$ and $\mathbf{f}_{u(n)}\cdot\mathbf{e}_{s(n)}=\mathbf{f}_{s(n)}\cdot\mathbf{e}_{u(n)}=0$. To stabilize the orbit, we require the next iteration of a trajectory point after falling into one of the small neighborhood around $\mathbf{X}_{O_n}$ to lie on the stable direction at $\mathbf{X}_{O_{(n+1)}}(p_0)$, i.e.,

$$[\mathbf{X}_{n+1}-\mathbf{X}_{O_{(n+1)}}(p_0)]\cdot\mathbf{f}_{u(n+1)}=0 \ . \qquad (5)$$

Substituting Eq. (3) into Eq. (5), we obtain the following expression for the parameter perturbations:

$$(\Delta p)_n=\frac{\{A[X_n-\mathbf{X}_{O_{(n)}}(p_0)]\}\cdot\mathbf{f}_{u(n+1)}}{-\mathbf{B}\cdot\mathbf{f}_{u(n+1)}} \ . \qquad (6)$$

We emphasize that parameter perturbations calculated from Eq. (6) are applied to the system at *each* time step, thus minimizing the effect of external noise [3,11]. It should also be noted that there are other ways to stabilize unstable periodic orbits, which are applicable to higher-dimensional systems, one of them is the "pole-placement method" [12].

III. CONTROLLING THE GASPARD-RICE SCATTERING SYSTEM

We illustrate our control algorithm Eq. (6) via a simple model of chaotic scattering due to Gaspard and Rice [9]. Originally, the model was used to study the dynamics of fragmenting molecules. The model describes a free one-dimensional particle subject to periodic kicks. To achieve the control, it is necessary to have some externally adjustable parameter. The parameter that quantifies the kicking strength is a natural choice. We also allow for the kicking frequency to be used as a control parameter. The system can be described by the following Hamiltonian:

$$H(X,P)=P^2/2+T_0G(X)\sum_{i=-\infty}^{\infty}\delta(t-T_i) \ , \qquad (7)$$

where the mass of the particle is unit, T_0 is a constant, the sequence $\{T_i\}$ is the occurrence time of the kicks, and $T_0G(X)$ is the amplitude of the kick at position X in the Hamiltonian. Let (X_n,P_n) be the position and momentum of the particle (dynamical variables) before the nth kick. Immediately before the $(n+1)$th kick, the dynamical variables are given by the following area-preserving map:

$$\begin{aligned}P_{n+1}&=P_n-T_0\frac{dG(X_n)}{dX_n} \ ,\\ X_{n+1}&=X_n+T_nP_{n+1} \ .\end{aligned} \qquad (8)$$

In order for the toy model Eq. (8) to exhibit scattering dynamics, it is necessary to choose $G(X)$ such that $dG(X)/dX$ vanishes at large distance. Gaspard and Rice chose the following form [9] for $G(X)$:

$$G(X)=D(1-e^{-aX})^2 \ , \qquad (9)$$

where D and a are parameters. After the following scaling

$$P_n\to P_n/(aT_0) \ , \quad X_n\to X_n/a \ , \qquad (10)$$

the map becomes

$$\begin{aligned}P_{n+1}&=P_n-d(e^{-X_n}-e^{-2X_n}) \ ,\\ X_{n+1}&=X_n+\frac{T_n}{T_0}P_{n+1} \ ,\end{aligned} \qquad (11)$$

where $d=2a^2T_0^2D$ is proportional to the kicking strength. In Eq. (11), both d and T_n can serve as the control parameter.

