Abnormal cascading on complex networks

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In the study of cascading failures on complex networks, a key issue is to define capacities of edges and nodes as realistically as possible. This leads to the consideration of intrinsic edge capacity associated with laws governing flows on networks, which goes beyond the existing definitions of capacity based on the initial load as quantified by the betweenness centrality. Limited edge capacity (or bandwidth) and high flux or attack can trigger cascading processes, which we find as characteristically different from those reported in the literature. In particular, there can be an abnormal parameter regime where incrementally augmenting the edge capacity can counterintuitively increase the severeness of the cascading process. Another striking finding is that heterogeneous flow distribution tends to suppress the cascading process, in contrast to the current understanding that heterogeneity can make the network more vulnerable to cascading. We provide numerical computations and analysis to substantiate these findings.

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I. INTRODUCTION

The concepts of load and capacity are fundamental to dynamical processes on networks. An intuitive and apparently well-accepted definition of load is the betweenness, the number of shortest paths through a node or an edge [1]. This is based on the consideration that information is usually transmitted along the various shortest paths, e.g., in a computer network. Assume that, in each time unit, a node transfers one data packet along any shortest path through the node. The total number of packets that the node handles in one time unit is thus equal to the number of shortest paths through it, so the betweenness represents the load. This load definition and the assumption that node capacity [2,3] or edge capacity [4,5] is linearly proportional to the load have been used widely in the complex-network literature, and they have been the key to addressing fundamental issues such as cascading breakdown. However, the initial-load-based definition of node capacity may be idealized for realistic network systems supporting a variety of flows. For example, a recent work has reported that the proportional relation between load and capacity appears to be violated for a number of real-world networked systems [6]. We are thus interested in alternative ways to define node capacity in the study of cascading dynamics on networks. Moreover, in physical, chemical, and biological networks the quantities of interest are usually variables such as the electrical currents or chemical concentrations. In such a case the underlying rules governing the evolutions of the relevant variables are a significantly more important factor to determine the flow than merely the number of shortest paths (betweenness).

In this paper, we study cascading dynamics triggered by overloads along edges by incorporating the Ohm's law and the Kirchhoff conservation law instead of using topological betweenness. Our idea is inspired by Ref. [5] in which the authors have given a suggestion to use the laws to explore cascading failures. The advantages of this approach is then that the capacities of edges can be naturally defined according to the laws which governs flows. This overcomes the difficulty associated with the existing definition of the node capacity [2,3], which depends on the initial load and is therefore somewhat artificial. To be as general as possible, we shall study weighted complex networks to take into account heterogeneous node-to-node interactions. Our main finding is that heterogeneous flow distribution at nodes can significantly enhance the network's ability to counter cascading failures as triggered by intentional attacks. This is surprising, considering that heterogeneous networks being more vulnerable to cascading failures is a central result from models in the complex-network literature that use betweenness to define load [2,3]. We also find that, incorporating inherent edge capacity on the network, situations can arise where a quantity characterizing the degree of the cascading failure can exhibit a nonmonotonic behavior as a function of the capacity parameter [7]. This implies that, for a given network setting, there can be an interval in the capacity parameter where enhancing it can actually cause the network to be more vulnerable to cascading failures. While our findings are counterintuitive with respect to existing results, we will provide analyses and numerical computations to establish that they are the consequences of considering flow and capacity in a more realistic way, and we expect these phenomena to be generic for complex networks.

In Sec. II, we describe our cascading model with intrinsic edge capacity. In Sec. III, we provide simulation results and analysis for cascading dynamics on regular and small-world networks. In Sec. IV, we study weighted scale-free networks with respect to global cascading breakdown and present evidence for abnormal cascading region. In Sec. V, cascading failures as induced by attacks are discussed and compared with results from previous studies. Conclusion and a brief discussion are offered in Sec. VI.

