Transition to global synchronization in clustered networks

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A clustered network is characterized by a number of distinct sparsely linked subnetworks (clusters), each with dense internal connections. Such networks are relevant to biological, social, and certain technological networked systems. For a clustered network the occurrence of global synchronization, in which nodes from different clusters are synchronized, is of interest. We consider Kuramoto-type dynamics and obtain an analytic formula relating the critical coupling strength required for global synchronization to the probabilities of intra-cluster and intercluster connections, and provide numerical verification. Our work also provides direct support for a previous spectral-analysis-based result concerning the role of random intercluster links in enhancing the synchronizability of a clustered network.

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I. INTRODUCTION

Synchronization in complex networks has become an active area of research in network science [1–11]. Previous works have addressed the synchronizabilities of small-world networks [3,4], scale-free networks [7], weighted networks [8,9], and gradient networks [11]. More recently, synchronization in complex clustered networks has been investigated [12–14]. The approach employed in most existing studies is the master stability function [15], which enables the synchronizability of the network under study to be inferred through its spectral properties [16]. A requirement of this approach is that all individual nodes, when isolated, are governed by an identical set of dynamical equations. In realistic situations this may not be the case. Thus it is interesting to ask what can happen to synchronization when the oscillators are nonidentical. The problem can easily become intractable if the local oscillatory dynamics is complicated, due to the complex connecting topology of the network. Given the complexity of the system, a compromise is to make the local dynamics as simple as possible, yet nontrivial. The simplest oscillatory dynamics is that given by an ideal phase oscillator, $\dot{\phi}=\omega$, where ω is the frequency. Assuming each node is governed by such a phase evolution, heterogeneity in the node dynamics can be modeled by assigning different frequencies to different oscillators. In general, the frequencies can be assumed to be drawn from a random distribution. For the simple network topology where the coupling is all to all, the phase-coupled oscillator network is the classical Kuramoto model [17,18], which is analytically treatable in many aspects. Recently, the problem of transition to synchronization of phase-coupled oscillators in scale-free networks has been addressed [19]. In particular, an analytic formula for the critical coupling strength at the *onset* of synchronization, where a properly defined order parameter (see below) characterizing the degree of synchronization begins to increase from zero, has been obtained.

In this paper, we investigate the dynamics of phase-coupled oscillators on complex clustered networks. A clustered network typically consists of a number of sparsely connected subnetworks, or clusters, where the connectivity within each individual cluster is dense. Such a clustered structure is commonly seen in social [20] and biological [21] networks, and it also appears in certain technological networks [22]. Due to the dense connections in any individual cluster, synchronization within the cluster can be expected as the coupling strength is increased from zero. In particular, since the connections among nodes belonging to different clusters are sparse, for relatively small coupling strength the dynamics in distinct clusters can be regarded as being independent of each other. Roughly, this means that the onsets of synchronization in different clusters can be treated as independent of each other as well. For a complex clustered network, the more relevant and perhaps the more interesting issue concerns thus the occurrence of *global* synchronization among nodes in different clusters. Let p_o be the probability of an intercluster link, i.e., the probability for a pair of nodes, each belonging to a different cluster, to be connected. The critical coupling strength ε_c required for global synchronization depends on p_o. Intuitively, ε_c decreases as p_o is increased. The main contributions of this paper are an analytic formula relating ε_c and p_o and numerical support.

In Sec. II, we present our theoretical derivations that lead to an explicit dependence of ε_c on p_o. In Sec. III, we provide numerical support and also address the issue of the role of the nature of intercluster links on network synchronizability through direct calculation of ε_c. Conclusions are summarized in Sec. IV.

