Cascading dynamics on random networks: Crossover in phase transition

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In a complex network, random initial attacks or failures can trigger subsequent failures in a cascading manner, which is effectively a phase transition. Recent works have demonstrated that in networks with interdependent links so that the failure of one node causes the immediate failures of all nodes connected to it by such links, both first- and second-order phase transitions can arise. Moreover, there is a crossover between the two types of transitions at a critical system-parameter value. We demonstrate that these phenomena can occur in the more general setting where no interdependent links are present. A heuristic theory is derived to estimate the crossover and phase-transition points, and a remarkable agreement with numerics is obtained.

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A networked dynamical system typically consists of a large number of nodes or components interacting with each other in a complicated manner. The heterogeneous structure of the network and the intricate node-to-node interaction pattern render the system vulnerable to a cascading type of dynamics, where the failure of one or a few nodes can trigger failures of a large number of nodes in an avalanching manner [1]. Real-world examples of cascading failures include large-scale blackouts of power grids [2] and severe traffic jams on the Internet [3]. In the past decade, many models were proposed to understand the dynamics of cascading failures, and strategies for preventing or mitigating failures were articulated [4–8].

It has been recognized that the interplay between structural changes and the redistribution of traffic loads can play a significant role in cascading failures. For example, a frequently studied class of models is based on the redistribution of traffic loads on the network, the destructive effect of which can be magnified significantly by the highly heterogeneous topology typically seen in complex networks [5,9]. In such a case, disabling one or a few nodes that have overwhelmingly more connections than an average node in the network can reduce the network to small fragments. Cascading failures on a global scale can also be induced by a local dependence among neighboring nodes, as described by the sandpile model [10,11], the dynamical flow model [12], and the model of activation process [13–18]. In the local-dependence models, theoretical analysis is feasible to aid numerical simulations to probe the underlying dynamics. For example, it was found that the breakdown of a network depends sensitively on the initial disturbance even if it involves as low as a 0.1% fraction of nodes [14]. However, a complete theoretical understanding of cascading dynamics in complex systems in general remains a challenging problem.

Quite recently, cascading failures have been uncovered in complex systems of interdependent networks that are of a quite different nature but nonetheless interact with one another. An example is a communication network and a power grid [19], where the former relies on the latter to provide a source of electricity but the latter depends on the former for effective coordination of the electrical power distribution and transmission. A phenomenon in this type of interdependent networks pertinent to cascading dynamics is the *crossover* from a first-order to a second-order phase transition when the coupling strength among different networks is reduced [20]. Analogous behavior has been found in a single network composed of interdependent links, where the dynamics of a pair of nodes at the ends of such a link depend on each other and, as a result, the failure of one node can immediately trigger failures of the other [21]. Note that this mechanism of failure is different from the load-redistribution-based mechanism, where the failure of one node would not necessarily cause the immediate failure of those connected to it.

From the standpoint of physics, phase transitions are fundamental to many types of condensed-matter systems, and the crossover phenomenon, i.e., a change in the nature of the transition as a system parameter varies continuously, is interesting. Recent works [20,21] demonstrated that the crossover in phase transitions of cascading failures can arise in interdependent networks. In this paper, we study cascading dynamics on complex networks *in the absence of interdependent links*, and we find two types of phase transitions. In particular, we find that networks consisting of locally dependent nodes only can also exhibit first- and second-order phase transitions. The critical point between the two distinct types of phase transition is determined by the vulnerability of nodes, which is quantified by the minimum fraction of connections with neighbors required for survival (to be defined below). For an arbitrary value of the vulnerability, there exists a phase transition, first- or second-order, when a random attack occurs. The phase transition is measured by an order parameter, the normalized size of the giant

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component after the cascading dynamics ceases. There is then a critical value of the vulnerability that separates first- from second-order transitions. This value depends on the network characteristics, and we will demonstrate that it can be determined both numerically and theoretically. Our theory also predicts, for each value of vulnerability in the regime of first-order phase transition, the critical point of the transition for different network structures. Our finding indicates that the phenomenon of crossover in phase transitions can be more general beyond the framework of interdependent networks, and our computations and analysis provide a solid base for observing the phenomenon in complex dynamical systems.

