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Detecting unstable periodic orbits in high-dimensional chaotic systems from time series: Reconstruction meeting with adaptation

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Detecting unstable periodic orbits (UPOs) in chaotic systems based solely on time series is a fundamental but extremely challenging problem in nonlinear dynamics. Previous approaches were applicable but mostly for low-dimensional chaotic systems. We develop a framework, integrating approximation theory of neural networks and adaptive synchronization, to address the problem of time-series-based detection of UPOs in *high-dimensional chaotic systems*. An example of finding UPOs from the classic Mackey-Glass equation is presented.

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The most fundamental building blocks of any chaotic set, attracting or nonattracting, are unstable periodic orbits (UPOs) [1]. Consider, for example, a chaotic attractor. The motion of a typical trajectory can be regarded as consisting of intermittent "epochs" of visits to the neighborhoods of various UPOs and, as a result, the natural measure of the attractor is determined by the unstable eigenvalues of the UPOs [2]. A similar picture arises for nonattracting chaotic sets leading to transient chaos [3], in that the natural measure of such a set can be characterized by UPOs in a way similar to that for chaotic attractors [4]. UPOs are also pivotal for many other areas of research, such as controlling chaos [5], where a central task is to stabilize the system about some UPO that gives rise to desirable performance. In the field of quantum chaos, the celebrated Gutzwiller formula expresses the quantum density of states in terms of classical periodic orbits [6]. It is no surprise then that investigations of UPOs played an extremely important role in the development of nonlinear dynamics and chaos.

In the experimental study of nonlinear systems, a common situation is that the system equations are not known but one is interested in detecting UPOs. Consequently, one must rely on measured time series to accomplish this task, and there has been a significant amount of previous work on this topic [7–12], where some pioneering approaches were based on the recurrence of chaotic trajectories in the reconstructed phase space [7–9], including the approach articulated by Kostelich and Lathrop (LK) [8]. In particular, given a time series, one first reconstructs a phase-space trajectory by using Takens delay-coordinate-embedding method [13]. One next follows the phase-space evolution and records the recurrence time, the time that it takes for the trajectory to return to a small neighborhood of some recurrent point. Statistical significance test can then be conducted to determine whether the recurrent point belongs to some UPO. This approach is not only applicable to chaotic attractors but also to detecting UPOs from transiently chaotic systems where only short segments of informative time series are available [10]. In spite of its wide usage, a basic limitation of the LK method lies in the difficulty with high-dimensional chaotic systems. This is especially the case when detection of UPOs of long periods is attempted, due to the difficulty to identify long recurrences. In fact, due to the basic characteristics of the dynamical recurrences in chaotic systems, the LK method is best suited for detecting UPOs from low-dimensional chaotic systems. The problem of detecting UPOs in high-dimensional chaotic systems remains outstanding in applied nonlinear dynamics.

In this paper, we articulate a general method to detect UPOs in high-dimensional chaotic systems by integrating the approximation theory of neural networks [14,15] and adaptive delayed feedback control [16,17]. In particular, our method consists of three steps: (1) reconstructing from time series the phase space of the underlying system using the standard delay-coordinate embedding technique, (2) adaptively training proper neural networks in a bounded region in the phase space to obtain an estimate of the vector field of the underlying system, and (3) using adaptive control or synchronization to detect UPOs. We demonstrate that our method is capable of detecting UPOs from high-dimensional chaotic systems modeled, for example, by delay-differential equations. We expect our method to find applications in experimental study of high-dimensional nonlinear dynamical systems.

We consider nonlinear dynamical systems described by $\dot{\boldsymbol{x}} = F(\boldsymbol{x})$, where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector function. We assume that the system has a chaotic attractor $\mathcal{A}$ contained in some bounded set $\Omega \subset \mathbb{R}^n$ in the phase space and the n-dimensional state variable obeys the constraint $\boldsymbol{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^\top \in \Omega \subset \mathbb{R}^n$. The output of the system is defined as $\boldsymbol{y}(t) = \boldsymbol{h}(\boldsymbol{x}(t))$, where $\boldsymbol{h}$ is a smooth observable function. In the special case where $\boldsymbol{h}$ is an identity function, each component of the state variable can be measured: $\boldsymbol{y}(t) = \boldsymbol{x}(t)$. In experimental situations, however, not all but only a subset of the dynamical variables can be measured; i.e., $\boldsymbol{y}(t) = h(x_{j_1}(t), x_{j_2}(t), \ldots, x_{j_q}(t))$, where the arguments in h are $q(<n)$ components selected from the n-dimensional state variable. In this case, we use the classic Takens embedding theory to reconstruct the phase space. To be illustrative, we consider the case where h is a scalar function, i.e., only a scalar time series is available, which is denoted by $y(t) = h(\boldsymbol{x})$. The reconstructed vector is $\boldsymbol{z}(t) = [y(t), y(t+\tau), \ldots, y(t+(L-1)\tau)]^\top \in \mathbb{R}^L$, where L is the embedding dimension and τ is a properly chosen delay time. The vector time series $\boldsymbol{z}(t)$