II. THEORY

We consider the following modified Kuramoto model for a clustered network of N nodes:

$$\dot{\phi}_n = \omega_n + \varepsilon \sum_{m=1}^{N} a_{nm} \sin(\phi_m - \phi_n), \tag{1}$$

where $0 \le \phi_n \le 2\pi$ is the phase variable of node n, ω_n is the corresponding frequency, ε is a coupling parameter, and a_{nm}

are the elements of the adjacency matrix determined by the connecting topology of the clustered network. Specifically, $a_{nm}=1$ if nodes n and m are connected, $a_{nm}=0$ otherwise, and $a_{nn}=0$. The frequencies are drawn from a probability distribution $g_0(\omega)$, say Gaussian or uniform distribution, which can be conveniently chosen to center about zero. We assume that the network consists of M identical clusters, each of L nodes. For convenience we choose N such that $N=ML$. Without loss of generality we further assume that an arbitrary node has probability p_i to connect with other nodes inside the cluster. The network is thus characterized by two probabilities: the probability of intracluster links p_i, and the probability of intercluster links p_o. For typical clustered networks, usually $p_i \gg p_o$.

The standard approach to analyzing Eq. (1) is to introduce a proper order parameter. Since our focus is on synchronization among nodes in different clusters, it is necessary to distinguish such nodes. We thus follow Ref. [19] to consider the following local order parameter r_n:

$$r_n e^{i\Psi_n} \equiv \sum_{m=1}^{N} a_{nm}\langle e^{i\phi_m(t)}\rangle_t, \tag{2}$$

where $\langle\cdot\rangle_t$ denotes a time average, r_n is real valued and positive, and Ψ_n is the time-averaged phase of node n. Using Eq. (2), Eq. (1) becomes

$$\dot{\phi}_n = \omega_n - \varepsilon r_n \sin[\phi_n(t) - \Psi_n] + \varepsilon h_n(t), \tag{3}$$

where $h_n(t)=\mathrm{Im}[e^{-i\phi_n(t)}\Sigma_{m=1}^{N} a_{nm}(e^{i\phi_m(t)}-\langle e^{i\phi_m(t)}\rangle_t)]$ and $\mathrm{Im}[\cdot]$ stands for the imaginary part. We are interested in the coupling regime where all oscillators are nearly synchronized. In this case, $r_n=|\Sigma_{m=1}^{N} a_{nm}\langle e^{i\phi_m(t)}\rangle_t| \approx |\langle e^{i\phi_m(t)}\rangle_t|\Sigma_{m=1}^{N} a_{nm} \approx |\langle e^{i\phi_m(t)}\rangle_t| d_n$, where $d_n=\Sigma_{m=1}^{N} a_{nm}$ is the degree of node n. For the term $h_n(t)$ note that $\Delta_m(t) \equiv e^{i\phi_m(t)}-\langle e^{i\phi_m(t)}\rangle_t$ is in fact the amount of fluctuation of $e^{i\phi_m(t)}$ with respect to its time average. Thus, roughly, two such fluctuation terms $\Delta_m(t)$ and $\Delta_n(t)$ where $n\neq m$, can be regarded as uncorrelated. This means that $h_n(t)$ is approximately the sum of d_n uncorrelated random terms. We have $h_n(t)\sim\sqrt{d_n}$. Since nodes are densely connected (at least within the same cluster), we have $d_n \gg 1$. Thus $r_n \gg |h_n(t)|$ and we can neglect $h_n(t)$ in Eq. (3). This leads to

$$\dot{\phi}_n \approx \omega_n - \varepsilon r_n \sin[\phi_n(t) - \Psi_n]. \tag{4}$$

Note that the same approximation has been used in Ref. [19] for scale-free networks. In that case, each node in the network is assumed to be densely connected. In our case of clustered networks, as we have argued, the assumptions leading to Eq. (4) are automatically satisfied due to the defining nature of the networks.