For illustrative purposes, we study the phase transition of cascading dynamics on the standard Erdős-Rényi random network with the Poisson degree distribution $p_k = e^{-\langle k\rangle}\langle k\rangle^k/k!$, where $\langle k\rangle$ characterizes the average connectivity of the network. In general, there is physical or information flow from the neighbors of a node back to the node itself, such as the power supply in electric-power grids, investment in economic networks, and packets in a computer network. Insufficient support from neighbors can cause a node to fail or malfunction. From a topological point of view, the number of neighbors of a node can be used to characterize its supporting strength. We are thus led to introduce a tolerance parameter β to characterize the survival ability of a node in terms of the number of its surviving neighbors. Note that the cascade model is subject to the locally dependent dynamical process, in contrast to the cascade model based on load redistribution. To be concrete, in the model, an initial random attack removes a fraction $1-p$ of nodes and their links from the network of N nodes, so p is the fraction of surviving nodes from the initial attack. At each iteration, a node i with all its links is removed if $k_i'/k_i < \beta$, where k_i is the original degree of node i, k_i' is its current degree in this iteration so that k_i'/k_i is the fraction of the preserved links of i, and β characterizes the tolerance of the node to failures of its neighbors. An identical value of β is assumed for all nodes. Failures of some nodes may trigger the removal of their neighbors and their neighbors' neighbors, and so on, generating a cascading process. This model resembles the activation-process model proposed by Watts [13], which is effectively governed by a type of cascading dynamics. After a number of iterations of the dynamical process so described, the fraction of the preserved links for each node is at least β, therefore a further cascade of failures will not occur and the network reaches a steady state that consists of several isolated components. The connectivity of the residual network is monitored, and the normalized size S_1 of the giant (largest) component as well as the final fraction S of the surviving nodes are calculated.

Figure 1(a) shows S_1 versus p for different values of β. For the particular network setting, we observe that S_1 approaches zero continuously for $\beta = 0.3$ and 0.49 as p is decreased from 1 to 0. However, for $\beta = 0.51$ and 0.7, S_1 exhibits an abrupt transition from a finite value to zero at a critical value of p. These numerical results indicate that the behavior of S_1 versus p could be a second-order phase transition for $\beta = 0.49$ and a first-order phase transition for $\beta = 0.51$. In order to validate this hypothesis, we plot the log-log plot of S_1 versus