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can be regarded as being generated by the following "virtual" dynamical system: $\dot{z} = \bar{\boldsymbol{F}}(z)$, where $\bar{\boldsymbol{F}} : \mathbb{R}^L \to \mathbb{R}^L$ is a vector field. In general, for faithful reconstruction, the vector time series $z(t)$ needs to cover the underlying chaotic attractor $\bar{\mathcal{A}} \subset \Psi$, where Ψ is a bounded set in $\mathbb{R}^L$.

We next estimate the vector field $\bar{\boldsymbol{F}}$ by using the approximation theory of neural networks [14,15], which stipulates that, for each component of $\bar{\boldsymbol{F}}$, say $\bar{F}_i (i = 1,\ldots,L)$, and arbitrary small $\delta > 0$, there exists a radial-basis function (RBF) neural network $\mathcal{N}_i(z)$, such that the inequality $|\mathcal{N}_i(z) - \bar{F}_i(z)| < \delta$ holds for all $z \in \Psi$. The vector field can then be approximated by all the properly weighted RBF neural networks. The weights of the neural networks can be obtained through a standard training process using the method of adaptive synchronization and parameter estimation [17]. To be concrete, we write $\mathcal{N} = [\mathcal{N}_1,\mathcal{N}_2,\ldots,\mathcal{N}_n]$, which can be further written in a compact way as $\mathcal{N}(\boldsymbol{\zeta},\boldsymbol{P},z) = \boldsymbol{P} \times G(\boldsymbol{\zeta},z)$, where $\boldsymbol{\zeta} = [\zeta_1,\zeta_2,\ldots,\zeta_m]$ with all $\zeta_j \in \mathbb{R}^n$ distributed evenly in the phase-space area Ψ, $\boldsymbol{P} = \{p_{ij}\}_{L\times m}$ is a weight matrix, and $G(\boldsymbol{\zeta},z) = [g_1(z),g_2(z),\ldots,g_m(z)]^\top$ with each $g_j(z) = \exp(-\lambda_j\|z-\zeta_j\|^2)$ being a Gaussian type of RBF centered at ζ_j. The problem of training the weight matrix $\boldsymbol{P}$ thus becomes a problem of adaptive synchronization and parameter estimation for the following system:

$$\dot{\boldsymbol{u}} = \boldsymbol{P} \times G(\boldsymbol{\zeta},\boldsymbol{u}) + \boldsymbol{K}\cdot(\boldsymbol{u} - z), \tag{1}$$

where $\boldsymbol{K}\cdot(z - \boldsymbol{u})$ is a feedback coupling term and each element of $\boldsymbol{K} = \mathrm{diag}\{k_1,\ldots,k_n\}$ represents a dynamic coupling strength. The adaptive rules for the dynamic coupling strengths and weight parameters can be taken as

$$\begin{aligned} \dot{k}_i &= -\delta_i(u_i - z_i)^2, \quad \dot{p}_{ij} = -r_{ij}g_j(\boldsymbol{u})(u_i - z_i), \\ i &= 1,2,\ldots,L, \quad j = 1,2,\ldots,m, \end{aligned} \tag{2}$$

where δ_i, r_{ij} are constants that can be adjusted to achieve an optimal convergence rate. According to the approximation theory and the adaptive approach, the weight matrix $\boldsymbol{P}$ will fluctuate about a constant matrix $\bar{\boldsymbol{P}} = [\bar{p}_{ij}]_{n\times m}$ when the system $\boldsymbol{u}(t)$ nearly synchronizes with the measured time series $z(t)$. After the adaptive synchronization training, the vector field $\bar{\boldsymbol{F}}$ can be reconstructed in the following sense: For some small $\delta > 0$, there exists a positive time t_0 such that for $t > t_0$,

