Emergence and scaling of synchronization in moving-agent networks with restrictive interactions

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In fields such as robotics and sensor networks, synchronization among mobile and dynamic agents is a basic task. We articulate an effective strategy to achieve synchronization in dynamic networks of moving chaotic agents. Our counterintuitive idea is to restrict agents' ability to interact with each other, which can be implemented by designating a finite number of *fixed* zones in the space, in which agents are allowed to interact with each other but agents outside the zones are deprived of the ability of mutual interaction. Our setting is thus different from the one used in existing works on synchronization of mobile agents where each agent is associated with an interacting zone that moves with the agent. We find, through a mathematical analysis, that an optimal interval exists in the interaction probability, where stable synchronization emerges. An inverse square-root scaling law is uncovered which relates the interval with the system size, i.e., the total number of moving agents. Extensive numerical support for physical spaces of one, two, and three dimensions is provided.

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I. INTRODUCTION

Consider a number of dynamic agents that can move freely in a large region in the physical space. By "dynamic" we mean that each agent by itself can be represented by a dynamical system, such as a nonlinear oscillator. Agents are allowed to interact with each other through certain coupling mechanisms. In certain applications the interactions among the agents may be restricted. For example, in a wireless communication system, essentially a system of moving agents in a two-dimensional region, there are "blind" areas that communication signals cannot reach. Can synchronization still occur?

In this paper, we propose a simple but effective scheme to synchronize an ensemble of moving dynamical agents. In particular, let Σ be a region within which the agents are confined but they can move freely. Intuitively, one might think that allowing the agents to interact with each other throughout Σ can promote synchronization. However, our idea is to introduce certain "blind" subregions in which agents are not allowed to interact with each other. In the case of a robotic network, this can be realized through some external control signal. Equivalently, we assume that there are a number of subregions (zones), in which interactions among agents are possible but outside the zones, agents cannot interact with each other. To be concrete, we assume Σ to be a region of size L^d in $\mathbb{R}^d$, where d is the dimension of the physical space in which the agents move. Each zone has radius r, and there are m such zones. By construction we have $S(Z) < L^d$, where $S(Z)$ is the total volume of the zones. The quantity $P \equiv S(Z)/L^d$ is then the *probability of interaction*. Intuitively, one might expect synchronization to be more probable as P is increased, because mutual interactions among the agents are more likely. However, we find that, for typical chaotic agents, there exists an interval in P: $P_l \leqslant P \leqslant P_u$, in which stable synchronization can be achieved. The lower and upper bounds of the interaction probability for synchronization depend on the number N of agents, where N is effectively the size of the moving-agent network. We find the following scaling relations:

$$P_l \sim N^{-1/2}, \quad P_u \sim N^{-1/2}. \tag{1}$$

We shall provide a mathematical argument and numerical support for the size-scaling relations. While it is reasonable that, as the number of moving agents is increased, synchronization is more difficult, the relations in (1) indicate that the degree of difficulty is characterized by a power-law scaling between the synchronization interval $\Delta P \equiv P_u - P_l$ and the size of the mobile network.

The problem of synchronization of moving agents under restrictive interactions, as formulated above, is different from the extensively studied problem of synchronization in complex networks [1–14], where most previous efforts focused on issues such as the effect of network structure on synchronization. Our problem is also distinct from the existing works on synchronization in moving-agent networks [15–19], where interactions are unrestricted and can occur whenever agents are sufficiently close. In particular, in these works each agent "carries" with itself a zone of interaction that moves in time, and the number of such zones is equal to the number of agents, or the size of the network. The relevant parameter is then the density of agents, and it was found that the density interval within which synchronization can be achieved does not depend on the system size [16]. In contrast, our setting is that of *fixed* zones of interaction, where a (small) number of such zones are prespecified in the space, and at any given time, only those agents within the fixed zones are allowed to interact with each other. As we demonstrate, the ability for the network of moving agents to achieve synchronization depends strongly on the system size: Synchronization is more difficult for larger systems, as quantified by the scaling relations in (1).