Toward the global synchronization point, all oscillators become increasingly phase locked. The locked phase is given by the stable fixed point of Eq. (4), which is given by

$$\phi_n^0 = \Psi_n + \sin^{-1}\left(\frac{\omega_n}{\varepsilon r_n}\right). \tag{5}$$

Near the global synchronization point, i.e., when $\varepsilon=\varepsilon_c$, we have $\Psi_m \approx \Psi_n$. (We have verified numerically that this approximation indeed holds for clustered networks.) Following the argument in Ref. [19], we can write the order parameter as

$$r_n \approx \sum_{m=1}^{N} a_{nm}\sqrt{1-[\omega_m/(\varepsilon_c r_m)]^2}. \tag{6}$$

Since d_n is large, we expect the frequency distribution of all nodes connected to node n to be representative of the frequency distribution of all oscillators in the network. The summation can thus be replaced by an integral. We have

$$r_n \approx \varepsilon_c \sum_{m=1}^{N} a_{nm} r_m \int_{-\infty}^{+\infty} g(z\varepsilon_c r_m)\sqrt{1-z^2}\,dz, \tag{7}$$

where $z \equiv \omega/(\varepsilon_c r_m)$ and the frequency distribution function $g(\omega)$ is determined by the threshold value of the global order parameter r, defined as $r=e^{i\psi}=\frac{1}{N}\Sigma_{m=1}^{N} e^{i\phi_m}$, for characterizing global synchronization, which in general *differs from the initial distribution* $g_0(\omega)$ for $\varepsilon=0$ when oscillators are independent of each other. Since, as the global synchronization state is approached, the frequency distribution becomes increasingly narrow, the integral in Eq. (7) is effectively independent of intercluster probability p_o and tends to a constant. In addition, in a clustered network, the connections inside each cluster are dense, and connections among clusters are random, thus we have $r_n \approx r_m$. To evaluate the summation in Eq. (7), we note that a node is connected with $(L-1)$ nodes in the same cluster with probability p_i, and simultaneously the node is connected with a node from a different cluster with probability p_o. There are $(N-L)$ such nodes. We thus have $\Sigma_{m=1}^{N} a_{nm}=p_i(L-1)+p_o(N-L)$. These considerations finally lead to

$$\varepsilon_c = \frac{C}{p_i(L-1)+p_o(N-L)}, \tag{8}$$

where the constant C depends on the threshold value of the global order parameter r for defining global synchronization, which can be regarded as a fitting parameter. Equation (8) is our main result. It says that given the intracluster probability p_i for a certain clustered network, as the probability p_o of an intercluster link is increased, the critical coupling strength required for global synchronization is decreased. In particular, for a network that contains many clusters (i.e., $N \gg L$), if p_o is not close to zero, ε_c is approximately inversely proportional to p_o.

III. NUMERICAL SUPPORT

We now provide numerical support for our main result. We first demonstrate that, for small probability p_o of an intercluster link, synchronous clusters occur, as shown in Figs. 1(a)–1(d) for $\varepsilon=1.0$. The network has $N=300$ nodes grouped into M clusters locating on a ring, each being connected to its nearest-neighbor clusters. In this case, probability p_o is close to zero. The initial frequencies ω_i of oscillators are drawn from a Gaussian distribution of zero mean and unit variance. In particular, Fig. 1(a) shows the final frequency versus oscillator index for $M=3$ and Fig. 1(b) displays the time evolution of the global order parameter r. We observe

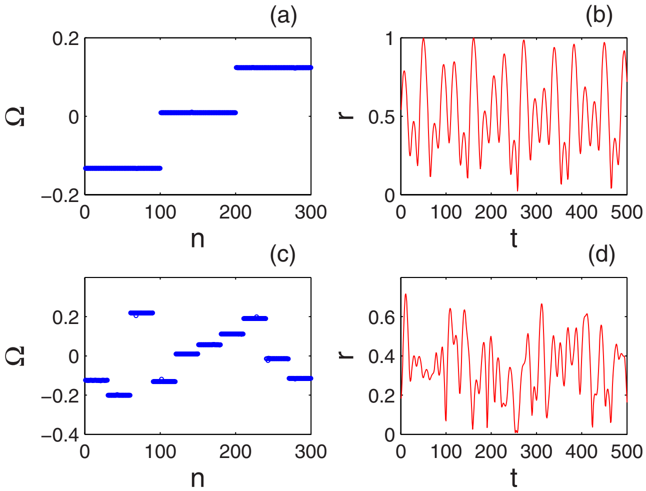

FIG. 1. (Color online) For a clustered network of $N=300$ nodes, $p_i=0.7$, and $p_o \approx 0$, (a) final frequency distribution and (b) evolution of the network-averaged order parameter, both for $M=3$. (c), (d) Respective plots for $M=10$. The coupling parameter is $\varepsilon=1.0$ for (a)–(d).

distinct frequencies associated with each cluster, and the average value of the order parameter is below unity. Similar behaviors occur for $M=10$, as shown in Figs. 1(c) and 1(d), respectively. The formation of the synchronization clusters is apparently independent of the initial frequency distribution, as we have verified numerically.

