Enhancing synchronization based on complex gradient networks

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The ubiquity of scale-free networks in nature and technological applications and the finding that such networks may be more difficult to synchronize than homogeneous networks pose an interesting phenomenon for study in network science. We argue and demonstrate that, in the presence of some proper gradient fields, scale-free networks can be more synchronizable than homogeneous networks. The gradient structure can in fact arise naturally in any weighted and asymmetrical networks; based on this we propose a coupling scheme that permits effective synchronous dynamics on the network. The synchronization scheme is verified by eigenvalue analysis and by direct numerical simulations using networks of nonidentical chaotic oscillators.

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Complex networks have attracted a great deal of interest since the discoveries of the small-world [1] and scale-free [2] properties. Roughly, small-world networks are characterized by a locally highly regular connecting structure and a globally small network distance, while the defining characteristic of scale-free networks is a power-law distribution in the number of links or the degree variable. Signatures of small-world and scale-free networks have been discovered in many natural and man-made systems [3–5], and they constitute the cornerstones of modern network science.

At the network level, synchronization is one of the most common dynamical processes. For instance, in biology, synchronization of oscillator networks is fundamental [6]. In a computer network designed for large scale, parallel computation, achieving synchronous timing is essential. Recent studies of the synchronizability of complex networks have revealed that small-world and scale-free networks, due to their small network distances, are generally more synchronizable than regular networks [7–9]. A somewhat surprising finding is that a scale-free network, while having smaller network distances than a small-world network of the same size, is actually more difficult to synchronize [9]. This counterintuitive phenomenon can be explained heuristically as due to the blockade of communication, or interaction, among nodes due to the highly heterogeneous degree distribution seen in scale-free networks. Considering the ubiquity of scale-free networks and the importance of synchronization in network functions, the finding seems to have generated a paradox. Since the networks considered in the original study [9] are unweighted and undirected, recent efforts have been focused on searching for network configurations incorporating weights and directionality, to achieve more efficient synchronization in scale-free networks [10–12]. For instance, in Ref. [10], the coupling strength for a given node from other connected nodes (incoming coupling strength) in the network is determined by the local degree of this node. In this case, the average degree of the network is the key to synchronization and, under certain conditions, scale-free networks can indeed be synchronized more easily as compared with homogeneous networks [10]. In Ref. [11], it has been proposed that high synchronizability can be achieved when the incoming coupling strength to a node is matched by the betweenness centrality of the node. Since knowing the betweenness centrality requires knowledge about the entire network connection topology, this scheme may be said to be based on global information. In the situations considered [10,11], the couplings are directed and asymmetrical.

In this paper, we propose a scheme to address the synchronizability of asymmetrical and weighted complex networks. The setting is quite general, incorporating, for any pair of nodes in the network, both the directionality and the asymmetry of the coupling. The basic idea is to regard such a network as the "superposition" of a symmetrically coupled network and a directed network, both being weighted. A weighted, directed network is actually a *gradient network* [13,14], a class of networks for which the interactions or couplings among nodes are governed by some gradient field on the network. Hypothesizing an appropriate gradient field based on a few elementary considerations of realistic networks, we are able to come up with a coupling scheme and demonstrate that it can lead to networks that are more synchronizable than those from previous schemes. (Our construction of the new coupling scheme can also be regarded as a detailed derivation of the "optimal" scheme proposed in Ref. [15], where a similar configuration is briefly introduced based on empirical observations.) Indirect synchronizability analysis based on eigenvalues of the coupling matrix and direct simulation of oscillator networks provide support for the effectiveness of our scheme.