Besides wireless communication systems, another situation where our work may be relevant and useful is coordination in robotic networks. Intuitively if robots are allowed to interact with each other more, especially when they get close together, one would expect synchronization to occur. However,

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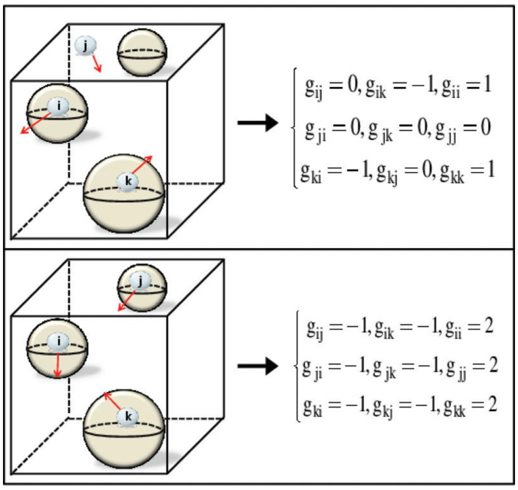

FIG. 1. (Color online) Representative configurations of matrix G for a network of moving agents in $\mathbb{R}^3$.

our analysis indicates that imposing certain restrictions on their interactions may counterintuitively generate and enhance synchronization. For example, one can define certain zones of interaction, where robots are programed to turn on their transmission signals to interact with each other only when they enter these zones, while if they are outside the zones, their transmission devices are off so that no interaction can occur. Our results indicate that this kind of restriction can in fact be beneficial to synchronization.

In Sec. II, we describe our scheme to achieve synchronization. In Sec. III, we present a mathematical analysis for the emergence of synchronization and size scaling. Numerical support is provided in Sec. IV. A brief conclusion and discussion are presented in Sec. V.

II. SYNCHRONIZATION SCHEME

In our scheme, N agents are randomly distributed in a region Σ of size L in the d-dimensional physical space that contains a number of fixed spheres, each of radius r, in which interactions among agents are enabled, as shown in Fig. 1 for $d = 3$. We call the spheres *coupling zones*. In particular, coupling between the dynamics of any pair of agents is activated only when both agents move into some coupling zones (they do not have to be in the same zone). In Σ, each agent can be regarded as a random walker that moves with velocity $\mathbf{v}_i(t)$, where $\mathbf{v}_i(t) = \pm v$, $\mathbf{v}_i(t) = [v \cos \theta_i(t), v \sin \theta_i(t)]$, and $\mathbf{v}_i(t) = [v \sin \phi_i(t) \cos \theta_i(t), v \sin \phi_i(t) \sin \theta_i(t), v \cos \phi_i(t)]$ for one-, two-, and three-dimensional space, respectively. In one dimension, an agent takes on the velocity $+v$ or $-v$ with equal probability. In two dimensions, the angles θ_i $(i = 1, \ldots, N)$ are N independent random variables chosen uniformly from the interval $[-\pi,\pi]$. In three dimensions, the angles θ_i are chosen randomly from the interval $[-\pi,\pi]$ and the angles ϕ_i are drawn from the interval $[-\pi/2,\pi/2]$. The dynamical updating rule of an agent's motion is

$$\mathbf{y}_i(t + \Delta T) = \mathbf{y}_i(t) + \mathbf{v}_i(t)\Delta T, \quad (2)$$

where ΔT is the time unit and $\mathbf{y}_i(t)$ is the position of agent i at time t. To be concrete but without loss of generality, we assume that the dynamics of each agent is chaotic and governed by the classical Rössler dynamics: $(\dot{x}_1^i, \dot{x}_2^i, \dot{x}_3^i) = [-(x_2^i + x_3^i), x_1^i + 0.2x_2^i, 0.2 + x_3^i(x_1^i - 7.0)] \equiv \mathbf{F}(\mathbf{x}^i)$. Taking into account coupling dynamics, the dynamical equation of agent i can be written as

$$\dot{\mathbf{x}}^i = \mathbf{F}(\mathbf{x}^i) - K \sum_{j=1}^{N} g_{ij}(t)\mathbf{H}(\mathbf{x}^j), \quad i = 1,2,\ldots,N, \quad (3)$$