II. CASCADING MODEL

To incorporate the natural laws governing the flow on the network, we assume that each edge ij has a resistance R_{ij} against the flow, which is determined, for example, by the conductance associated with the edge in an electrical network, or by the size of the vessel in a biofluid network, or by the condition of the road in a transportation network. The bandwidth w_{ij} of edge ij can be defined as being inversely proportional to the resistance: $w_{ij} \sim 1/R_{ij}$, which can be the conductance of an electrical wire, the cross-sectional area of a vessel, or the number of lanes in a transportation network. A flow should obey the Ohm's law: $f_{ij}=(P_i-P_j)/R_{ij}$, where f_{ij} is the flux passing through edge ij, P_i and P_j represent various quantities that can be voltages in an electrical network, hydraulic pressures in a biofluid network, or the amount of traffic in a transportation network. For a networked system, the total flow at any node obeys the Kirchhoff conservation law: $F_i=\Sigma_{j=1}^{N}A_{ij}f_{ij}=\Sigma_{j=1}^{N}A_{ij}(P_i-P_j)/R_{ij}=0$, where A is the adjacency matrix of the network. We then have $\mathbf{GP}=\mathbf{F}$, where $\mathbf{G}$ is the Laplacian matrix of the underlying network whose elements, when taking into account edge resistances, are given by

$$\mathbf{G}_{ij} = \begin{cases} \sum_{l=1}^{N} A_{il}/R_{il}, & i=j \\ -A_{ij}/R_{ij}, & i \neq j. \end{cases} \quad (1)$$

If we randomly choose a pair of nodes i and j as input and output, respectively, with a unit flux, the flow vector $\mathbf{F}$ becomes $F_i=1$, $F_j=-1$, and $F_k=0$ for $k \neq i,j$. The vector $\mathbf{P}$ can then be solved by $\mathbf{GP}=\mathbf{F}$ and the flux of each edge can be calculated by the Ohm's law. The actual flow passing through an edge ij is the sum of flows from all possible input nodes a and output nodes b,

$$F_{ij} = \sum_{a=1}^{N} \sum_{b=1,b<a}^{N} |f_{ij}^{a\to b}|. \quad (2)$$

The capacity of an edge ij can be defined to be the bandwidth of this edge, which is the maximum flux that the edge can handle without congestion or damage. This way, the capacity (bandwidth) of an edge is an inherent characteristic and is inversely proportional to the edge resistance. We write

$$C_{ij} = \alpha/R_{ij}, \quad (3)$$

where α is a capacity parameter. When the capacities are sufficiently large, all edges are functional and the network stays in a free-flow state. However, when a number of edges is disabled, the Laplacian matrix is changed and the flux at some of the remaining edges will also be changed. If the flux at an edge exceeds its capacity, the edge fails, which can induce further edge failures, leading to a cascading process. An important issue is how network topologies affect the cascading dynamics in our model.

If the network is broken into several disconnected components after cascading failures, the input and output can be obtained only within the same component. In this case, the total flux will be considerably reduced compared to the case of a single connected component. As a result, the cascading process will usually stop.

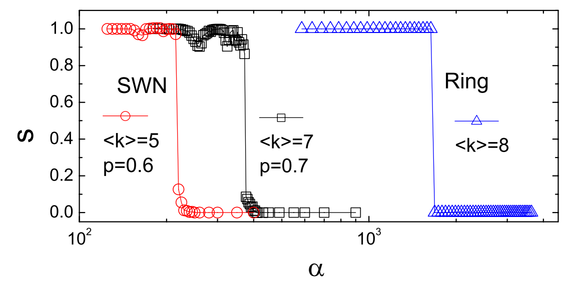

FIG. 1. (Color online) Normalized cascading size s as a function of the capacity parameter α for small-world and regular ring networks. The small-world networks are constructed by randomly rewiring a regular ring network [8]. The average degree $\langle k \rangle$ of the regular ring and the small-world networks, and the rewiring probability p of the small-world networks are shown in the figure.