$$\left|\sum_{j=1}^{m} \bar{p}_{ij}g_j(z(t)) - \bar{F}_i(z(t))\right| < \delta, \quad i = 1,2,\ldots,L, \tag{3}$$

for $z \in \bar{\mathcal{A}}$. Consequently, the estimated system for z can be expressed as

$$\dot{z} = \bar{\boldsymbol{P}} \times G(\boldsymbol{\zeta},z). \tag{4}$$

In general, the vector field associated with system (4) satisfies the condition (3) only with respect to the measured time series. However, because of ergodicity of chaotic trajectories in the region of $\bar{\mathcal{A}}$, the validity of the condition (3) in $\bar{\mathcal{A}}$ is guaranteed.

To detect UPOs in the estimated system (4), we utilize the adaptive delayed feedback control (ADFC) method [17], which requires no knowledge about the periods of the UPOs in advance. In particular, we introduce an ADFC term into system (4):

$$\dot{z} = \bar{\boldsymbol{P}} \times G(\boldsymbol{\zeta},z) + C(t), \tag{5}$$

where $C(t)$ is a feedback control term chosen to be $C(t) = \boldsymbol{\Gamma}(t)\cdot[z(t-\tau) - z(t)]$. For convenience, all components of the control term except one can be set to zero, i.e., $\boldsymbol{\Gamma} = \mathrm{diag}\{0,\ldots,0,\gamma_i,0,\ldots,0\}$. In order to maintain boundedness of the controlled system and noninvasiveness of the ADFC method [17], we need to consider some truncated function and

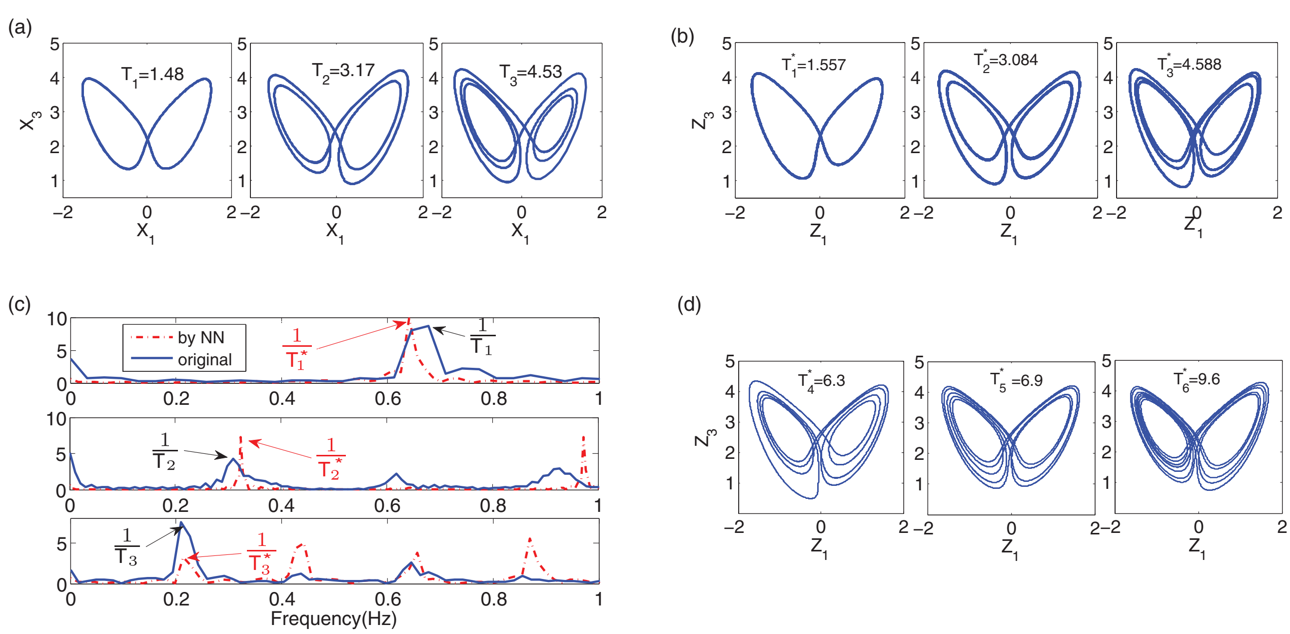

FIG. 1. (Color online) (a) Three UPOs of periods $T_1 = 1.48$, $T_2 = 3.17$, and $T_3 = 4.53$, computed directly from the original Lorenz system. (b) The corresponding UPOs of approximately the same periods ($T_1^* = 1.557$, $T_2^* = 3.084$, and $T_3^* = 4.588$, respectively), detected adaptively from the measured data based on system (6). (c) Spectra of the UPOs in panels (a) and (b). (d) Three detected UPOs of longer periods: $T_4^* = 6.3$, $T_5^* = 6.9$, and $T_6^* = 9.6$.