Depending on the value of d and T_n, the map (11) displays different dynamical behaviors. For $T_n=T_0$, Gaspard and Rice demonstrated that the map produces both nonhyperbolic and hyperbolic chaotic scattering [9] when d is in different ranges. In particular, for $0<d<d_c \approx 4.58$, there are both KAM surfaces and a nonscattering chaotic invariant set in the phase space (nonhyperbolic chaotic scattering). While for $d>d_c$, all the KAM surfaces are apparently destroyed and the scattering is hyperbolic. To perform the control we choose the system when $d=1.8$. At this d value, there is a large KAM island surrounded by a chaotic set, as shown in Fig. 2(a). The stable and unstable manifolds of the chaotic invariant set are shown in Figs. 2(b) and 2(c), respectively. The chaotic invariant set and its stable and unstable manifolds are numerically obtained by the "sprinkler method" [13,4]. By this method, we first choose a grid of 1500×1500 initial conditions in a region determined by $X \in [-1,6]$ and $P \in [-1.5,1.5]$. We then set a critical time n_0 (integer) and keep track of particles with trajectories $\{X_n, P_n\}$, $n=0,1,\ldots,n_0$ that escape the scattering region in time longer than n_0. When the dynamical variables satisfy $X \geq 10$ and $P>0$ we consider that the particles have escaped the scattering region. The chaotic invariant set and its stable and unstable manifolds are approximately the sets determined by the points $[X_n,P_n]|_{n=n_0/2}$, $[X_n,P_n]|_{n=0}$, and $[X_n,P_n]|_{n=n_0}$, respectively, of all trapped trajectories (n_0 is chosen to be 40 in our numerical experiments). Generally, particles initiated from $X=+\infty$ with $P<0$ approach the scattering region along the stable manifold, wander in the chaotic set, and eventually leave the scattering region ($P>0$) along the unstable manifold. When looking for an appropriate periodic orbit to stabilize, we first observe that there is one single fixed point in the system ($X=0, P=0$). Its existence does not depend on any parameter. The origin is, therefore, uncontrollable by parameter perturbation. We thus have to take a longer periodic orbit to control, which immediately involves the problem of the occurrence of complex-conjugate local eigenvalues on the unit circle. The shortest unstable periodic orbits found are 5-cycles. Also shown in Figs. 2(a)–2(c) are locations of a period-5 orbit that we set out to stabilize. The numbers 1–5 denote the order in which the components of the periodic orbit are visited under the map. We find that the eigenvalues of the Jacobian matrices at the components 1 and 4 are complex conjugates to each other and are on the unit circle. Therefore, it is preferable to use the control method described in Sec. II and Ref. [7].

To achieve control, we take an ensemble of particles simulating what is going on in a physical scattering process: we launch a large number of particles towards the scattering region by choosing different P values (negative) at a fixed position $X=8$. Most of the particles wander in the scattering region for a finite time and then escape. Among all the particles, any one entering the neighborhood of the period-5 orbit (defined to be a ball of radius $\epsilon=0.01$ around each component) will be stabilized. Figures 3(a) and 3(b) show an example, where a particle starting from $X=8$ and $P=-4.398\,117$ is stabilized by perturbing the parameter d according to Eq. (6). The transient time for the particle to enter the neighborhood of the period-5 orbit is 21. The parameter control based on Eq. (6) is turned on at the same time ($n=21$). The range of the parameter perturbation is chosen to be $\delta=0.01$. Specifically, Figs. 3(a) and 3(b) show the X_n-versus-n and P_n-versus-n time series, respectively, where the stabilized X-versus-n curve shows only three distinct X_n values because two pairs of the period-5 orbit are degenerate in the X axis [see Fig. 2(a)]. Most of the parti-