III. CASCADING DYNAMICS ON REGULAR AND SMALL-WORLD NETWORKS

To gain insights, we first consider regular networks. First all edge resistances are set to be unity and the capacity parameter α is fixed. Then the flow F_{ij} is calculated and compared to C_{ij}. An edge fails if $F_{ij}>C_{ij}$, which changes the network topology and leads to changes in F_{ij} and failures of edges whose loads exceed their capacities at the next time step. This process will continue until all remaining links satisfy $F_{ij}<C_{ij}$. The severeness of an edge-cascading process can be characterized by the ratio s of the number of broken edges at the end of the cascading process to the total number of edges in the network. We shall focus on the relation between s and the capacity parameter α. For simplicity, we set the resistance of all edges to be unity. As shown in Fig. 1, our computations with regular ring networks indicate the existence of a generic first-order phase transition at some critical value α_c, where $s=0$ (free-flow state) for $\alpha>\alpha_c$. For $\alpha<\alpha_c$, all edges are damaged and $s=1$. The transition is thus abrupt in the sense that the value of s changes suddenly from 0 to 1 as α is decreased through α_c. The same phenomenon has been found on two-dimensional lattice networks of different coordination numbers. For ring networks with coordinate number 1, the value of α_c can be calculated analytically. In particular, the sum of fluxes at all edges is given by

$$\sum_{i=1}^{N} \sum_{j=1,j>i}^{N} A_{ij}F_{ij} = \frac{N}{L}\int_0^L (L-l)l\,dl = \frac{N}{6}L^2, \quad (4)$$

where L is the length of the ring. Since the ring topology has a circular symmetry, the fluxes through all edges are the same. We have

$$\alpha_c = \frac{1}{N}\sum_{i=1}^{N} \sum_{j=1,j>i}^{N} \frac{A_{ij}F_{ij}}{R_{ij}} = \frac{1}{6}L^2. \quad (5)$$

In general, the sharp transition in regular networks is due to the narrow distribution of edge fluxes. For instance, in an eight-neighbor lattice, there are only two different fluxes. When α is less than the higher flux, half of all edges with the higher flux will be broken and the higher flux will be distributed to the other half of the edges with the lower flux, lead-

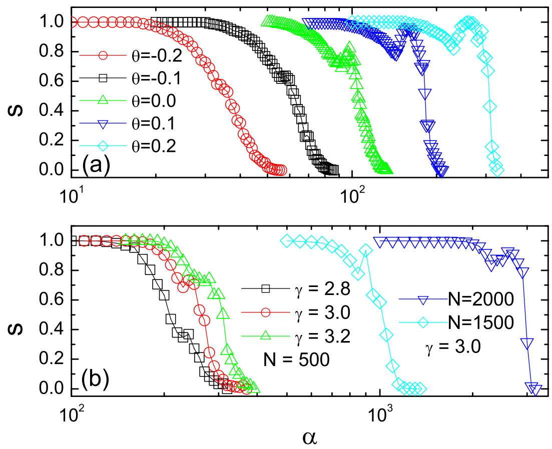

FIG. 2. (Color online) For weighted scale-free networks, normalized cascading size s as a function of the capacity parameter α for (a) different values of the weighting parameter θ and (b) different values of exponent γ of the power-law degree distribution and network size N for $\theta=0$. Barabási-Albert (BA) networks [9] with $N=500$ have been used in (a) and random scale-free networks have been used in (b).

ing to a complete breakdown. The critical value α_c is thus determined by the highest flux.

For a small-world network, the edge fluxes are narrowly distributed due to the homogeneous degree distribution. Thus a cascading phenomenon similar to that on regular networks is expected. As shown in Fig. 1, there exists a sharp transition from a full cascading state to a cascading-free state at some critical value of α_c for two different values of rewiring probability of the small-world network.