We next increase p_o, the probability of intercluster links. For a fixed value of p_o, as the coupling parameter ε is increased from zero, synchronization clusters form, followed by a transition to global synchronization. Figures 2(a)–2(c) show the final frequency distributions for $\varepsilon=0.01$, 0.5, and 1.0, respectively, where the system parameters are $N=300$, $M=10$, $p_i=0.7$, and $p_o=0.002$. We see that for $\varepsilon=1.0$, global synchronization, as characterized by one common frequency for all oscillators in the network, has already been achieved for this ten-cluster network even when the intercluster linking probability is quite small.

Since for clustered networks, the intercluster probability p_o plays a dominant role in shaping the network topology, we now fix the intracluster probability p_i to verify the dependence of ε_c on p_o. To do so we calculate, for a number of fixed values of p_o, the global order parameter r as a function of ε, as exemplified by Fig. 3 for five fixed values of p_o. We observe that, as ε is increased from zero, r increases almost immediately from a small residual value for $\varepsilon \approx 0$, indicating the occurrence of synchronous clusters. This reinforces our reasoning that the onset of local synchronization, on which almost all previous works on Kuramoto-type models focus, is a relatively straightforward (if not trivial) issue for a network that is already structurally clustered. Instead, global synchronization is more relevant. Setting a threshold value near 1 for r yields ε_c, the critical coupling parameter for global synchronization. Figures 4(a) and 4(b) show ε_c versus p_o for $p_i=0.7$ and $p_i=1.0$, respectively. The clustered network has $N=500$ nodes with $M=5$ clusters, and the initial frequencies are drawn from a Gaussian distribution. In Fig. 4, the circles and crosses are for $r_c=0.999$ and $r_c=0.99$, respectively, and the solid curves are theoretical predictions. We observe an excellent agreement between theory and numerics. Furthermore, comparing Figs. 4(a) with 4(b), we find that increasing probability p_i of intracluster links generally enhances global synchronization, as predicted by Eq. (8). Similar agreement has been obtained for uniform initial frequency distribution (in $[-1,1]$), as shown in Figs. 4(c) and 4(d).

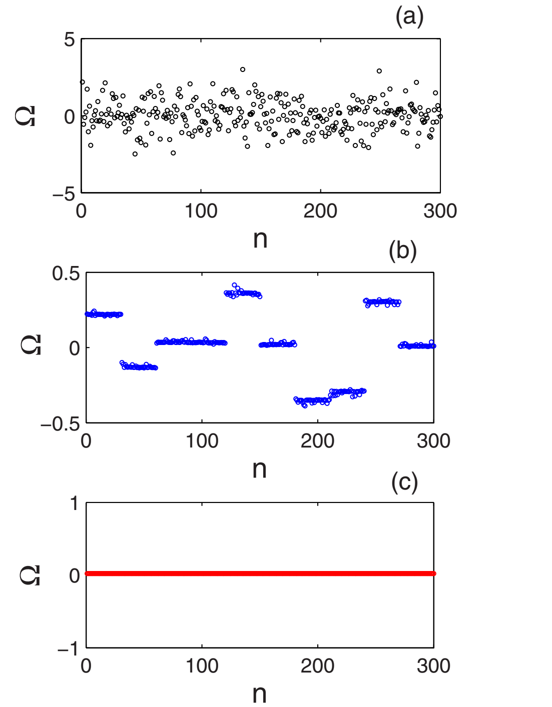

FIG. 2. (Color online) For a clustered network of $N=300$, $M=10$, $p_i=0.7$, and $p_o=0.002$, (a)–(c) final frequency distribution for $\epsilon=0.01$, 0.5, and 1.0, respectively. Global synchronization has already occurred for $\varepsilon=1.0$.