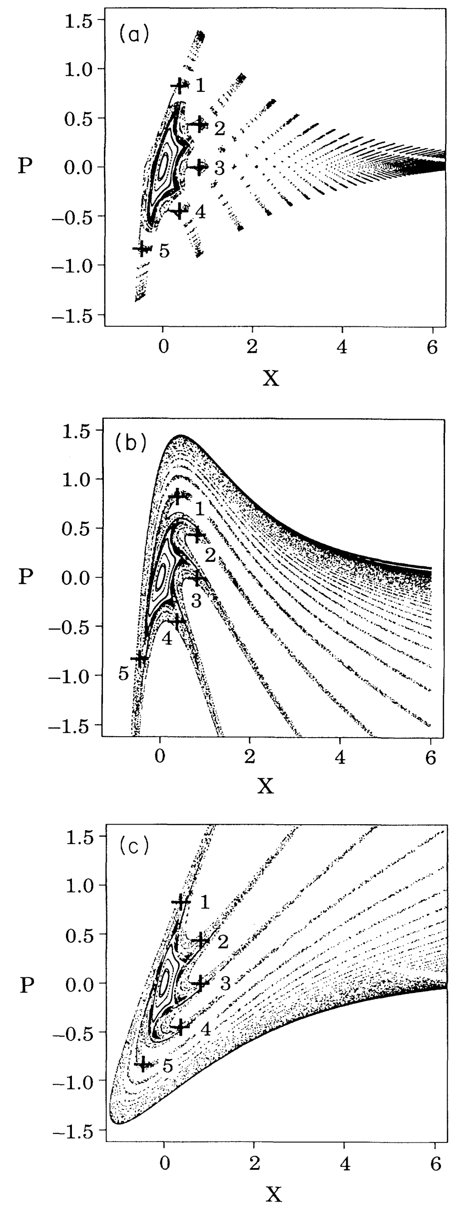

FIG. 2. For the Gaspard-Rice scattering system of Eq. (11) with $T_n=T_0$ and $d=1.8$, (a) the chaotic invariant set, (b) the stable manifold, and (c) the unstable manifold of the chaotic set. In (a)–(c) there is also a KAM island. Crosses denote the components of the period-5 orbit chosen to be stabilized. The numbers 1–5 denote the sequence of the component under the map. Component 1 is sitting at ($X_0 \approx 0.369\,97, P_0 \approx 0.828\,66$). Its Jacobian matrix (as well as that of component 4) possesses complex-conjugate pairs of eigenvalues on the unit circle. The unstable eigenvalue of the entire cycle has been found to be 16.318, and, correspondingly, $\lambda_u=2.7923$.

cles that can enter the neighborhood of the period-5 orbit have similar short transient times $n \leq 40$. This indicates that those particles have not experienced the stickiness of the KAM surfaces (see Sec. IV for details). However, long transient times are also possible, although these cases are rare for this period-5 orbit. Figures 3(c) and 3(d) show such a situation, where the transient time is $n=133$ for a particle with initial condition $(X,P)=(8,-9.072\,119)$.

Using d as a control parameter corresponds to an amplitude modulation at a fixed frequency T_0^{-1}. A control via frequency modulation is also possible by keeping in Eq. (8) $T_n = T_0$ as long as the particle is outside a given neighborhood of the period-5 orbit, and by choosing $\Delta p_n \equiv T_n - T_0$ at fixed d as the perturbation according to Eq. (6) when the particle is inside the neighborhood. This leads to the same results.

Although the above strategy to stabilize unstable periodic orbits works in the case of chaotic scattering (transient chaos), we stress that we still face two major problems. The first is the influence of the external noise, which can kick the already stabilized trajectory out of the neighborhood of the periodic orbit. In the case of controlling chaotic attractors, this is not as severe because the particle trajectory reenters the stabilizing region after a finite transient time and can then be controlled again [3]. However, in the case of controlling chaotic scattering, the effect of noise could be disastrous because, when the noise kicks the particle out of the stabilizing region, it is most likely that the particle will escape the system and never return. Increasing the size of the stabilizing region around the periodic orbit (or equivalently, the magnitude of the parameter perturbation) might alleviate the problem, because by doing so we reduce the probability for the noise to kick out the particle. Another possible solution is to induce some feedback mechanism to reinject the particle into the scattering region when it is kicked out. The effect of noise, which could be so drastic for a single trajectory, is much weaker for the entire ensemble considered. It might happen that noise kicks such particles into the stabilizing region that would not be controlled in the deterministic case. This shows again that the control of chaotic scattering can only be realistic by using beams containing large numbers of particles.