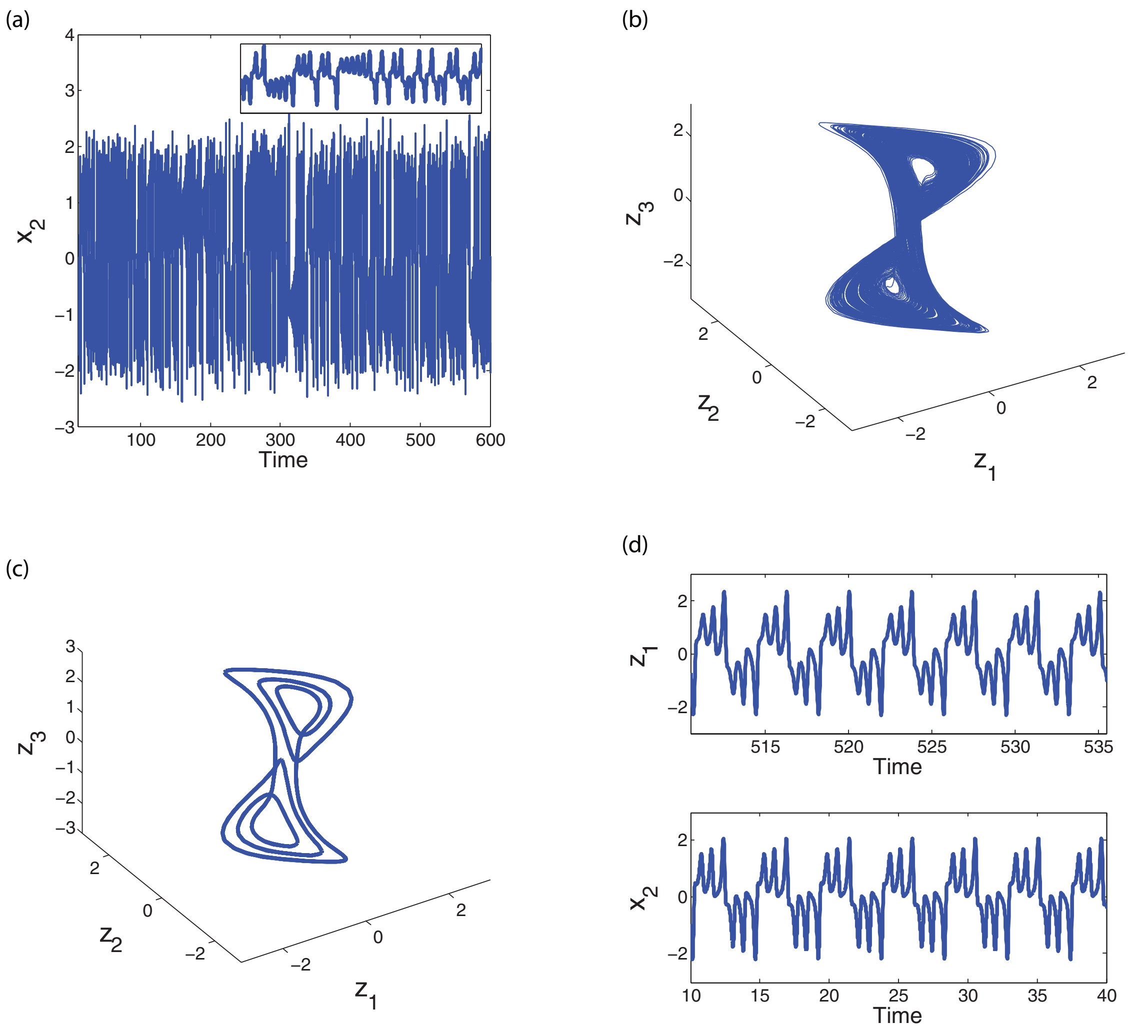

FIG. 2. (Color online) (a) Time series $x_2(t)$ from the Lorenz system, (b) reconstructed attractor, (c) an UPO detected from the reconstructed attractor, and (d) time evolution of UPO trajectories, where the upper panel exhibits the detected UPO in panel (c) and the lower panel is the directly computed period-T_3 UPO shown in Fig. 1(a).