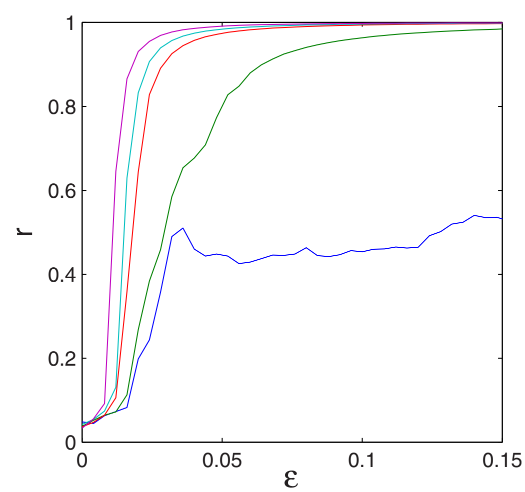

FIG. 3. (Color online) The global order parameter versus the coupling parameter for $p_o=0.001$, 0.01, 0.05, 0.1, and 0.2, corresponding to the curves from bottom up. For very small p_o, only clustering occurs. Global synchronization can be achieved with increase of p_o. The network parameters are $N=500$, $M=5$, and $p_i=0.7$.

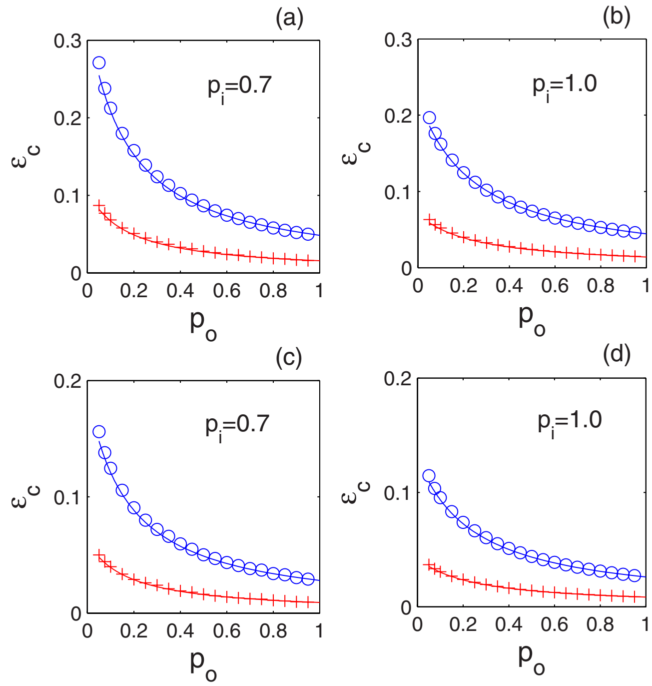

FIG. 4. (Color online) For a network of $N=500$ nodes and $M=5$ clusters, ε_c versus p_o for Gaussian (a), (b) and uniform (c), (d) initial frequency distribution. The circles and crosses are for $r_c=0.999$ and $r_c=0.99$, respectively. The solid curves are from theory.

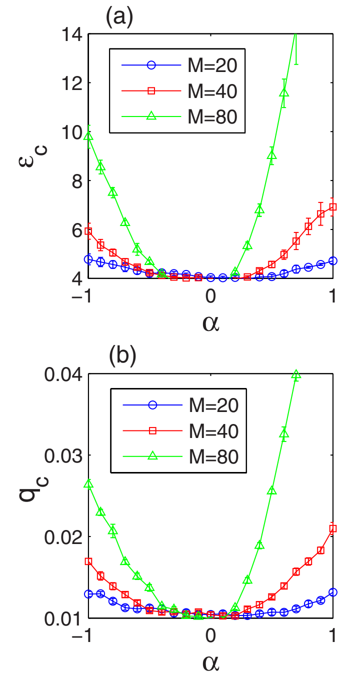

FIG. 5. (Color online) Effect of varying the nature of intercluster links on synchronizability for a network with ring-cluster topology: (a) ε_c versus α for $q=0.02$ (the link ratio) and (b) q_c versus α for $\varepsilon=1.0$. The data points are results of averaging over 40 network realizations.