IV. CASCADING ON WEIGHTED SCALE-FREE NETWORKS

We now investigate scale-free networks [9]. Since interactions among nodes can be heterogeneous in real-world networks, we incorporate weights into the network connectivity. For example, since hubs in a transportation or in a communication network usually play a key role in handling flow, heavier traffic can be expected for edges connected to various hubs. To make the network less vulnerable to edge cascading, it is reasonable to broaden the bandwidth of those edges or, equivalently, to reduce their resistances to flow. These considerations lead to the following assumption for edge resistance: $R_{ij}=(k_i k_j)^{\theta}$, where k_i and k_j are the degrees of the nodes at the ends of edge ij. This assumption is supported by empirical evidence of weighted networks found in the real world [10]. For $\theta<0$, edges connected to hub nodes have less resistances and broader bandwidths. Figure 2(a) shows, for a scale-free network of $N=500$ nodes, normalized cascading size s as a function of α for different values of θ. We observe the following: (1) there exists a critical value α_c such that for $\alpha>\alpha_c$, $s=0$ so that no cascading failures occur; (2) for $\alpha<\alpha_T$, the value of s approaches unity so that cascading breakdown occurs on a global scale, and (3) for $\alpha \lesssim \alpha_c$, there exists an abnormal regime where s, as a function of α, exhibits a nonmonotonic behavior. In this abnormal

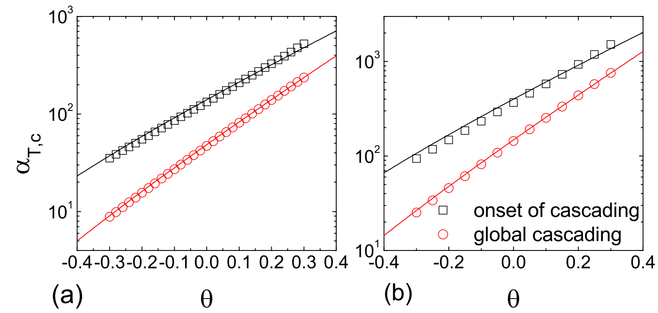

FIG. 3. (Color online) Critical values α_T (lower trace) and α_c (upper trace) as a function of the weight parameter θ for BA scale-free networks of size (a) $N=200$ and (b) $N=500$, on a semilogarithmic scale. Solid lines are analytical predictions from Eqs. (10) and (12).

region, cascading can be more severe for a larger value of the capacity parameter. The abnormal region is more pronounced for larger value of the weighting parameter θ. We have also investigated scale-free networks for different values of the degree-distribution exponent γ and different network sizes N. The abnormal phenomenon, as shown in Fig. 2(b), appears general. (Here, the scale-free networks are constructed by using the configuration model with a pre-existed degree sequence [11].)

To understand the cascading phenomenon in a quantitative manner, we have carried out a heuristic analysis, aided by numerical computations. Figure 3 shows, on a semilogarithmic scale, critical values α_c and α_T as a function of the weighting parameter θ. We observe that both $\alpha_c(\theta)$ and $\alpha_T(\theta)$ are exponential functions. In particular, decreasing θ can delay the occurrence of edge cascading in terms of adjusting the capacity parameter. To explain the exponential functions, we examine the total flux at a node defined as $\bar{F}_i=\sum_{j=1}^{N} A_{ij}F_{ij}$. The degree-dependent total flux $\bar{F}(k)$ can be obtained by averaging over all nodes of degree k. Figure 4 shows that $\bar{F}(k)$ can be approximately fit by an algebraic function of k for different values of θ. A striking phenom-

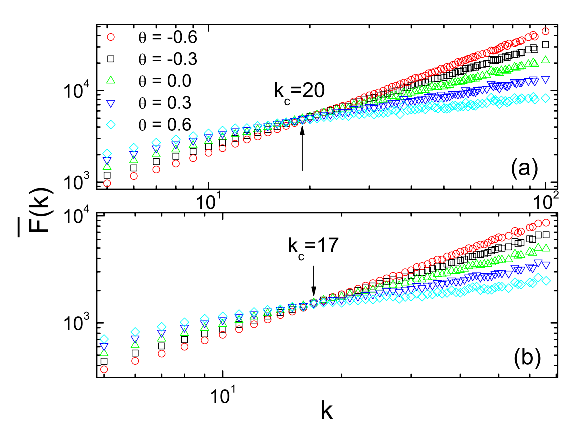

FIG. 4. (Color online) Degree-dependent total flux $\bar{F}(k)$ for different weight parameter θ on BA scale-free networks of size (a) $N=200$ and (b) $N=500$. The critical degrees k_c at the intersecting point are marked by arrows. Our analysis gives that $k_c=\langle k^2\rangle/\langle k\rangle$, which agrees well with the numerically obtained values.