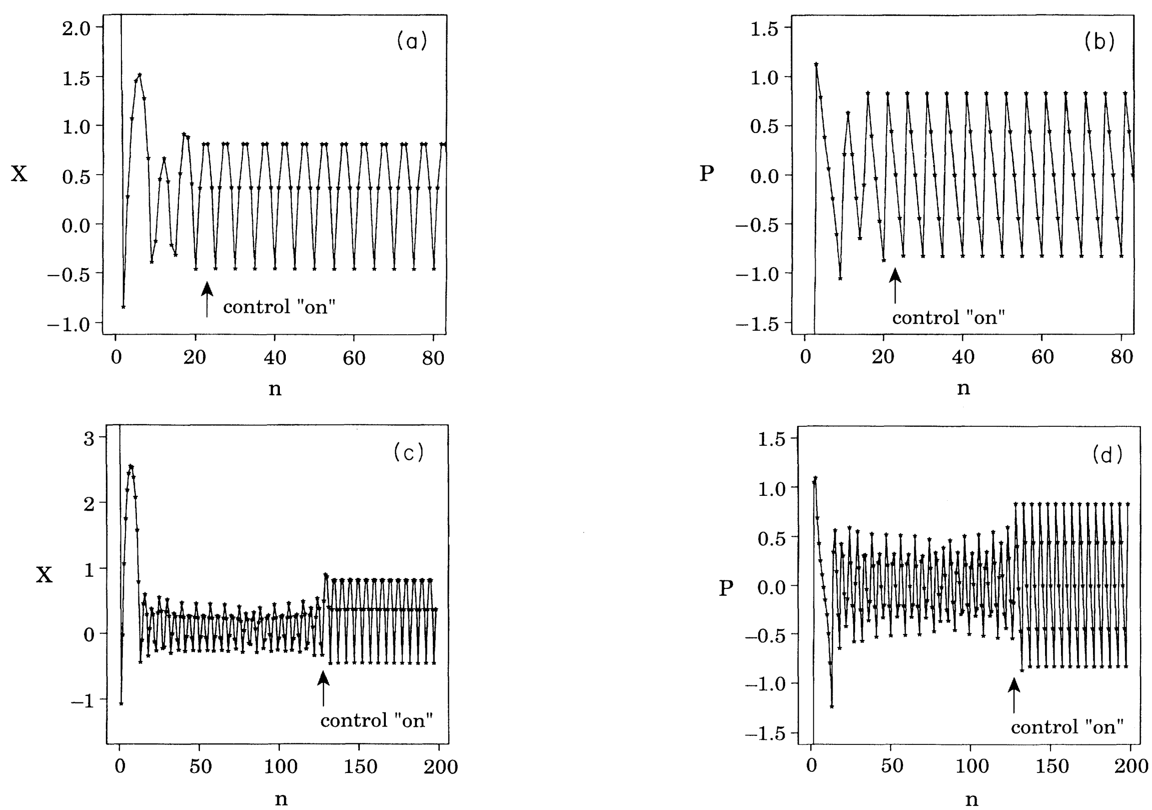

FIG. 3. Two examples of stabilizing an unstable period-5 orbit for the Gaspard-Rice scattering system, as shown in Figs. 2(a)–2(c). In the first example, a particle starting from $X=8$ and $P=-4.398\,117$ is stabilized around the periodic orbit by perturbing d in Eq. (11). The transient time is 21 and the maximum allowed parameter perturbation is 0.01. (a) The X_n-vs-n and (b) P_n-vs-n times series before and after control. This rather short transient time indicates that the particle has not experienced the KAM surfaces before being controlled (see Sec. IV). In the second example, a particle starting from $X=8$ and $P=-9.072\,119$ is stabilized around the periodic orbit. The transient time is 133. The maximum allowed parameter perturbation for d is still 0.01. (c) The X_n-vs-n and (d) P_n-vs-n time series before and after control. This relatively long transient time reflects the influence of the KAM surfaces.

The second problem concerns the probability that a particle can be stabilized. As we mentioned in Sec. I, in the case of a chaotic attractor virtually any initial condition can be stabilized, while in controlling chaotic scattering only a tiny fraction of initial conditions can be stabilized. Depending on the size of the stabilizing region, the number of trajectories entering this region obeys a scaling relation (to be discussed in Sec. IV). Devising a scheme that uses a feedback mechanism (like the case of controlling a chaotic attractor [3]) in a noisy environment and at the same time increases significantly the probability that a particle can be controlled has yet to be achieved.

IV. INFLUENCE OF KAM SURFACES

In controlling dissipative transient chaos, the number of trajectories that fall in some ϵ neighborhood of a periodic orbit scales with ϵ algebraically [5],

$$N(\epsilon)\sim\epsilon^{\alpha(\gamma)}\ , \tag{12}$$

where $\alpha(\gamma)$ is a scaling exponent that depends on the (exponential) escape rate γ of particles. The escape rate γ is the reciprocal value of the averaged scattering time. In the case where the neighborhood is chosen as a ball of radius proportional to the magnitude of the parameter perturbation δ, the scaling exponent is given by [3,5]

$$\alpha(\gamma)=1-\frac{\lambda_u-\gamma}{\lambda_s}\ , \tag{13}$$

where λ_s and λ_u are the negative and positive Lyapunov exponents of the periodic orbit to be stabilized, respectively. For a Hamiltonian system $\lambda_s+\lambda_u=0$, and Eq. (13) becomes

$$\alpha(\gamma)=2-\frac{\gamma}{\lambda_u}\ . \tag{14}$$

In nonhyperbolic chaotic scattering, due to the coexistence of a chaotic invariant set and KAM surfaces, particles decay from the scattering region exponentially initially and then algebraically [6]. The reason is that, initially, particles only wander in the chaotic set without experiencing KAM surfaces, and consequently they decay exponentially, as in hyperbolic chaotic scattering. For large times, it is more probable that particles wander in the vicinity of KAM surfaces. The stickiness effect of KAM surfaces then gives rise to the algebraic decay law. Figures 4(a) and 4(b) show the exponential and algebraic decay in short- and long-time scales, respectively, corresponding to the phase-space structure of Figs. 2(a)–2(c) for the Gaspard-Rice map. To obtain Figs. 4(a) and 4(b), we launch 10^5 particles at $X=8$ with P uniformly distributed in $[-10,0]$ and record the particles that have not exited the system ($X\leq 10$) at time t. The short-time-scale exponential decay rate is found to be $\gamma\approx 0.042$.