impulsive strategy by adaptively adjusting the control strength and the delay variable τ according to

$$\dot{\tau} = -r_1[z_i(t-\tau) - z_i(t)], \quad \dot{\gamma}_i = r_2[z_i(t-\tau) - z_i(t)]^2,$$

where r_1 and r_2 are positive constants that can be adjusted to achieve an optimal convergence rate. The controlled system (5) will asymptotically converge to one of the UPOs and, at the same time, τ approaches the period of the UPO.

To test our method, we first conduct benchmark test using the classic Lorenz system [18] for which many UPOs have been reported [19]: $[\dot{x}_1,\dot{x}_2,\dot{x}_3] = [10(x_2 - x_1), -10x_1x_3 + 28x_1 - x_3, 10x_1x_2 - 8x_3/3]$. The system has a chaotic attractor in the compact set $\Psi = [-2,2] \times [-3,3] \times [0,5]$. Assume that time series from the state variable $\boldsymbol{x}(t) = [x_1(t), x_2(t), x_3(t)]^\top$ are available. We choose the centers ζ_j of the RBF neural networks $\mathcal{N}$ as the grid points of Ψ with grid size 1 and construct the networks $\mathcal{N}(\boldsymbol{\zeta}, \boldsymbol{P}, \boldsymbol{x})$ by setting $g_j(\boldsymbol{x}) = \exp(-\|\boldsymbol{x} - \zeta_j\|^2/8)$. With the measured data of $\boldsymbol{x}(t)$ and the configuration of the RBF neural networks $\mathcal{N}(\boldsymbol{\zeta}, \boldsymbol{P}, \boldsymbol{x})$, we train the weight matrix $\boldsymbol{P}$ according to the rules in Eq. (2). After obtaining the trained networks $\mathcal{N}(\boldsymbol{\zeta}, \bar{\boldsymbol{P}}, \boldsymbol{x})$, we utilize the ADFC method to find the UPOs. Specifically, the trained network system with ADFC is

$$\begin{aligned} \dot{z}_1 &= \sum_{j=1}^{m} \bar{p}_{1j} g_j(z), \quad \dot{z}_2 = \sum_{j=1}^{m} \bar{p}_{2j} g_j(z) + C(t), \\ \dot{z}_3 &= \sum_{j=1}^{m} \bar{p}_{3j} g_j(z), \quad \dot{\tau} = -r_1\{z_2[t - \tau(t)] - z_2(t)\}, \qquad (6) \\ \dot{\gamma} &= r_2\{z_2[t - \tau(t)] - z_2(t)\}^2. \end{aligned}$$

To guarantee the boundedness of the controlled system, we use the impulsive strategy proposed in Ref. [17] and take the truncated function $C(t) = I_{\{|S(t)| < C_0\}} S(t) + C_0 I_{\{S(t) > C_0\}} - C_0 I_{\{S(t) < -C_0\}}$, where I_B represents the indication function of the set B, $S(t) = \gamma(t)\{z_2[t - \tau(t)] - z_2(t)\}$, and $C_0 = 0.1$ is an arbitrarily small constant. Starting from different initial conditions, various UPOs can be found from system (6). A comparison between these detected UPOs with the UPOs computed directly from the original Lorenz system indicates no significant difference, as shown in Figs. 1(a) and 1(b). The similarity between the detected and the original UPOs can also be seen in the frequency domain, as shown in Fig. 1(c). UPOs of much longer periods can also be found, as illustrated in Fig. 1(d).

We next assume that only a scalar time series from the Lorenz system is available, say $y(t) = h(\boldsymbol{x}(t)) = x_2(t)$, as shown in Fig. 2(a). Application of the delayed embedding method leads to a reconstructed vector with components $z_1(t) = y(t)$, $z_2(t) = y(t - \tau_0)$, and $z_3(t) = y(t - 2\tau_0)$, where we choose $\tau_0 = 0.1$. A reconstructed attractor is shown in Fig. 2(b). We set up and adaptively train the RBF neural networks using $z = [z_1, z_2, z_3]^\top$. A detected UPO is shown in Fig. 2(c). Figure 2(d) shows that there is no qualitative difference between the detected UPO and the corresponding one computed directly from the original Lorenz system. This result indicates that our UPO-detection strategy is applicable to common experimental situations where only partial information about the dynamical variable of the underlying system is available.