enon is that all curves intersect at a single point. The critical degree at the crossover point k_c is approximately given by $k_c = \langle k^2\rangle / \langle k\rangle$. These numerical findings provide insights into the behavior of the function $\alpha_c(\theta)$. For example, according to Fig. 4, we can deduce that the flux at node i is $\bar{F}_i = ak_i^{\beta}$. The flux of edge ij can be estimated as

$$F_{ij} \approx \frac{1}{2}\left[\bar{F}_i \frac{R_{ij}^{-1}}{\sum_{l=1}^{N} A_{il}R_{il}^{-1}} + \bar{F}_j \frac{R_{ij}^{-1}}{\sum_{l=1}^{N} A_{jl}R_{jl}^{-1}}\right],$$

which states that the flux at two edges connecting to the same node is approximately inversely proportional to the resistances of the edges, and the contributions of both nodes at the ends of an edge to the edge flux should be taken into account. If for any edge ij, the inequality $F_{ij} < C_{ij} = \alpha / R_{ij}$ is satisfied, the network will be in a free-flow state. Noting that the onset critical capacity parameter α_c is determined by the first set of failed edges as α is decreased, we have

$$\frac{1}{2}\left[\frac{ak_i^{\beta}(k_ik_j)^{-\theta}}{\sum_{l=1}^{N} A_{il}(k_ik_l)^{-\theta}} + \frac{ak_j^{\beta}(k_ik_j)^{-\theta}}{\sum_{l=1}^{N} A_{jl}(k_jk_l)^{-\theta}}\right] \approx \alpha_c (k_ik_j)^{-\theta}. \quad (6)$$

Based on the mean-field approximation, we have

$$\sum_{l=1}^{N} A_{il}(k_ik_l)^{-\theta} = k_i^{1-\theta} \sum_{k'=k_{min}}^{k_{max}} P(k'|k_i)k'^{-\theta} = k_i^{1-\theta} \sum_{k'=k_{min}}^{k_{max}} \frac{k'^{1-\theta}P(k')}{\langle k\rangle} = \frac{k_i^{1-\theta}\langle k^{1-\theta}\rangle}{\langle k\rangle}, \quad (7)$$

where the conditional probability $P(k'|k_i) = k'P(k')/\langle k\rangle$ and the identity

$$k_c \approx \sum_{k'=k_{min}}^{k_{max}} k'^{1-\theta}p(k') = \langle k^{1-\theta}\rangle$$

have been used. We can write

$$\alpha_c \approx \frac{a\langle k\rangle}{2\langle k^{1-\theta}\rangle}[k_i^{\beta+\theta-1} + k_j^{\beta+\theta-1}]. \quad (8)$$

For θ close to zero and the degree distribution is $P(k) = 2k_{min}^2 k^{-3}$, we have

$$\frac{\langle k\rangle}{\langle k^{1-\theta}\rangle} = \frac{\int_{k_{min}}^{k_{max}} kP(k)dk}{\int_{k_{min}}^{k_{max}} k^{1-\theta}P(k)dk} = (1+\theta)k_{min}^{\theta}. \quad (9)$$

Since the onset of cascading is determined by the edges between the nodes with the lowest degree, α_c can thus be obtained by setting $k_i = k_j = k_{min}$. We finally have

$$\alpha_c = A(\theta+1)k_{min}^{2\theta}, \quad (10)$$

where the coefficient A is independent of θ and given by $A = ak_{min}^{\beta-1}$. For $\alpha < \alpha_T$, all edges have been damaged. The crossover point k_c can be estimated by the average degree of the two nodes at both ends of a broken edge,