where, for simplicity, we assume that the coupling is through the x variable of the Rössler dynamics so that the coupling function is $\mathbf{H}(\mathbf{x}^j) = (x_1^j, 0, 0)$, K is a coupling constant, and $g_{ij}(t)$ are the elements of a time-varying matrix $G(t)$ that specifies the whereabouts of all agents in the network at time t with respect to the coupling zones. In particular, we have $g_{ij}(t) = g_{ji}(t) = -1$ if agents i and j both are in some coupling zones at time t, $g_{ii}(t) = h$, and h is the number of agents in the zones at time t. Note that, the diagonal matrix elements g_{ii} do not represent any kind of self-interaction. The requirement $g_{ii}(t) = h$ is imposed to ensure that the synchronous state $\{\mathbf{x}^i(t) = \mathbf{s}(t), \forall i\}$ is a solution of Eq. (3) so that synchronization is one of the possible states of the dynamical system as a whole. The matrix $G(t)$ is thus similar to the standard Laplacian matrix used in previous studies of network synchronization [1–14], except in our case where it is time dependent. For the simple case of three agents, some possible configurations of the matrix G are illustrated in Fig. 1.

III. EMERGENCE OF SYNCHRONIZATION AND SIZE SCALING: A MATHEMATICAL ANALYSIS

Consider the system described by

$$\dot{\mathbf{x}}^i = \mathbf{F}(\mathbf{x}^i) - K \sum_{j=1}^{N} \overline{g}_{ij}\mathbf{H}(\mathbf{x}^j) \quad (4)$$

with fixed topology $\overline{G}$. A ground-breaking result [15] in the field of synchronization of time-varying networks is that, if system (4) admits a stable synchronization manifold and if there exists a constant T such that $(1/T)\int_t^{t+T} G(\tau)d\tau = \overline{G}$, then there exists a constant ϵ^* such that for all fixed $0 < \epsilon < \epsilon^*$, the time-varying system governed by the coupling matrix $G(t/\epsilon)$:

$$\dot{\mathbf{x}}^i = \mathbf{F}(\mathbf{x}^i) - K \sum_{j=1}^{N} g_{ij}(t/\epsilon)\mathbf{H}(\mathbf{x}^j),$$

also admits a stable synchronization manifold. This means that, for a networked dynamical system whose connection topology is time dependent, its synchronization behavior can be analyzed [15] by the time-averaged coupling matrix $G(t)$, denoted by $\overline{G} \equiv [\overline{g}_{ij}]$. The implication is that, if the time-averaged coupling matrix $\overline{G}$ supports synchronization of the whole system and if the switchings between all the possible network configurations are sufficiently fast, then the time-varying dynamical network will synchronize. Our subsequent analysis is based on this fundamental result. In particular, we shall calculate the time-averaged coupling matrix $\overline{G}$ for our setting where mobile agents can interact with each other but only in predefined coupling zones.

We now calculate the time-averaged coupling matrix $\overline{G}$ by adopting the method developed by Frasca *et al.* [16]. Let A denote the set of all agents, i.e., $A = \{1,2,\ldots,N\}$, and $B = \{j_1 j_2 \cdots j_k\} \subset A$ be the set of interacting agents. We denote $p_{j_1 j_2 \cdots j_k} = p_B$ as the probability that agents $j_1 j_2 \cdots j_k$ are coupled with each other and $G_{j_1 j_2 \cdots j_k} = G_B$ as the corresponding coupling matrix. Insights can be gained by considering the simple case of three agents ($N = 3$). For such a system, the time-averaged coupling matrix is given by $\overline{G} = p_{12}G_{12} + p_{13}G_{13} + p_{23}G_{23} + p_A G_A = p_{12}G_A + p_A G_A$, where the identities $p_{12} = p_{13} = p_{23}$ and $G_{12} + G_{13} + G_{23} = G_A$ have been used and $G_A = \left[\begin{smallmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{smallmatrix}\right]$. The probabilities p_{12} and p_A are $p_{12} = P^2(1-P)$ and $p_A = P^3$. We thus have

$$\overline{G} = p_{12}G_A + p_A G_A = P^2 G_A. \tag{5}$$

To calculate the time-averaged matrix $\overline{G}$ for the general case of $N \geqslant 4$, we need to calculate the coupling probabilities for all subsets of all N agents. For subsets B and C with the same cardinality, i.e., $|B| = |C|$, the coupling probabilities p_B and p_C are equal. We can thus sum over all coupling matrices generated by i agents. Let B_i be the resulting matrix. For instance, for $N = 4$ and $i = 3$, we have $B_3 = G_{123} + G_{124} + G_{134} + G_{234}$. In general, we can derive the following matrix representation of B_i:

$$B_i = \frac{i-1}{N-1}\binom{N-1}{i-1} G_A, \tag{6}$$

where $\binom{N-1}{i-1}$ is the number of $(i-1)$ combinations from a given set of $N-1$ elements. A proof of Eq. (6) can be found in the Appendix.