The scaling exponent α in Eq. (14) depends on the exponential decay rate γ. In the case of algebraic decay, however, such a decay rate is not defined. The fact that the unstable period-5 orbit lies in the chaotic invariant set (hyperbolic component) and it is not in the immediate vicinity of the KAM island [Figs. 2(a)–2(c)] suggests that the major contribution to the probability that a particle can be controlled is mainly determined by the short-time exponential decay rate γ. This is expected by noting in Figs. 4(a) and 4(b) that the number of particles has decreased by 99% at the crossover time $t_c\approx 40$ when the algebraic decay starts. Hence, it is most probable that particles getting close to the period-5 orbit only wander in the hyperbolic component without experiencing the KAM island. Although a particle may stay in the scattering region for $t>t_c$ wandering near the KAM island and then enter the neighborhood of the period-5 orbit, we expect such a probability to be extremely small. Figure 5(a) shows, on a logarithmic scale, the $N(\epsilon)$-versus-ϵ plot, where 10^7 particles are launched at $X=8$ with their momenta uniformly distributed in $(-10,0)$. Such a plot can be fit by a straight line with slope 1.92. Using the short-time-scale exponential decay exponent in Eq. (14), we found that $\alpha\approx 1.98$. So indeed, the major contribution to $N(\epsilon)$ comes from the chaotic component, as if there were

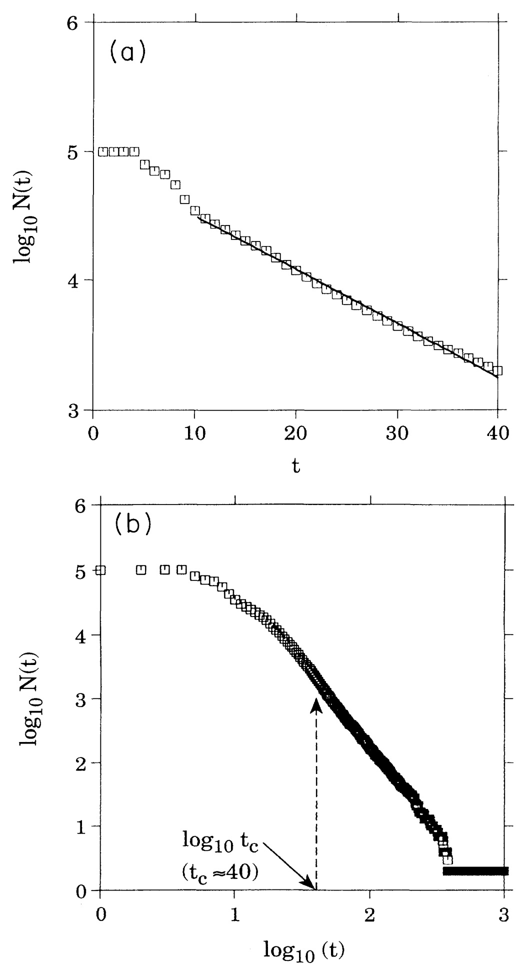

FIG. 4. (a) The short-time exponential decay and (b) the long-time algebraic decay corresponding to the phase-space structure [Figs. 2(a)–2(c)]. The short-time-scale exponential decay rate is approximately $\gamma=0.042$.

no KAM surfaces.

For controlling dissipative transient chaos, it was found [5] that the average time to achieve the control $\tau(\epsilon)$ satisfies

$$\tau(\epsilon) \leq 1/\gamma \ . \tag{15}$$

That is, $\tau(\epsilon)$ is bounded by the inverse of the exponential decay rate. For our case of nonhyperbolic chaotic scattering, this relation appears to hold, as shown in Fig. 5(b), with the average scattering time of the hyperbolic component as the upper bound. This further suggests that the stickiness effect of KAM surfaces plays virtually no role on the average time to achieve control.