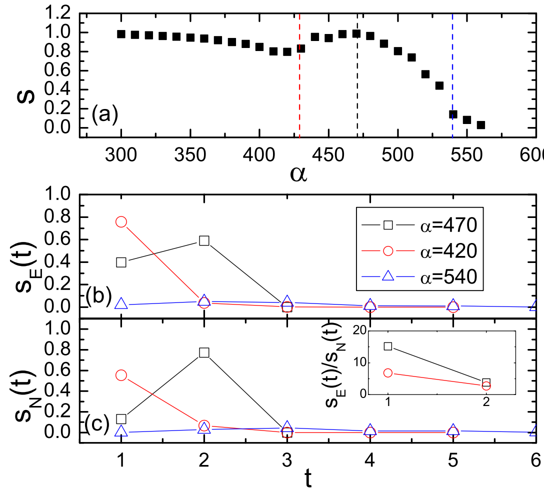

FIG. 5. (Color online) (a) Three representative points in the abnormal region as marked by colored dashed lines. (b) Edge failure size $s_E(t)$ and node failure size $s_N(t)$ as functions of time step t for the three representative values of α. The inset of (c) shows the ratio $s_E(t)/s_N(t)$. BA scale-free networks with $N=500$ and $\langle k\rangle = 5$ have been used.

$$k_c = \sum_i \sum_{j<i} A_{ij}\frac{k_i + k_j}{2}/(Nk_{min}) = \frac{\langle k^2\rangle}{\langle k\rangle}. \quad (11)$$

By setting $k_i = k_j = k_c$ in Eq. (8), the critical value of α_T for global cascading can be obtained as

$$\alpha_T \approx \bar{A}(1+\theta)(k_ck_{min})^{\theta}, \quad (12)$$

where $\bar{A} = ak_c^{\beta-1}$. Predictions from Eqs. (10) and (12) agree well with the numerics, as shown in Fig. 3.

Further insights into the existence of the abnormal region can be gained by examining the process of cascading, step by step, for three typical points in the abnormal region. As shown in Fig. 5(a), we choose three representative values of α, corresponding to the local minimum and maximum values of cascading size s and a lower value of s on the right-hand side of the local maximum, respectively. Both the edge failure size $s_E(t)$ and the node failure size $s_N(t)$ are then measured as functions of time t from the beginning of cascading to the end, where $s_N(t)$ is defined as the number of removed nodes normalized by N in the network at time t (a node is removed if all edges connecting to it are damaged). One can see that at the first step, s_E and s_N are inversely proportional to α, meaning that a lower capacity always leads to more severe damage at the first step, which is intuitive. However, the failure size at the second step triggered by the damage at the first step is not a monotonic function of α. We actually observe that $s_E(2)$ and $s_N(2)$ peak at $\alpha=470$, the local maximum, and their values are much higher than those for $\alpha=420$ and $\alpha=540$. For $t>2$, we find that the value of $s_E(t>2)$ for any value of α is insignificant relative to the total cascading size s. It is thus reasonable to focus on the contributions of $s_E(1)$ and $s_E(2)$ to s. By examining the behaviors of both $s_E(t)$ and $s_N(t)$, we can explain why smaller values of $s_E(1)$ can lead to larger values of $s_E(2)$ based on a compari-