We now calculate the time-averaged matrix:

$$\overline{G} = \sum_{i=2}^{N} p_i B_i,$$

where p_i is the probability that i agents are coupled with each other, which is given by $p_i = P^i(1-P)^{N-i}$. Using the probability property ($p_B = p_C$ if $|B| = |C|$) and Eq. (6), we obtain

$$\begin{aligned}
\overline{G} &= \sum_{i=2}^{N} P^i(1-P)^{N-i} B_i \\
&= \sum_{i=2}^{N} P^i(1-P)^{N-i} \frac{i-1}{N-1}\binom{N-1}{i-1} G_A \\
&= \sum_{i=0}^{N-2} P^{i+2}(1-P)^{N-i-2} \frac{i+1}{N-1}\binom{N-1}{i+1} G_A \\
&= P^2 G_A \sum_{i=0}^{N-2} P^i(1-P)^{N-i-2}\binom{N-2}{i} = P^2 G_A.
\end{aligned}$$

Note that, because G_A is a zero-sum matrix, $\overline{G}$ also possesses this property. The eigenvalues of $\overline{G}$ are then $\lambda_1 = 0$ and $\lambda_i = P^2 N$ (for $i = 2,3\ldots,N$).

Consider a generic class of oscillator systems [20,21] for which the master stability function (MSF) $\lambda_{\max}(\alpha)$ is negative for $\alpha_1 < \alpha < \alpha_2$. We get from the eigenvalues of $\overline{G}$ the following criterion for achieving stable synchronization:

$$\alpha_1 \leqslant KP^2N \leqslant \alpha_2. \tag{7}$$

In terms of the interaction probability P, there then exists an interval given by

$$P_l \equiv \sqrt{\frac{\alpha_1}{KN}} \leqslant P \leqslant \sqrt{\frac{\alpha_2}{KN}} \equiv P_u, \tag{8}$$

for which the mobile agents can synchronize with each other. Inequality (8) leads directly to the size-scaling relation (1).

We thus have

$$P_u - P_l \sim N^{-1/2}.$$

If P falls outside the interval (P_l, P_u), the system cannot be synchronized. An implication is that, to give all agents unlimited capability to interact with each other is not necessarily beneficial for synchronization. In fact, it is necessary to carefully choose the frequency of interaction to induce synchronization.

The reason that synchronization cannot be achieved for $P < P_l$ is simple: There is not sufficient interaction among the agents to generate coherent motion. As P is increased through P_l, synchronous behavior emerges. But why is synchronization lost for $P > P_u > P_l$? Here we offer an intuitive explanation for this behavior. Regarding the system as a network of coupled oscillators, we can associate P with some kind of coupling parameter, and the interactions among the agents are represented by various coupling terms. In general, the roles of the coupling terms are (1) to establish coherence among the oscillators, and (2) to act as perturbation to the dynamics of the individual oscillators, which is effectively random for chaotic oscillators. Whether synchronization can actually occur depends on the interplay between these two factors. For example, for small coupling, synchronization cannot occur because, although the perturbing effect of the coupling terms is small, the amount of coherence provided by them is small, too. For very large coupling, although the coupling terms can provide strong coherence, the effective perturbations are also large. As large perturbation requires a longer time for the system to reach an equilibrium state, e.g., synchronization, in this case, the system will have no time to relax into the equilibrium state in response to the perturbations and consequently is unable to synchronize. Thus, synchronization may not occur if the coupling is too strong, e.g., for $P > P_u$. While our analysis follows the standard approach of MSF [20] for network synchronization and especially its variant for networks of moving agents as in Refs. [15,16], there is a detailed difference between our analysis and that, for example, in Ref. [16]. In particular, in Ref. [16], the average matrix $\overline{G}$ is a kind of rescaled all-to-all matrix G_A whose eigenvalues do not depend on the system size. In our case, the nontrivial eigenvalues of the average matrix $\overline{G}$ depend explicitly on the system size, as reflected in the inequalities (8). Again, this key difference originates from the different settings of the synchronization problem to start with: In Ref. [16] the interaction zones move in time with their host agents and their number is the same as the network size, whereas in our case the interaction zones are fewer and fixed in the physical space.