On the other hand, when an unstable periodic orbit is in the vicinity of the KAM surfaces, we expect the stickiness effect of the KAM surfaces to be important for particle trajectories that enter the neighborhood of the periodic orbit. In this case, naturally, the average time to achieve control can greatly exceed that determined by the hyperbolic component. Figures 6(a)–6(d) show an unstable period-9 orbit (denoted by crosses) near the KAM surfaces. Figure 6(a) has the same scale as in Figs. 2(a)–2(c) so that one can see the closeness of this period-9 orbit to the KAM surfaces, as compared with the period-5 orbit in Figs. 2(a)–2(c). Figure 6(b) is a blowup of Fig. 6(a), where the numbers denote the iterates of the orbit under the map. Figures 6(c) and 6(d) show the $N(\epsilon)$ and $\tau(\epsilon)$-versus-ϵ plots, respectively, where 10^8 particles are launched at $X=8$ with their momenta uniformly distributed in $(-10,0)$. Clearly, the scaling relation (12) still holds in this case. This is expected because, although the stickiness of the KAM surfaces can greatly increase the average time that a particle can stay in the scattering region, the probability that a particle can be controlled depends only on the number of particles entering the ϵ neighborhood of the periodic orbit, no matter how long it takes for each individual particle to enter this neighborhood. Note, however, that the exponent is less than $\alpha(\gamma)$ computed from Eq. (14) with the Lyapunov exponent $\lambda_u = 1.43$ of the period-9 orbit. The slope of the fitting (straight) line in Fig. 6(c) is 1.78, which is also less than that (1.92) in Fig. 5(a). This means that in this case, it is easier for the particles to enter the neighborhood of the periodic orbit. Nonetheless, the average time to achieve control is indeed much larger than $1/\gamma$, as can be seen in Fig. 6(d).

The effect of KAM surfaces in controlling nonhyperbolic chaotic scattering, as discussed above, depends on the particular, though typical, phase-space structure shown in Figs. 2(a)–2(c) in which the chaotic component and KAM surfaces appear to exist in the same "layer" [6]. However, in typical Hamiltonian systems, the phase space could also be divided into layered components that are separated from each other by Cantori [14]. In such a case, our observation that $N(\epsilon)$ and $\tau(\epsilon)$ depend only on the chaotic component for controlling short periodic orbits not close to the KAM surfaces should still apply, if the ensemble of initial conditions and the target unstable periodic orbit are in the same layer. On the other hand, if initial conditions and the target periodic orbit are separated from each other by Cantori, we expect the probability for some particle to penetrate the Cantori and enter the neighborhood of the target to be extremely small. How to achieve control in layered Hamiltonian phase-space structures remains an open question.

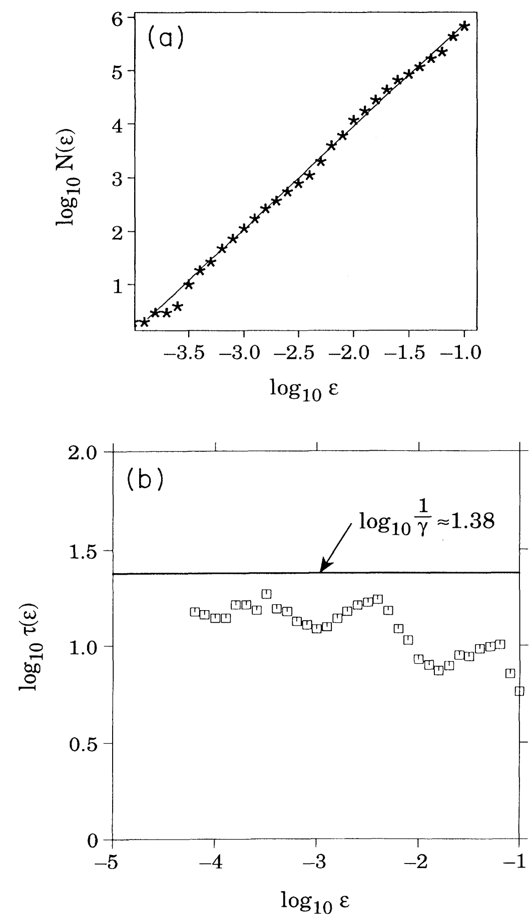

FIG. 5. (a) The number $N(\epsilon)$ of trajectories that enter an ϵ neighborhood of the unstable periodic orbit vs ϵ. (b) The average time to achieve the control vs ϵ. Note that this average time is bounded by the inverse of the short-time-scale exponential decay rate. In this case, we launch 10^7 particles at $X=8$ with their momentum values uniformly distributed in $(-10,0)$ and ϵ has been changed in the range $(10^{-4.2}, 10^{-1})$. Particles are considered to have exited the system if $X > 10$. Both (a) and (b) are on a logarithmic scale.