We consider oscillator networks of the form

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) - \varepsilon \sum_{j=1}^{N} G_{i,j}\mathbf{H}(\mathbf{x}_j), \quad i = 1, \ldots N, \tag{1}$$

where $\mathbf{F}(\mathbf{x}_i)$ governs the local dynamics of uncoupled node i, $\mathbf{H}(\mathbf{x})$ is a coupling function, ε is the coupling strength, and $G_{i,j}$ is an element of the coupling matrix $\mathbf{G}$ that is completely determined by the connecting topology of the underlying network. In general, $\mathbf{G}$ is asymmetrical. Let $G_{i,j}$ be the cou-

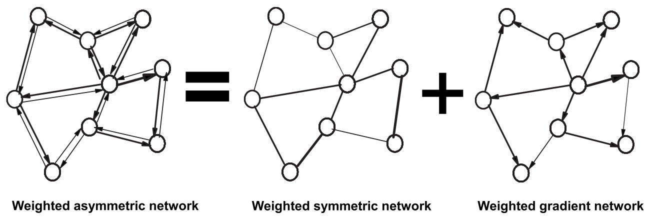

FIG. 1. Schematic illustration of how a general weighted, asymmetrical network may be regarded as a superposition of a symmetrical and a directed (or gradient) network, both weighted.

pling from node j to node i; we have $G_{i,j} \neq G_{j,i}$. Defining $\Delta G_{i,j} \equiv G_{i,j} - G_{j,i}$, we can write $G_{i,j} = (G_{i,j} + G_{j,i})/2 + \Delta G_{i,j}/2$, where the first term is a symmetrical coupling, and the second term represents a directed coupling. Since $\Delta G_{i,j} = -\Delta G_{j,i}$, the direction of the coupling is defined to be from node j to i if $\Delta G_{i,j} > 0$, and vice versa. The original network can thus be regarded as being composed of a symmetrical network characterized by the symmetrical coupling term, and a directed network represented by $\Delta G_{i,j}$. Both networks are weighted since the coupling value depends on the indices i and j. This "decomposition" idea is shown schematically in Fig. 1.

An important goal in the study of synchronization of complex networks is to figure out the appropriate coupling matrix $G_{i,j}$ to make the network as synchronizable as possible [10–12]. In this regard, one can examine the spread of the eigenvalue spectrum of the coupling matrix. A network is generally more synchronizable when the spread is narrower [8,9]. In particular, let $0 = \lambda_1 \leq \lambda_2 \cdots \leq \lambda_N$ be the eigenvalue spectrum of the coupling matrix. Then the smaller the ratio λ_N/λ_2, the more likely synchronous dynamics is to occur on the network.

Our idea to construct synchronizable networks is based on the concept of gradient networks [13,14]. To define a gradient network, consider a network denoted by $\Sigma(V,E)$, where V stands for the set of nodes (vertices) and E denotes the set of links (edges) that can be conveniently specified by the adjacency matrix $A = \{a_{i,j}\}$, where $a_{i,j} = 1$ if i and j are connected, $a_{i,j} = 0$ otherwise, and $a_{i,i} = 0$. Consider a scalar field denoted by $h = \{h_1, \ldots, h_N\}$, where h_i is the scalar assigned to node i. In practice, the scalar field can be generated by the potential or temperature in chemical systems, the information concentration in technological systems, or the rate of processing and adequacy in neuronal systems [13,14]. Regarding the problem of network synchronization, a natural choice is to define the scalar field on node degrees. Let k_i be the degree of node i. We define the neighbors of i as the set of nodes that are linked to it: $V_i = \{j \in V | a_{i,j} = 1\}$. This way a directed link pointing to i can be established from one of its neighbors, if this neighbor has the highest value of the scalar field. If several neighbors have the same scalar field, one is chosen randomly to have a link pointing to i. A gradient network is the collection of all the directed links [13]. Regardless of the topology of the originally undirected network, e.g., regular, random, or scale-free, the way in which the gradient is established stipulates that there be no loops in the network except self-loops. Previous work has shown that, for a homogeneous network of random scalar distribution, the degree distribution of gradient network follows a power-law scaling, $P(k) \sim k^{-\varsigma}$, where $\varsigma \approx -1$ [13].