We summarize briefly the key assumptions and their validity used in our analysis, as follows.

(1) Emergence of synchronization is determined by the time-averaged coupling matrix $\overline{G}$. This is valid when the switchings between all the possible network configurations are sufficiently fast. This idea was introduced and validated in Ref. [15] and used in Ref. [16].

(2) Calculation of $\overline{G}$ is done in terms of a binomial distribution due to the appearance of two groups of agents with distinct behaviors: one interacting and another noninteracting. No approximation is needed for this step.

(3) The assumption that the MSF is negative in a finite parameter interval is then used to arrive at the main scaling law (1). This assumption is justified based on a previous work that such a behavior of the MSF is generic [21].

IV. NUMERICAL SUPPORT

We now provide numerical support for the scaling relations in (1). We vary the number of mobile agents from 2 to 100 and calculate the following synchronization index [16] for different values of P:

$$\langle\delta\rangle \equiv \left\langle \sum_{i=2}^{N} \frac{\left(\left|x_1^i - x_1^1\right| + \left|x_2^i - x_2^1\right| + \left|x_3^i - x_3^1\right|\right)}{3(N-1)} \right\rangle, \quad (9)$$

where the total time of simulation is chosen to be $T = 1000$ (arbitrarily) and the average is performed on the time interval $[9T/10, T]$. The initial conditions are chosen randomly in the phase-space region where the chaotic attractor resides. The number of statistical realizations used to calculate the synchronization index is 100. Figure 2 shows the synchronization index versus the interaction probability for $K = 10$

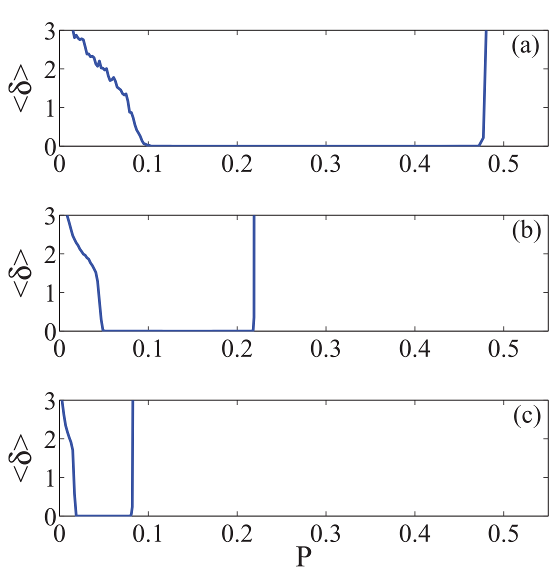

FIG. 2. (Color online) For $d = 2$ and different number N of agents, synchronization index $\langle\delta\rangle$ versus the probability of interaction P for (a) $N = 2$, (b) $N = 10$, and (c) $N = 100$.