V. CONCLUSION

In this work, we demonstrate the applicability of the OGY controlling chaos method to the more complicated case (as compared with hyperbolic chaotic scattering) of nonhyperbolic chaotic scattering. We incorporate an algorithm that makes use of the stable and unstable directions at unstable periodic orbits to resolve the difficulty associated with the existence of complex-conjugate eigenvalues (on the unit circle) arising in Hamiltonian systems.

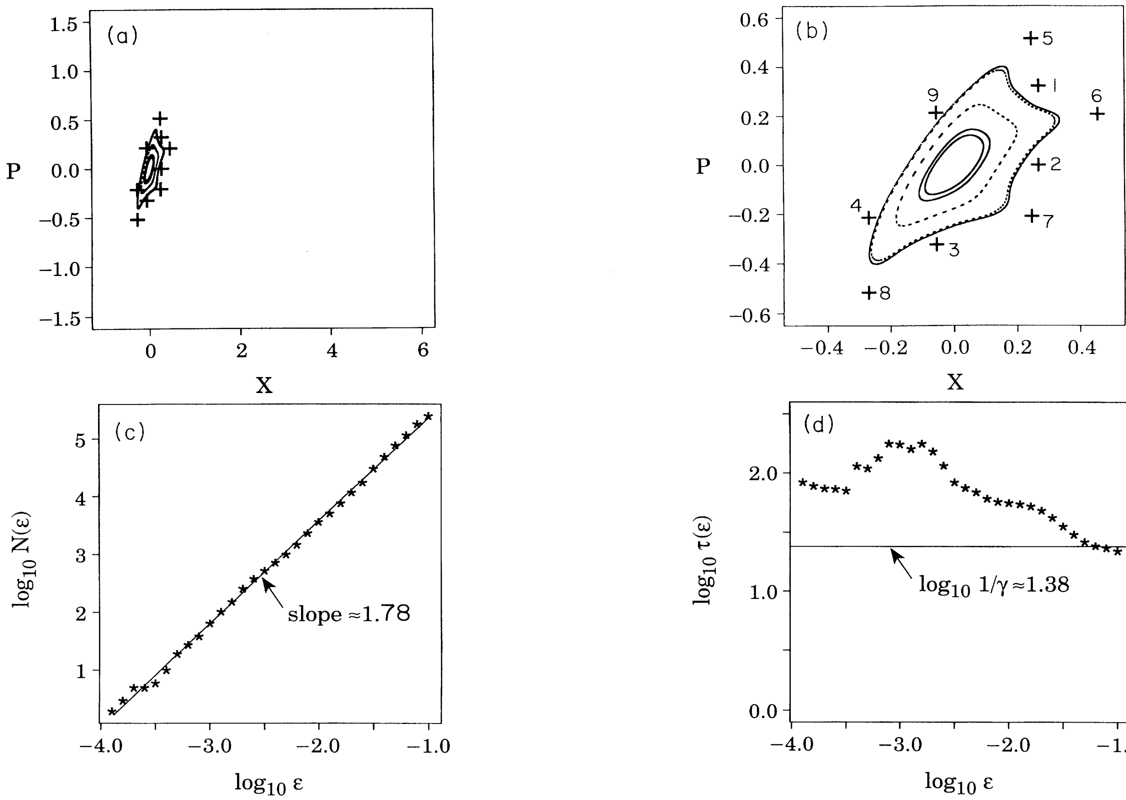

FIG. 6. (a) A period-9 orbit in the vicinity of the KAM surfaces. The scale is the same as in Figs. 2(a)–2(c). (b) A blowup of (a). The numbers denote the iterates of the orbit under the map. The positive Lyapunov exponent of the orbit is $\lambda_u = 1.4334$. (c) The number $N(\epsilon)$ of trajectories that fall in an ϵ neighborhood of the period-9 orbit vs ϵ. The slope of the fitting line is approximately 1.78. (d) The average time to achieve control $\tau(\epsilon)$ vs ϵ. In this case, $\tau(\epsilon)$ can greatly exceed $1/\gamma$. In this figure, we launch 10^8 particles at $X = 8$ with their momentum values uniformly distributed in $(-10,0)$, and ϵ has been changed in the range $(10^{-4}, 10^{-1})$. Particles are considered to have exited the system if $X > 10$. Both (c) and (d) are on a logarithmic scale.

Our method thus provides us with the possibility of stabilizing intermediate complexes of driven classical scattering systems in time-periodic states. Our major finding is that if the unstable periodic orbit to be stabilized is not close to the KAM surfaces, the probability that a particle can be controlled and the average time to achieve the control depend mainly on the dynamical properties of the hyperbolic component, as if there were no KAM surfaces, while for periodic orbits in the vicinity of the KAM surfaces, the stickiness effect of the KAM surfaces can be very influential for control.