The conventional way [13,14] of constructing a gradient network has the drawback of stipulating the equality of the number of directed links and the number of nodes. This is not compatible with the directed network component that can be extracted from an arbitrary oscillator network, as in Fig. 1. Thus a generalized definition of the gradient network is needed. A simple remedy is to consider a pair of linked nodes and direct the link according to a prescribed scalar field. This way the number of links in the directed network is the same as the number of links in the original network, in consistency with the decomposition scheme in Fig. 1.

We now present heuristic considerations that lead us to a class of asymmetrical network possessing synchronizability superior to that of previous networks reported in the literature. To propose a gradient field suitable for synchronization, we assume that any given node can access only local information about its neighbors. This consideration is more of a practical nature, as global information about the whole network is usually not readily available for an arbitrary node in the network. Thus the value of the gradient field at node i is determined by, for instance, its degree and the degree information of its neighbors. Our choice is

$$h_i = k_i^{\beta} \sum_{l \in V_i} k_l^{\beta}, \tag{2}$$

where β is a control parameter (the function of β will be explained later). The adoption of Eq. (2) is partially motivated by wide observations in real networks, including scientific collaboration networks, airport networks, and metabolic networks, where weighted asymmetric links are reported and the gradient between nodes has been found to exhibit a strong correlation with the corresponding degrees [17]. Now consider an arbitrary pair of *connected* nodes, say i and j, where $A_{i,j} \neq 0$. We have

$$\Delta G_{j,i} \sim h_i - h_j = k_i^{\beta} \sum_{l \in V_i} k_l^{\beta} - k_j^{\beta} \sum_{l \in V_j} k_l^{\beta}. \tag{3}$$

If both i and j are hubs, we have $k_i \approx k_j$, $\Sigma_{l \in V_i} k_l^{\beta} \approx \Sigma_{l \in V_j} k_l^{\beta}$ and, hence, $\Delta G_{i,j} \sim 0$. Thus the interactions between two hub nodes are mostly nondirectional. The recent finding shows that, during the process of network synchronization, hub nodes are usually synchronized first and the synchronized hubs act as the "core" in propagating the synchronous state over the entire networks [18]. This finding indicates that, to achieve synchronization, it is advantageous to establish efficient coupling between the hubs. Since symmetric coupling is more efficient in achieving partial synchronization among the hubs than directional couplings [18], it is reasonable to set nondirectional coupling between the hub nodes. On the other hand, if i is a hub node and j is not, then $h_i \gg h_j$ and the interaction between them is strongly directed. To obtain an explicit expression for $\Delta G_{j,i}$, we need to include a normalization constant in Eq. (3). For convenience, we write

$$\Delta G_{i,j} = \frac{1}{C_{i,j}} \left(k_i^{\beta} \sum_{l \in V_i} k_l^{\beta} - k_j^{\beta} \sum_{l \in V_j} k_l^{\beta} \right).$$

A natural requirement for the normalization constant $C_{i,j}$ is that it be symmetrical with respect to nodes i and j: $C_{i,j} = C_{j,i}$. We choose

$$C_{i,j} = \sum_{l \in V_i} \sum_{l' \in V_j} k_l^{\beta} k_{l'}^{\beta}, \tag{4}$$

which leads to

$$\Delta G_{j,i} = G_{j,i} - G_{i,j} = \frac{k_i^{\beta}}{\sum_{l \in V_j} k_l^{\beta}} - \frac{k_j^{\beta}}{\sum_{l' \in V_i} k_{l'}^{\beta}}.$$

Incorporating the definition of the adjacency matrix, we have thus arrived at the following choice for the coupling matrix $G_{i,j}$:

$$G_{i,j} = -\frac{A_{i,j} k_j^{\beta}}{\sum_{j=1}^{N} A_{i,j} k_j^{\beta}} \quad \text{for } i \neq j. \tag{5}$$

For convenience, we choose $G_{i,i}=1$. Note that, regardless of the value of β, the total coupling cost of the network remains constant. Different values of β simply correspond to different distributions of the coupling. [It is worthy of note that the coupling scheme Eq. (5), the one we have derived from the gradient-network point of view, has essentially the same configuration as the empirical scheme proposed in Refs. [15,16]].