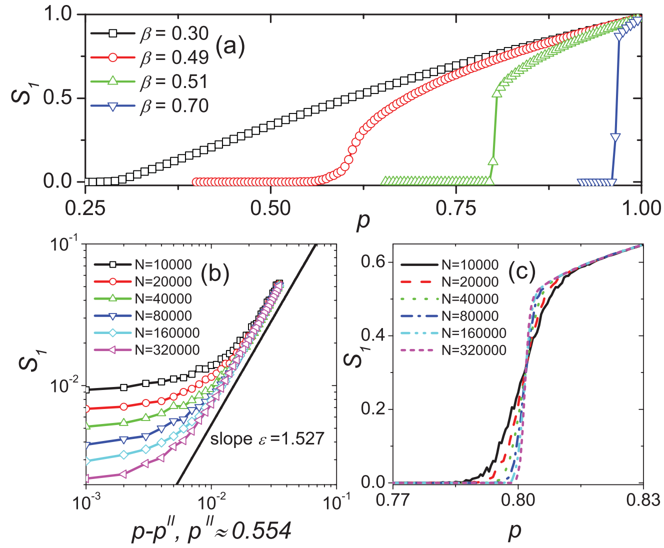

FIG. 1. (Color online) Simulation results demonstrating first- and second-order phase transitions on ER networks. Shown in panel (a) are the fraction of nodes in the giant component after cascade, denoted by S_1, as a function of p for $\beta = 0.3, 0.49, 0.51$, and 0.7, respectively, where the network size is 10^5. Shown in panel (b) are the log-log plot of S_1 vs the distance between p and the second-order phase-transition point p^{II} for different network sizes when $\beta = 0.49$. Shown in panel (c) are the curves of S_1 vs p depending on the network size N for $\beta = 0.51$. All the results are obtained by averaging over 1000 independent network realizations, and the average degree is $\langle k\rangle = 4$.

the distance between p and the second-order phase transition point p^{II} for different network sizes when $\beta = 0.49$, as shown in Fig. 1(b) (the method of locating phase transition points will be introduced later in the paper, and $p^{II} \approx 0.554$ for $\beta = 0.49$). We see that $S_1 \sim (p - p^{II})^{\varepsilon}$ in the limit $N \to \infty$, with the critical exponent $\varepsilon = 1.527 \pm 0.006$. In Fig. 1(c), we see that when $\beta = 0.51$, all the curves intersect at one point, and thus we can expect that S_1 will jump to zero from that point as $N \to \infty$. These numerical results demonstrate that the parameter β, which characterizes the node vulnerability, plays an important role in both the robustness and breaking form of the network against external perturbations. When β assumes a relatively large value, even a small fraction of failing nodes is able to disintegrate the network in the form of a first-order phase transition, while for a relatively small value, the network disintegrates continuously in the form of a second-order phase transition.

In order to locate the transition point accurately to further support our findings, we use the method developed by Parshani *et al.* [21]. For a first-order phase transition, we use the number of iterations (NOI) in the cascading process required for the system to reach a steady state. For a second-order phase transition, we calculate S_2, the normalized size of the second largest component after the cascading process is complete. For a finite but large network, these quantities tend to exhibit exceptionally large values at the transition points, providing a way to estimate these points numerically. Figure 2(a) shows the values of NOI versus p, from which the first-order phase-transition point p^I can be identified. Analogously, a peak in S_2 versus p is observed, as shown in Fig. 2(b), which gives the

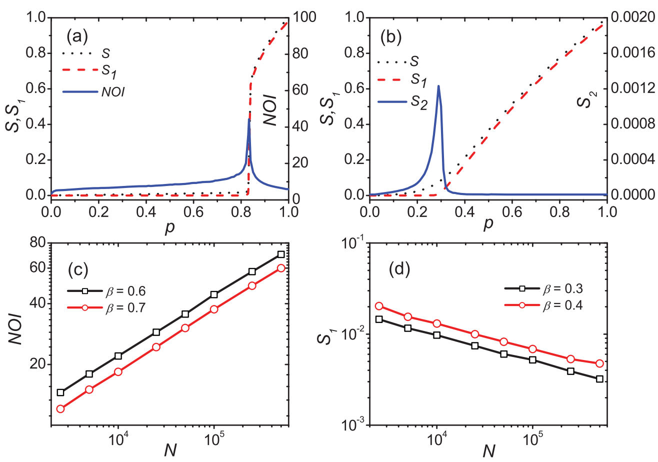

FIG. 2. (Color online) (a) For $\beta = 0.6$ (first-order phase transition), normalized size S of the final fraction of preserved nodes (dotted line), normalized size of the giant component S_1 (dashed line) in the steady state, and NOI (full line) as a function of p. (b) For $\beta = 0.3$ (second-order phase transition), S (dotted line), S_1 (dashed line), and the normalized size of the second largest component S_2 (full line) in the steady state versus p. Panels (c) and (d) display the scaling behaviors of NOI and S_1 for different parameter values of β at criticality, respectively. The results are obtained by averaging over 1000 independent network realizations, where the average degree of networks is 4 and the network size is 10^5 for panels (a) and (b).