We then consider the realistic situation where time series are measured at lower sampling rate and are subject to noise by using the Hindmarsh-Rose system [20] with a chaotic attractor: $[\dot{x}, \dot{y}, \dot{z}] = [2y - x^3/4 + 5.84x^2/2 - 0.2z + 5.98, 1.0 - 5x^2/4 - y, 0.4(0.5x + 1.6) - 0.01z]$, which describes the spiking-bursting behavior of the membrane potential x of a single neuron, where y and z represent the transport rates of ions across the membrane through the ion channels. Figure 3(a) shows the chaotic attractor. We first use $\hat{\Box}(t) = \Box(t)[1 + n(t)]$ to generate the time series perturbed by multiplicative noise, where $n(t)$ is a white noise term of strength 0.001, and $\Box \equiv x, y$, or z. In addition, in each unit time interval, we sample 10 points from $[\hat{x}(t), \hat{y}(t), \hat{z}(t)]$ to generate the measured data, denoted by $[\tilde{x}(t), \tilde{y}(t), \tilde{z}(t)]$, as shown in Fig. 3(b). The corresponding sampling rate is 10 Hz, which is far below the frequency of about 10^3 Hz of the original time series in Fig. 3(a). As shown in Figs. 3(c)–3(d), the detected UPO is qualitatively consistent with the UPO computed directly from the original Hindmarsh-Rose neuron model. To assess the robustness of our method against additive noise, we consider $\bar{\Box}(t) = \Box(t) + n_\Box(t)$, where $n_\Box(t)$ obeys the normal distribution of strength proportional to the signal rms. Figure 3(e) shows the mean square error (MSE) between the vector field functions of the estimated neural-network model and the original. The MSE assumes much lower values, below 0.1, until the noise level increases through a threshold, which is about 1.4% of the signal rms. Similar results have been obtained for the noise-perturbed time series from the Lorenz system [21]. Thus, the noise tolerance of our approach is approximately twice as that of the classic LK algorithm [8].

Finally, we demonstrate the power of our method to detect UPOs in high-dimensional chaotic systems. We consider the time-delayed Mackey-Glass system [22]:

$$\dot{x} = \frac{a x(t-\tau)}{1 + [x(t-\tau)]^b} - cx, \tag{7}$$

where τ is the time delay and a, b, and c are parameters. The model was originally introduced to describe the dynamics of regeneration of blood cells, but it has become a paradigmatic model for studying higher-dimensional chaos. In particular, due to the time delay, the right-hand side of the equation becomes a functional, so in principle the system is an infinitely dimensional dynamical system. To be concrete, we set $a = 2$, $b = 10$, $c = 1$, and $\tau = 3.18$. The system thus exhibits a hyperchaotic attractor, as shown in Fig. 4(a), with multiple positive Lyapunov exponents [21]. A representative UPO embedded in the attractor [23] is also shown, which is obtained by using the ADFC method applied directly to the original Mackey-Glass system. To detect the UPO from the time series $x(t)$, we need to choose RBF neural networks with the ability to reconstruct functionals. This can be accomplished [14] by replacing the basis functions with