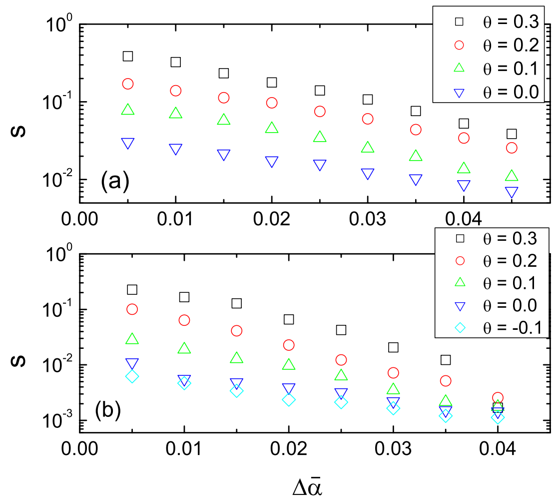

FIG. 6. (Color online) Cascading size s as a function of normalized capacity parameter $\Delta\bar{\alpha}$ for different values of θ for BA networks with size (a) $N=200$ and (b) $N=500$. The cascading event is triggered by a single attack on the largest degree node when the network is in the free-flow state.

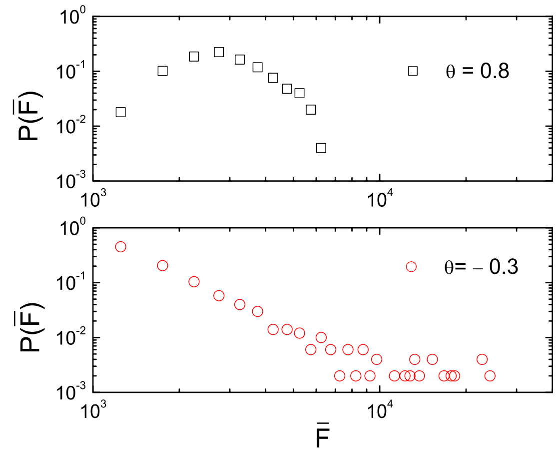

FIG. 7. (Color online) Distribution of flux at nodes $P(\bar{F})$ for different values of θ. BA scale-free networks with $N=500$ have been used.

son between the $\alpha=470$ and the $\alpha=420$ cases. Since there is a unit flow between each pair of nodes, the total flux at all edges, $\Sigma_{i=1}^{N}\Sigma_{j=1,j>i}^{N}A_{ij}F_{ij}$, is positively correlated with the number of nodes N. For a fixed value of N, fewer edges carry larger average flux so that the capacities of the edges are more likely to be exceeded. Consequently, one can expect that more damaged edges with less removed nodes at the first step will trigger more severe failures subsequently. We can use the ratio $s_E(t)/s_N(t)$ to quantify the influence of the network state at time step t to failures at $t+1$, where higher values of $s_E(t)/s_N(t)$ result in higher values of $s_E(t+1)$ and $s_N(t+1)$. As shown in the inset of Fig. 5(c), $s_E(1)/s_N(1)$ for the local maximum ($\alpha=470$) is much larger than that for the local minimum, leading to larger values of $s_E(2)$ and $s_N(2)$ at the local maximum and then the larger cascading size. The existence of the abnormal behavior indicates that enhancing edge capacities may not make network more robust against edge cascading.

V. ATTACK-INDUCED CASCADING

We address the issue of attack-induced cascading. As can be seen from Fig. 4, the flux through a node is positively correlated with the node degree. Attacking the node with the highest degree is thus most likely to trigger a global cascading event. Before the attack, the network is assumed to be in a free-flow state so that the cascading event is due completely to the attack. To make an unbiased comparison, we introduce a normalized parameter, defined as $\Delta\bar{\alpha}=\Delta\alpha/\alpha_c$, where α_c is the onset of a cascading event. The actual capacity parameter becomes $\alpha=\alpha_c(1+\Delta\bar{\alpha})=\alpha_c+\Delta\alpha$. We examine cascading size s as a function of $\Delta\bar{\alpha}$ for different values of θ. As shown in Fig. 6, larger values of θ make the network more fragile against attack-induced cascading breakdown. Moreover, we find that larger values of θ lead to more homogeneous flux at nodes, as shown in Fig. 7. Due to the correlation between θ and s and that between θ and the flux heterogeneity at nodes, we see that heterogeneity in flux can actually make the network more robust against cascading failures. This phenomenon does not arise in any betweenness-centrality-based cascading model. For example, it has been found that intentional attack can be much more fatal to heterogeneous networks than homogeneous networks as a result of cascading breakdown. Disabling the node with the largest load in a heterogeneous network can result in more severe load redistribution to other nodes and consequently lead to more node failures. However, this observation is based on the assumption that the capacity of a node or an edge is linearly proportional to its initial load, which is somewhat artificial. In our model, the capacity of an edge does not depend on the amount of flow passing through it. Our flow model and the associated capacity hypothesis are thus more natural. In fact, we do not assume any linear relation between the initial flux and edge capacity. As a result, for a more heterogeneous network, flux redistribution caused by attack on the highest-flux node can be more easily absorbed by other nodes, effectively inhibiting cascading breakdown.