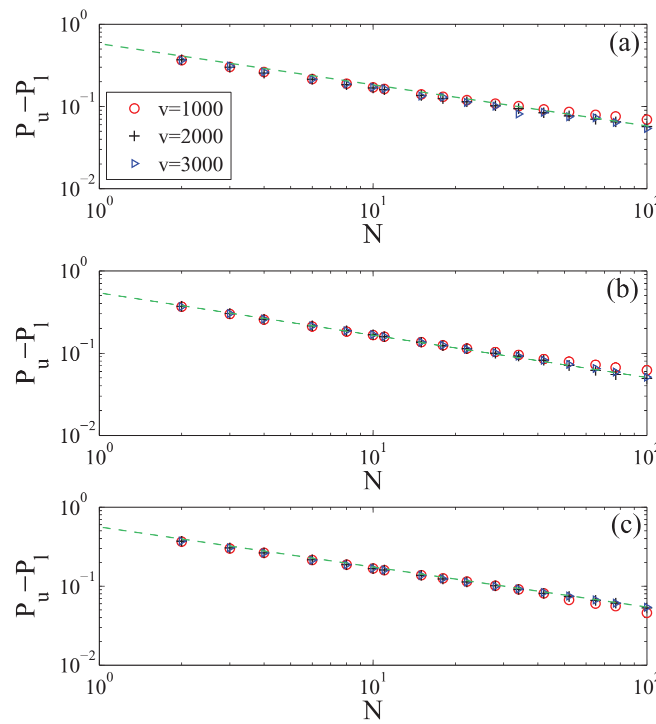

FIG. 3. (Color online) Numerically obtained size scaling of the synchronization interval: $P_u - P_l$ versus N for physical space of (a) one-dimension, (b) two dimensions, and (c) three dimensions, and for three values of the moving velocity.

and $N = 2, 10, 100$. We see that the synchronization parameter interval decreases as the number of agents is increased; so do the lower and upper bounds of the interval, providing qualitative evidence for (1). Quantitative support for the scaling can be found in Fig. 3, where numerically obtained size-scaling relations of $P_u - P_l$ are shown for physical space of (a) one dimension, (b) two dimensions, and (c) three dimensions. We see that the scaling law (1) holds regardless of the dimensionality of the physical space in which the agents move.

In previous works on moving-agent networks, especially those occurring in biological systems, the agent motility is an important character [22–24]. To explore the effect of motility on synchronization in our system, we calculate the size scaling for three different values of the speed of the moving agents, as shown in Fig. 3. We observe that the scaling law holds for all three cases. However, as we demonstrate and explain in Fig. 7, synchronization can be achieved only when the agent velocity is chosen from a suitable range. For example, for $m = 1, r = 1$, $N = 100$, and $d = 2$, this range is approximately $[10^3, 10^4]$. The reason for the scaling law to hold in some proper velocity range lies in the setting of our model: the interacting zones are fixed in space, versus those in the previous studies where the zones move with the agents, so the moving speed can have a significant impact on the dynamics.

To demonstrate that the scaling relations in (1) hold regardless of the radius and the number of interaction zones, we show in Fig. 4 the synchronization index for different values of r but fixed $N = 100$ and $m = 1$. We obtain nearly identical

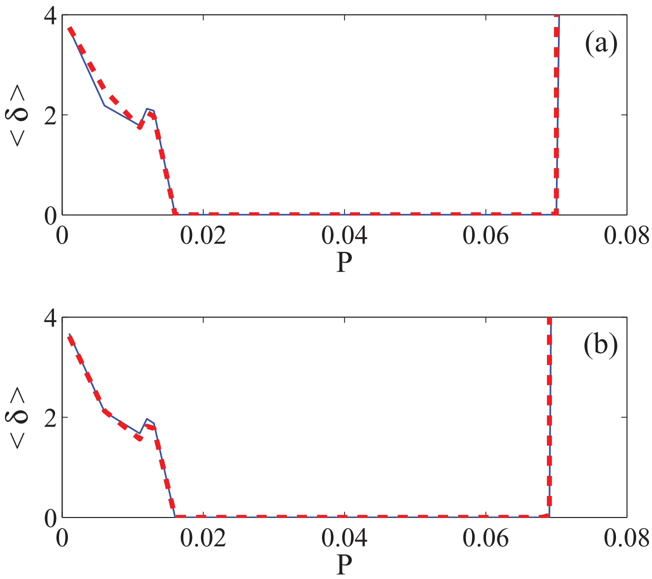

FIG. 4. (Color online) Synchronization index versus interaction probability P for different values of the radius of interaction zone for fixed $N = 100$, $d = 2$, and $m = 1$: (a) $r = 1$ (blue curve) and $r = 0.9$ (dashed red curve), and (b) $r = 0.8$ (blue curve) and $r = 0.7$ (dashed red curve).

values of P_l and P_u for different values of r. The scaling relations are also independent of the number of interaction zones, as shown in Fig. 5, where the synchronization index is plotted versus the interaction probability P for different values of m but fixed $N = 100$ and $r = 1$.

How does synchronization behavior change as model parameters are adjusted? Here we address this question by using specific examples. Figures 6(a) and 6(b) show the synchronization index $\langle\delta\rangle$ versus the interaction probability P for significant variations in the parameters r and m, respectively. Particularly, Fig. 6(a) indicates that, insofar as the radius of the interaction zone remains at the order of unity, synchronization can be achieved. Similarly, variations in m, the number of interacting zones, appear to have little effect on synchronization, as shown in Fig. 6(b).