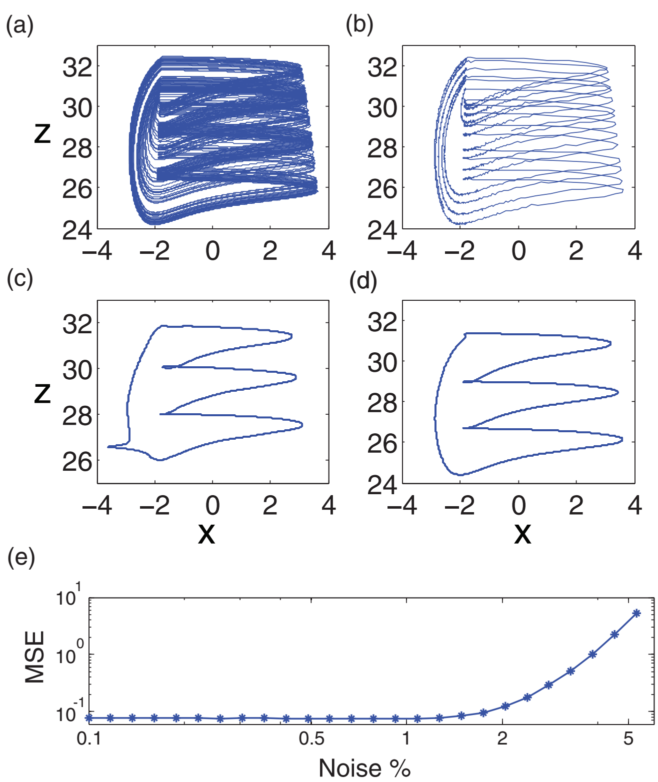

FIG. 3. (Color online) For the Hindmarsh-Rose neuron model, (a) a chaotic attractor, (b) noise-perturbed and sparsely sampled time series, (c) a detected UPO from the time series data in panel (b) with the period $T = 91.47$, (d) the corresponding UPO computed directly from the model with the period $T = 91.22$, and (e) MSE vs the noise level.

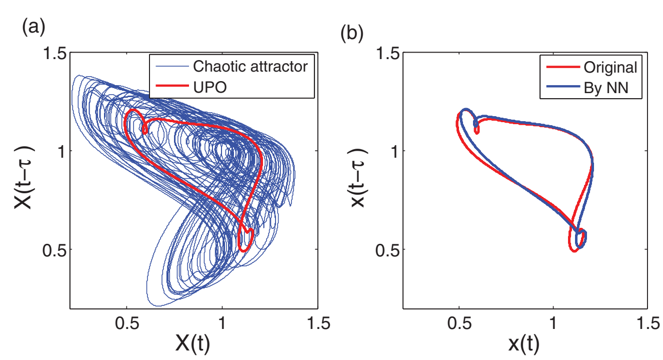

FIG. 4. (Color online) (a) A chaotic attractor of the Mackey-Glass system and an UPO of period about $T = 7.61$ computed directly from system equations. (b) UPO in panel (a) and the corresponding UPO (of period $T^* = 7.60$) detected from our reconstructed system of RBF neural networks.

the basis functionals $g_j(\boldsymbol{x}_t) = \exp\{-\lambda_j \|\boldsymbol{x}_t^M - \boldsymbol{\zeta}_j\|^2_{\mathbb{R}^M}\}$, where $\boldsymbol{x}_t^M = [x_t(\theta_1),\ldots,x_t(\theta_M)]^\top \in \mathbb{R}^M$ holds for any continuous function $x_t \in \mathrm{C}([-\tau,0],\mathbb{R})$. We choose $M = 2$, $\theta_1 = 0$, $\theta_2 = -\tau = -3.18$ and $\lambda_j = 1/0.32$. Grid points $\boldsymbol{\zeta}_j = [\zeta_{j1},\zeta_{j2}]^\top$ are chosen from the area $[0.2,1.4] \times [0.2,1.4]$ with grid size 0.2. Neural networks can then be trained and UPOs can be detected. Figure 4(b) shows an example of a detected UPO. The observed consistency between the two UPOs in Figs. 4(a)–4(b) indicates that our method is capable of detecting UPOs in high-dimensional chaotic systems.

In summary, we have articulated a general approach to detecting UPOs embedded in chaotic attractors from time series. Particularly, using the classic Takens embedding theorem to reconstruct vector time series from an available scalar time series, we construct neural networks to obtain an estimate of the system equations, based on which the adaptively delayed feedback control method can be used to detect UPOs. The basic philosophy underlying our approach is the following: The estimated system equations through the neural networks are not the exact equations of the original dynamical system, so it is in general not possible to, for example, assess and predict the exact dynamical evolution of the system. However, UPOs are robust building blocks of any chaotic systems, and they constitute the "skeleton" of the underlying chaotic set. As a result, detection of UPOs does not require a very precise equation of the system. The appealing feature of our method lies in its ability to detect UPOs from high-dimensional chaotic systems, as we have demonstrated using the Mackey-Glass delay-differential equation system. We remark that, in situations where the periods of the UPOs to be detected are known *a priori*, algorithms such as those developed in the field of chaos control [24] can lead to better performance in terms of accuracy and convergence.

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