VI. CONCLUSIONS

In conclusion, we have reexamined the concept of load in network dynamics and proposed a model for cascading failures based on the inherent edge capacities, independent of the initial loads at edges. We find that cascading dynamics can occur for limited edge capacity. For regular and small-world networks, the occurrence of cascading is abrupt but for networks with heterogeneous degree distribution, such as scale-free networks, the transition from free-flow state to global cascading breakdown is continuous. Between these two states there exists an abnormal regime where an increase in the edge capacity can counterintuitively make the cascading process more severe. We also find that heterogeneous flow distribution at nodes can enhance the network's ability to sustain edge cascading, in contrast to previous results.

These findings suggest that cascading behaviors based on the inherent edge capacity are quite different from cascading based on the linear relation that has been studied extensively in the complex-network literature. It is noteworthy that from an applied point of view, controlling cascading failures is an important issue of practical significance. Some controlling strategies have been proposed in Refs. [12,13] to enhance network robustness against random failures and targeted attacks. The key ingredient in our approach is to find the value of θ for optimal robustness by taking into account the investment cost. A comparison study with previous approaches can then be carried out.

ACKNOWLEDGMENTS

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[1] M. E. J. Newman, Phys. Rev. E **64**, 016132 (2001).
[2] A. E. Motter and Y.-C. Lai, Phys. Rev. E **66**, 065102(R) (2002).
[3] L. Zhao, K. Park, and Y.-C. Lai, Phys. Rev. E **70**, 035101(R) (2004); L. Zhao, K. Park, Y.-C. Lai, and N. Ye, *ibid.* **72**, 025104(R) (2005); L. Huang, L. Yang, and K. Yang, *ibid.* **73**, 036102 (2006); L. Huang, Y.-C. Lai, and G. Chen, *ibid.* **78**, 036116 (2008); W.-X. Wang and G. Chen, *ibid.* **77**, 026101 (2008).
[4] M. Anghel, K. Werley, and A. Motter, e-print arXiv:physics/0609217.
[5] I. Simonsen, L. Buzna, K. Peters, S. Bornholdt, and D. Helbing, Phys. Rev. Lett. **100**, 218701 (2008).
[6] D.-H. Kim and A. E. Motter, J. Phys. A: Math. Theor. **41**, 224019 (2008).
[7] Such a function is typically monotonic in the existing literature that defines load according to betweenness.
[8] D. J. Watts and S. H. Strogatz, Nature (London) **393**, 440 (1998).
[9] A.-L. Barabási and R. Albert, Science **286**, 509 (1999).
[10] A. Barrat, M. Barthélemy, R. Pastor-Satorras, and A. Vespignani, Proc. Natl. Acad. Sci. U.S.A. **101**, 3747 (2004); P. J. Macdonald, E. Almaas, and A.-L. Barabási, Europhys. Lett. **72**, 308 (2005).
[11] M. Molloy and B. Reed, Random Struct. Algorithms **6**, 161 (1995).
[12] A. E. Motter, Phys. Rev. Lett. **93**, 098701 (2004).
[13] M. Schäfer, J. Scholz, and M. Greiner, Phys. Rev. Lett. **96**, 108701 (2006).