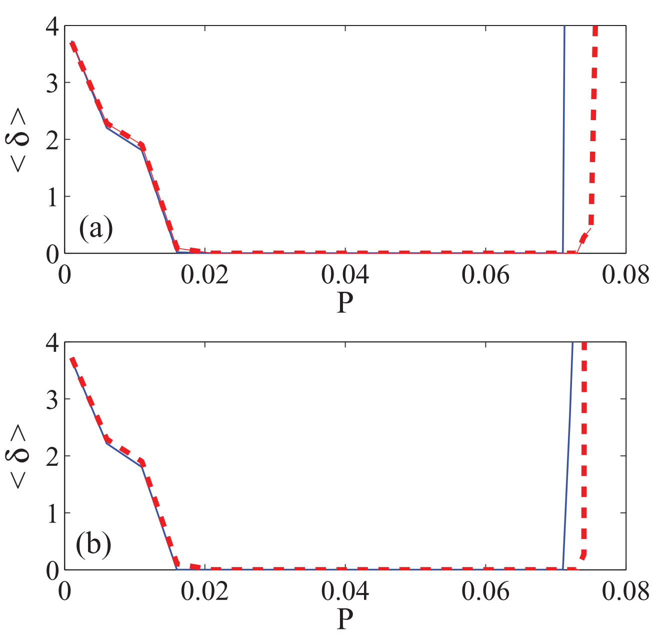

FIG. 5. (Color online) Synchronization index versus interaction probability P for different values of the number of interaction zones for fixed $N = 100$, $d = 2$, and $r = 1$: (a) $m = 2$ (blue curve), $m = 3$ (dashed red curve), and (b) $m = 4$ (blue curve), $m = 5$ (dashed red curve).

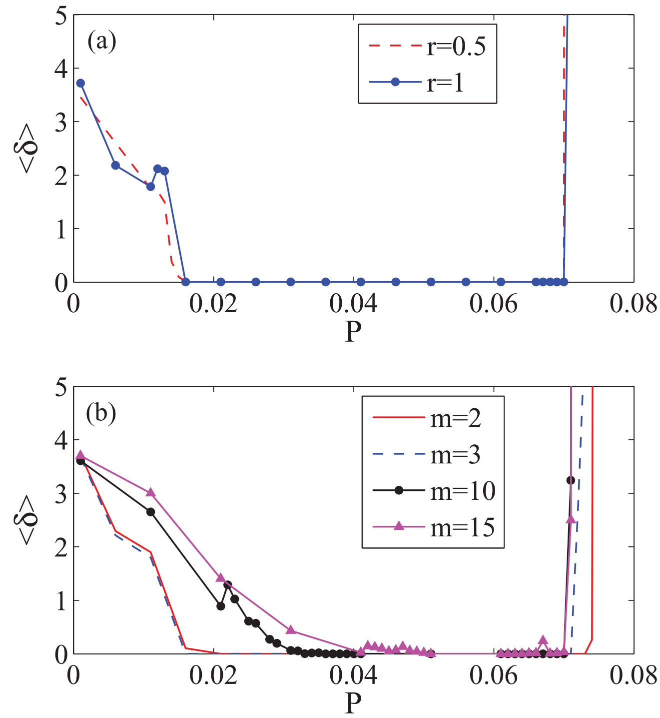

FIG. 6. (Color online) Synchronization index versus the interaction probability P for significant variations of the parameters r and m: (a) a number of values of r chosen from the range between 0.5 and 1 for fixed $m = 1$ and $v = 10^3$, and (b) a number of m values for fixed $r = 1$ and $v = 10^3$. Other parameters are $N = 100$ and $d = 2$.

The interplay between synchronization and velocity is, however, quite intriguing. Numerically, we have observed that the measure of synchronization, $P_u - P_l$, is not zero only when the velocity is chosen from some "optimal" interval, as shown in Fig. 7. Outside this velocity interval, synchronization is lost. This can be understood from the perspective of effective coupling with respect to different velocity values. In particular, since agent interactions are allowed only when they are in certain zones, a small velocity means that, on average, they can stay in the zones for a longer time, leading to strong coupling. In contrast, high velocity values correspond to weak coupling. As discussed in Sec. III, generically synchronization can be achieved when the coupling is neither too weak nor too strong.