More generally, we can say that the invariant set underlying chaotic scattering appears to be the union of a hyperbolic component, equivalent to a chaotic saddle, and another part consisting of the KAM surfaces and their close neighborhoods. It might often happen that trajectories escaping the scattering region relatively soon do not feel the influence of KAM surfaces. The scattering behavior is then different for different time scales. For times shorter than some crossover value t_c the effect of the chaotic saddle dominates and the system behaves as if it were hyperbolic, while the really asymptotic behavior $t > t_c$ is governed by the nonhyperbolic KAM surfaces. Our observation shows that the control of periodic orbits lying away from the KAM surfaces is a short-time effect on chaotic scattering, and the control of periodic orbits in the vicinity of the KAM surfaces can be a long-time effect.

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[1] D. L. Rod, J. Diff. Eqs. **14**, 129 (1973); R. C. Churchill, G. Pecelli, and D. L. Rod, *ibid.* **17**, 329 (1975); M. C. Gutzwiller, Physica D **7**, 341 (1983); D. W. Noid, S. Gray, and S. A. Rice, J. Chem. Phys. **84**, 2649 (1986); C. Jung, J. Phys. A **19**, 1345 (1986); M. Hénon, Icarus **66**, 536 (1986); Physica D **33**, 132 (1988); B. Eckhardt and H. Aref, Trans. R. Soc. London Ser. A **326**, 655 (1988); B. Eckhardt, Europhys. Lett. **61**, 329 (1988); P. Gaspard and S. A. Rice, J.

Chem. Phys. **90**, 2225 (1989); Z. Kovács and T. Tél, Phys. Rev. Lett. **64**, 1617 (1990).

[2] C. Jung and H. J. Scholz, J. Phys. A **20**, 3607 (1987); **21**, 2301 (1988); S. Bleher, E. Ott, and C. Grebogi, Phys. Rev. Lett. **63**, 919 (1989); S. Bleher, C. Grebogi, and E. Ott, Physica D **46**, 87 (1990); M. Ding, C. Grebogi, E. Ott, and J. A. Yorke, Phys. Rev. A **42**, 7025 (1990).

[3] E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990); in *Chaos: Soviet-American Perspectives on Nonlinear Science,* edited by D. K. Campbell (AIP, New York, 1990).

[4] T. Tél, "Transient Chaos," in *Directions in Chaos 3,* edited by Bai-Lin Hao (World-Scientific, Singapore, 1991).

[5] T. Tél, J. Phys. A **24**, L1359 (1991); T. Tél, Int. J. Bif. Chaos (to be published).

[6] C. F. F. Karney, Physica D **8**, 360 (1983); B. V. Chirikov and D. L. Shepelyansky, *ibid.* **13**, 395 (1984); J. D. Meiss and E. Ott, Phys. Rev. Lett. **55**, 2741 (1985); Y. C. Lai, M. Ding, C. Grebogi, and R. Blümel, Phys. Rev. A **46**, 4661 (1992).

[7] Y. C. Lai, M. Ding, and C. Grebogi, Phys. Rev. E **47**, 86 (1993).

[8] Z. M. Lu, M. Valliéres, J. M. Yuan, and J. F. Heagy, Phys. Rev. A **45**, 5512 (1992).

[9] P. Gaspard and S. A. Rice, J. Phys. Chem. **93**, 6947 (1989).

[10] Y. C. Lai, C. Grebogi, J. A. Yorke, and I. Kan (to be published).

[11] Note that a trajectory originally in the neighborhood of a period-m orbit leaves the orbit exponentially like $\exp(\lambda_u m)$, where $\exp(\lambda_u) > 1$ is the unstable Lyapunov number of the orbit. Under the influence of noise, if control is applied at every mth time step, the trajectory will deviate from the orbit rapidly before the next parameter control can be applied, and one may lose control. Hence, it is desirable that the control be applied at *each* time step.

[12] F. J. Romeiras, C. Grebogi, E. Ott, and W. P. Dayawansa, Physica D **58**, 165 (1992).

[13] H. Kantz and P. Grassberger, Physica D **17**, 75 (1985); G. Hsu, E. Ott, and C. Grebogi, Phys. Lett. A **127**, 199 (1988).

[14] R. S. Mackay, J. D. Meiss, and I. C. Percival, Phys. Rev. Lett. **52**, 697 (1984).