To demonstrate the synchronizability of the class of networks as defined by Eq. (5), we have carried out a series of numerical tests. Please note that the coupling matrix can be written as $G=QLD^{\beta}$, with $D=\text{diag}\{k_1, k_2, \ldots, k_N\}$ the diagonal matrix of degrees and $Q=\text{diag}\{1/\Sigma_j L_{1,j} k_j^{\beta}, \ldots, 1/\Sigma_j L_{N,j} k_j^{\beta}\}$ the normalization factors on rows of G. From the identity

$$\det(QLD^{\beta} - \lambda I) = \det(Q^{1/2} D^{\beta/2} L D^{\beta/2} Q^{1/2} - \lambda I), \tag{6}$$

we can see that the eigenvalues of the asymmetric matrix G are equal to those of the symmetric matrix $H = Q^{1/2} D^{\beta/2} L D^{\beta/2} Q^{1/2}$, which are real and non-negative. We consider an ensemble of scale-free networks of $N=1024$ nodes with average degree $\langle k \rangle = 6$. Figure 2(a) shows the eigenratio R versus the control parameter β, for $\gamma=3.0$. We observe a continuous decrease of R as β is increased, indicating improved network synchronizability for large values of β. Of particular interest is the region where $\beta > 0$. In this case, the coupling in the gradient-network component is from large-degree to small-degree nodes. Incorporating the symmetrical-network component, this means that large-degree nodes have more significant influence than small-degree nodes, an attribute that can be expected in realistic networks.

The decrease of the eigenratio as β is increased can be explained heuristically, as follows. For the limiting case of $\beta \to \infty$, the only contribution to the coupling that a node receives is from the node with the largest degree among all the neighboring nodes. Every node in the network, except for the largest-degree node, receives coupling from another node but provides coupling to a *different* node. This is effectively a one-way coupling scheme, which corresponds to a "treelike" structure. The largest-degree node, however, can receive coupling from and provide coupling to the same node, the one in its neighboring set with the largest degree. There is then a "loop" structure, but it is associated only with the largest-degree node in the network. For such a network, we have $\lambda_N=2$ (associated with the loop structure only), and $\lambda_i=1$ ($i=2, \ldots, N-1$) (associated with the other nodes which are one-way coupled). The eigenratio is thus 2. As β is increased from zero, we expect to observe a continuous decrease of the eigenratio toward this limiting value [19].

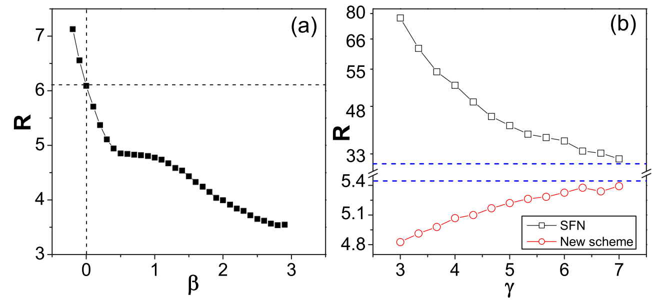

FIG. 2. (Color online) Ensemble of scale-free networks with $N=1024$ and $\langle k \rangle = 6$ under the coupling scheme defined by Eq. (5). (a) Eigenratio R versus the control parameter β. There is a continuous decrease of R as β is increased, indicating improved network synchronizability in the large-β regime. (b) Eigenratio R versus the degree exponent γ for unweighted and symmetrical networks (upper trace) and for networks under the coupling scheme defined by Eq. (5) by setting $\beta=1.5$ (lower trace). The two dashed lines represent the corresponding synchronizability of homogeneous networks under the situation of unweighted symmetrical coupling (the upper line) and weighted asymmetrical coupling (the lower line). When the networks are weighted and the interactions among nodes are directed, heterogeneity in the degree distribution actually helps improve the synchronizability. Each data point is the result of averaging over 50 network realizations.