V. CONCLUSION

In conclusion, we have articulated an efficient scheme to realize synchronization in dynamic networks of mobile agents. The key is to impose restriction on the moving agents' interactions by depriving them of the ability to interact with

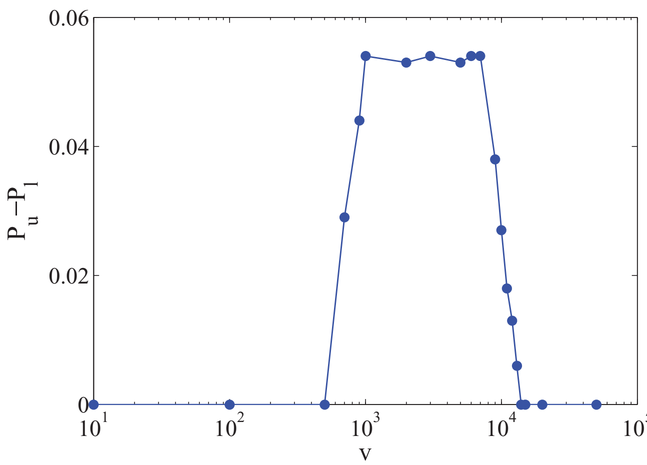

FIG. 7. (Color online) Synchronization measure $P_u - P_l$ as a function of the agent velocity. Only when the velocity is chosen from an optimal interval can synchronization be observed. See text for an explanation.

each other in certain regions of the physical space in which they move. That is, granting agents unlimited capability to interact with each other can be detrimental to synchronization. While this is somewhat counterintuitive, we have discovered, through a mathematical analysis, the existence of an optimal interval in the interaction probability in which synchronization is stable.

Our analysis also leads to scaling relations revealing that the length of the optimal synchronization interval scales with the number of agents according to the inverse square-root law. We have provided strong numerical support for these results. A field where our results may be relevant is robotics. There are applications in which synchronization among moving robots is desired. In such a case, our finding suggests the strategy of externally imposing proper control of the agents' ability to interact with each other. Another field of relevance is wireless communication, where signals are accessible only in limited geographical regions. Our results may provide hints to achieving maximum communication efficiency.

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APPENDIX

Here we provide a proof of the matrix representation Eq. (6). The coupling matrix for all agents A is given by

$$G_A = \begin{bmatrix} N-1 & -1 & -1 & \cdots & -1 \\ -1 & N-1 & -1 & \cdots & -1 \\ \vdots & & \ddots & & \vdots \\ 1 & \cdots & -1 & N-1 & -1 \\ -1 & \cdots & \cdots & -1 & N-1 \end{bmatrix}.$$

To get the right-hand side of Eq. (6), we consider entries of matrix B_i. For diagonal entries of B_i $(k = 1,2,\ldots,N)$, $B_i(k,k)$, the number of coupling matrices G_B including the agent k, i.e., $k \in B$ and $|B| = i$, is the number of choosing $(i-1)$ agents from all agents except agent k, which is $\binom{N-1}{i-1}$. Since $G_B(k,k) = i - 1$, we have

$$B_i(k,k) = (i-1)\binom{N-1}{i-1}.$$

Using $G_A(k,k) = N - 1$, we get

$$B_i(k,k) = \frac{i-1}{N-1}\binom{N-1}{i-1}G_A(k,k).$$

For off-diagonal entries (j,k) of B_i with $j \neq k$, we need to consider agent subsets B with $|B| = i$ and $j,k \in B$. The number of these agent subsets is $\binom{N-2}{i-2}$ and we thus have $G_B(j,k) = -1$. It implies that

$$B_i(j,k) = -1 \times \binom{N-2}{i-2}.$$

Using the combinatorial property

$$\binom{N-2}{i-2} = \frac{i-1}{N-1}\binom{N-1}{i-1}$$

and $G_A(j,k) = -1$, we get

$$B_i(j,k) = -\binom{N-2}{i-2} = \frac{i-1}{N-1}\binom{N-1}{i-1}G_A(j,k).$$

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