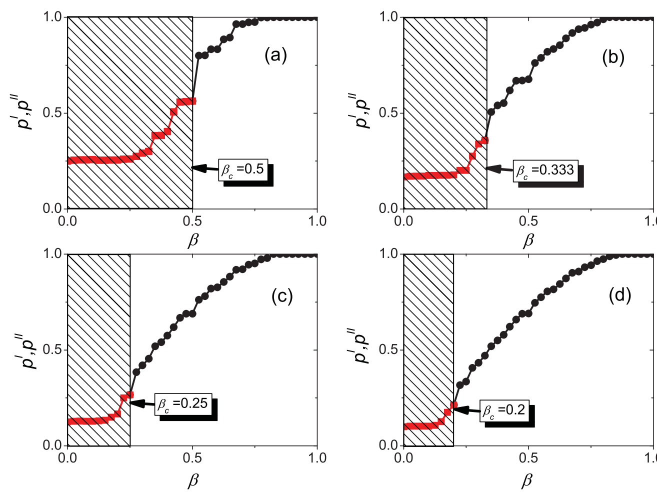

FIG. 3. (Color online) Diagram of the phase-transition points, p^I in the first-order region (black dot) and p^{II} in the second-order region (red square) as a function of nodal vulnerability β for ER networks with different values of $\langle k \rangle$: 4, 6, 8, and 10 for panels (a)–(d), respectively. The striped region denotes the second-order regime predicted by our theory. The simulation results are obtained by averaging over 1000 independent network realizations and the error bars are comparable to the size of the symbols.

estimated value of the second-order phase-transition point p^{II}. From our simulations, we find that, at the first-order transition point, NOI scales as N^{γ}, with $\gamma = 0.293 \pm 0.003$, as shown in Fig. 2(c), while for a second-order phase transition, S_1 scales as N^{δ}, with $\delta = -0.302 \pm 0.005$, as shown in Fig. 2(d). Note that the scaling exponents γ and δ are slightly different for different parameter values of β. Using the scaling behaviors of NOI and S_1, respectively, we can locate the points of first- and second-order phase transitions more accurately. These results further validate the critical crossover phenomenon indicated in Fig. 1.

To provide more support for the phenomenon of crossover in a phase transition, we compute the phase diagram associated with cascading dynamics for networks with different values of $\langle k \rangle$, as shown in Fig. 3. In each case, a critical value β_c can be identified, at which the phase transition switches from second-order for $\beta \leqslant \beta_c$ to first-order for $\beta > \beta_c$. From Fig. 3(a), we also observe some staircase structures, which can be attributed to the relatively narrow spectrum of node degrees in random networks. Due to the fact that most node degrees are about the mean value, and the vulnerability of nodes in terms of their degrees is stepwise, most nodes are not sensitive to small changes in β so that the robustness of the whole network is not sensitive to β and exhibits staircase structures. For larger average degrees, e.g., $\langle k \rangle = 8$ and 10 [Figs. 3(c) and 3(d)], the staircase structures become obscure because a larger average degree tends to smooth out the stepwise function of node vulnerability. Note that for $\beta = 0$, cascading dynamics does not occur as our model reduces to the model of site percolation on random graphs. In this special case, the second-order phase-transition point associated with percolation has been analytically derived in Ref. [22], which is given by $p^{II} = 1/\langle k \rangle$. This has indeed been observed in our computations. In addition, we find that β_c tends to be negatively correlated with $\langle k \rangle$.