In Ref. [12], it is emphasized that for a clustered network, the *random* intercluster links can enhance the network synchronizability most effectively. If the intercluster links are placed not completely randomly, even when their number is large, the network synchronizability may not be strong. This result is obtained based on analyzing the behavior of the eigenratio of the coupling matrix, which is meaningful only for identical oscillators. Here, the simplicity of the node dynamics (i.e., simple phase oscillator) allows us to address the effect of the nature of intercluster links in a direct and quantitative manner, for nonidentical oscillators. To be concrete and to compare results with those in Ref. [12], we use the same model of clustered network, i.e., a ring topology for clusters with intracluster probability $p_i=1$, as in Ref. [12]. The probability of intercluster links can be written as $p(l)$, where l is the distance between two clusters. In Ref. [12], $p(l)$ is assumed to be exponential: $p(l)\sim e^{-\alpha l}$, where $\alpha>0$ is a parameter that determines the nature of the intercluster links. In particular, for $\alpha>0$, short-range intercluster links are favored, while long-range links are more likely for $\alpha<0$. Random intercluster links are favored for $\alpha\approx 0$. For a fixed value of the link ratio q, the ratio of the number of intercluster links to the total number of links in the network, we can calculate ε_c, the critical coupling strength for global synchronization as a function of α. Since a smaller value of ε_c indicates stronger network synchronizability, we expect ε_c to reach minimum for $\alpha\approx 0$ when intercluster links are completely random. This behavior has indeed been observed, as shown in Fig. 5(a) for three cases: $M=20$, 40, and 80, where $q=0.02$ and $L=5$ are fixed. Likewise, we can fix the coupling parameter ε and investigate how q_c, the critical link ratio, varies with α. Since a smaller value of q_c implies stronger network synchronizability, we expect q_c to be minimal for $\alpha\approx 0$ as well. Again, this behavior has been numerically observed, as shown in Fig. 5(b), where $\varepsilon=1.0$. These results thus confirm and generalize the previous finding concerning the interplay between the network synchronizability and the nature of the intercluster links.

IV. DISCUSSIONS

In summary, we have addressed the occurrence of global synchronization in clustered networks of heterogeneous phase oscillators. As the probability of intercluster links is increased, the network becomes more synchronizable, as characterized by smaller values of the critical coupling parameter required for global synchronization. While this result can be intuitively expected, we have obtained an explicit theoretical formula relating the dynamical to the topological properties of the network, and provided strong numerical support. The relevance of clustered networks to real-world situations, particularly in biology and social science, has been increasingly recognized. Our work provides a quantitative criterion for predicting under what conditions global synchronization may occur in these networks.

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[16] A network of N coupled, identical oscillators permits a synchronization manifold in general, in which the states of all oscillators evolve identically with time. Synchronization can be physically achieved if the manifold is stable with respect to perturbation transverse to it. This requires all transverse Lyapunov exponents to be negative. The master-stability function is the largest Lyapunov exponent in every transverse subspace. Topologically, the network can be characterized by an $N \times N$ matrix, which usually has a spectrum of eigenvalues $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \ldots \leq \lambda_N$, where $\lambda_1 = 0$ is associated with the synchronization manifold itself, and the remaining $N-1$ eigenvalues are with the $N-1$ transverse subspaces. Each transverse subspace can thus be described by an effective coupling parameter proportional to its eigenvalue. To guarantee synchronization, the master-stability function evaluated for the effective coupling parameters must be negative. Since the function can be negative for a finite range of its argument, synchronization requires that all effective coupling parameters fall in this range. This is more difficult to achieve if the range of the eigenvalues is large, and vice versa. The synchronizability of the network is thus determined by the spread of the eigenvalue spectrum of the coupling matrix. In particular, a network is more synchronizable if the spread is narrower, and vice versa.
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