We now derive a heuristic theory to explain the numerical findings. Consider an arbitrary, uncorrelated complex random network with degree distribution p_k in the infinite-size limit. The network can be regarded as having a purely branched structure so that the probability that the subclusters are connected by cycles is negligibly small. In this case, each subcluster can be treated independently of the others. Following the approach in the zero-temperature random-field Ising model [23], we build up our theory based on a level-by-level updating process on a hierarchical structure [14]. An arbitrary node can be chosen and assigned to the top level, whose nearest neighbors constitute the next lower level of the hierarchical structure, and the neighbors of the nearest neighbors belong to the following lower level, and so on. The nodes of the entire network can then be placed at different levels of this hierarchical structure. The bottom level is labeled level 1, and the top level is labeled ∞. The updating process proceeds from the bottom to the top level. For a random node in the level that has not been updated, it is removed with probability $1 - p$ and preserved with probability p. We define the probability x_n that a random node at level n is removed if this level has already been updated. When the updating process reaches level $n + 1$, we consider a preserved node at level $n + 2$ and randomly choose a preserved node from its offspring in level $n + 1$. For this chosen node with probability $\widetilde{p}_k$ it has k neighbors, where $\widetilde{p}_k$ is the degree distribution of a node at the end of a randomly chosen link: $\widetilde{p}_k = k p_k / \langle k \rangle$. The probability that the chosen node has m removed nodes in its $k - 1$ offspring (at level n) follows a binomial distribution:

$$P(m,k) = C_{k-1}^{m} x_n^m (1 - x_n)^{k-1-m}. \quad (1)$$

We then obtain the probability x_{n+1} that a random node at level $n+1$ of the hierarchical structure is removed, as follows:

$$x_{n+1} = 1 - p + p\sum_{k=1}^{\infty}\widetilde{p}_k\sum_{m=0}^{k-1}C_{k-1}^{m}x_n^m(1-x_n)^{k-1-m}F(m,k)$$
$$\equiv g(x_n,\langle k\rangle,\beta,p),$$

where $F(m,k)$ is the response function: $F(m,k)=1$ for $m/k > 1-\beta$ and $F(m,k)=0$ otherwise. For $n\to\infty$, x_∞ reaches a fixed point. We then have $x_\infty = g(x_\infty,\langle k\rangle,\beta,p)$. Since the top node has degree k with probability p_k, the average final fraction S of the preserved nodes is given by

$$S = p\left[1-\sum_{k=1}^{\infty}p_k\sum_{m=0}^{k}C_k^m x_\infty^m(1-x_\infty)^{k-m}F(m,k)\right]. \tag{2}$$

To estimate the first-order transition point, we make use of the observation that S_1 depends on the size S of the final fraction of preserved nodes. If the value of S changes abruptly, S_1 will also change sharply (cf. Fig. 2), since S_1 is the giant cluster size in S [24]. Hence, the behavior of S versus p gives the transition point. In particular, the value of x_∞ can be graphically solved by constructing the following system [19,20]:

$$\begin{aligned} y &= x,\\ y &= g(x,\langle k\rangle,\beta,p). \end{aligned} \tag{3}$$

The solutions are presented by the intersections of these two equations in the (x,y) plane. Since the degree distribution is Poissonian, the curve $y = g(x,\langle k\rangle,\beta,p)$ can be obtained by neglecting the probability of high degrees. There is a trivial solution $x_\infty = 1$, corresponding to the situation in which all nodes are removed at each level, i.e., all nodes of the network are removed. Figure 4(a) shows the solutions of Eq. (3) for different values of p for $\beta = 0.55$. We see that there is a tangent point between the curve and the line at $p_c \approx 0.8028$. For $p > p_c$, there are three intersections on the plane and the solution is given by the lowest one; if $p = p_c$, the solution is given by the tangent point; while for $p < p_c$, there is only one intersection at $x = 1$, which determines the solution. By reducing the value of p from p_c, the solution of Eq. (3) can change abruptly to nearly 1 from a nonzero

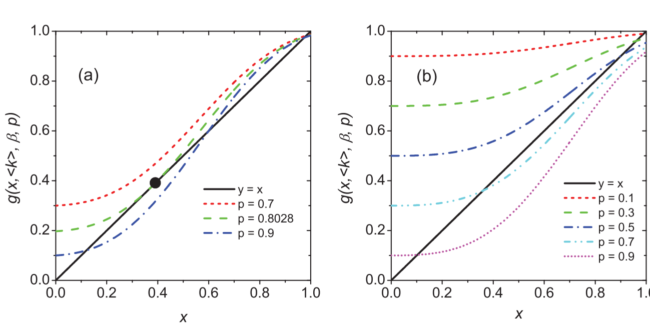

FIG. 4. (Color online) For $\langle k\rangle = 4$, illustrations of graphical solutions of Eq. (3). (a) First-order phase transition ($\beta = 0.55$), where the black dot denotes the tangent point. (b) Second-order phase transition ($\beta = 0.5$).

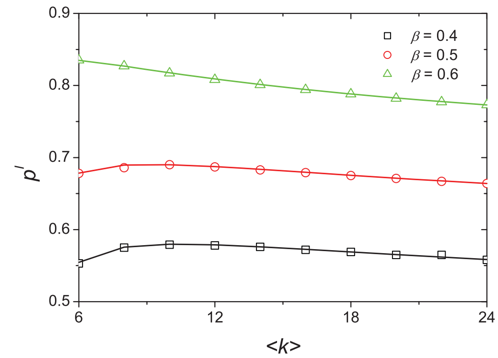

FIG. 5. (Color online) Comparison between the numerically obtained first-order phase-transition points and theoretical predictions for ER networks of different average degree $\langle k\rangle$ and nodal vulnerabilities β. The symbols denote the numerical results and the lines are from theory. The simulation results are obtained by averaging over 1000 independent network realizations and the error bars are comparable to the size of the symbols, where the network size is 10^5.

value at $p = p_c$, which corresponds to a first-order phase transition. The presence of the tangent point in the solution plane thus indicates the existence of a first-order transition and, simultaneously, gives the transition point. Figure 5 presents a comparison of the first-order transition points obtained by simulation and analysis. We observe a good agreement. To give an example of a second-order phase transition, Fig. 4(b) shows the solutions of Eq. (3) for different values of p for $\beta = 0.5$, from which we can observe that there is only one solution of Eq. (3) for any value of p. By reducing p, we observe that the solution of Eq. (3) increases continually to 1, which corresponds to a situation of second-order phase transition. The switch point β_c between the first- and second-order phase transition so estimated is shown for networks of different average connectivity $\langle k\rangle$ in Fig. 3, which is in good agreement with results from numerical simulations.

In summary, we have studied cascading failures associated with a local dependence on random networks without interdependent links. Such links were assumed in interdependent networks [19] and networks with interdependent groups [21], where the failure of one node can cause the immediate failures of its interdependent nodes. Systematic computations reveal the existence of both first- and second-order phase transitions and the crossover between the two in these models. However, in our model, a node fails when the fraction of its surviving neighbors is less than a tolerance parameter characterizing the node vulnerability. We find that both types of phase transitions exist and, as the tolerance parameter is increased through a critical point, there is a switch from second- to first-order phase transition associated with the cascading dynamics. Utilizing the classical Ising model, we derive a heuristic theory to estimate the various transition points, with good agreement with numerics. We have also used small-world [25] and scale-free [26] networks to reproduce the cascading dynamics with our model, and we found similar transitions between first- and second-order phase transitions. Our finding indicates that the phenomenon of crossover in phase transitions underlying cascading dynamics on complex networks is more general than previously thought.

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[24] S_1 is the giant component (the giant spanning cluster) in the residue network S. When the network size goes to infinity, if the parameter p is greater than the phase-transition point p_c, the giant component also becomes infinite; if the parameter p is less than the phase-transition point p_c, no spanning cluster exists and all clusters are finite. Hence, if all the nodes in each hierarchical level are removed, the spanning cluster cannot survive. In this regard, we claim that the transition point of S is the same as the first-order phase-transition point.

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