We also find that, for $\beta > 0$, the eigenratios of our networks are smaller than those from previously achievable eigenratios reported in the literature [10,11]. For example, in these previous works, for the same ensemble of scale-free networks, the minimally achievable eigenratio is about 6, while in our case, the ratio can be made smaller for almost all $\beta > 0$. In particular, the case $\beta \to \infty$ in our network is similar to the ideal model proposed in Ref. [12], if we decouple the loop structure associated with the largest-degree node.

To illustrate the advantage of network heterogeneity in promoting synchronization, when weight and asymmetry are taken into account, we show in Fig. 2(b) the dependence of the eigenratio on the degree exponent γ for unweighted, symmetrical networks (the upper trace) and for weighted, asymmetrical networks constructed from our coupling scheme (the lower trace). As references, the eigenratio of a homogeneous network, the case of $\gamma \to \infty$ in scale-free networks, under the situations of unweighted, symmetrical coupling (the upper dashed line) and weighted, asymmetrical coupling (the lower dashed line) are also plotted. As γ is increased, the network becomes less heterogeneous. Most realistic scale-free networks have values of γ around $\gamma_0=3$ [4].

We see that, if weight and asymmetry are not taken into account, the eigenratio increases as γ is decreased toward γ_0, indicating continuous deterioration of synchronizability as the network becomes more heterogeneous [9]. This is the origin of the so-called synchronization paradox for scale-free networks [10–12]. As indicated by the lower trace in Fig. 2(b), the paradox is naturally resolved when weight and asymmetry are present in the network, since the eigenratio decreases continuously as γ is decreased, suggesting that scale-free networks are more synchronizable than homogeneous networks under the new coupling scheme. This provides a justification for the ubiquity of scale-free networks in natural and technological systems.

The results exemplified by Figs. 2(a) and 2(b) are from numerical eigenvalue analysis. It is useful to examine the synchronous behavior of actual oscillator networks. For this purpose we use scale-free networks of nonidentical, chaotic Rössler oscillators, a typical model employed in detecting the collective behavior of complex networks [8,10,11,18] (similar phenomena to those we are going to report for the Rössler oscillator are also found in other models such as phase oscillators, Van der Pol oscillators, and logistic maps). The dynamics of a single oscillator is described by $\mathbf{F}_i(\mathbf{x}_i) = [-\omega_i y_i - z_i, \omega_i x_i + 0.15 y_i, z_i(x_i - 8.5) + 0.4]$, where ω_i is the natural frequency of the ith oscillator. In simulations we choose ω_i randomly from the range [0.9, 1.1], so as to make the oscillators nonidentical. The coupling function is chosen to be $\mathbf{H}(\mathbf{x}) = \mathbf{x}$. The degree of synchronization can be characterized by monitoring the amplitude A of the mean field $X(t) = \sum_{i=1}^{N} x_i(t)/N$ [10,18]. For small coupling strength ε, $X(t)$ oscillates irregularly and A is approximately zero, indicating lack, or a lower degree, of synchronization. As the coupling parameter is increased, synchronization sets in. We expect to observe a relatively fast increase of A as the coupling is increased through a critical value, as shown in Fig. 3(a) for an ensemble of 100 scale-free networks of 1024 nodes and $\langle k \rangle = 10$, and for three values of the control parameter ($\beta = 0, 1, 5$). We have also tested this oscillator network model for previous coupling schemes [10–12], and have found that, to achieve the same value of the amplitude A, the coupling strength needed in our scheme is generally smaller. Evidence for the improvement of synchronization in our scheme with increasing parameter β is shown in Fig. 3(b) for $\varepsilon = 0.1$.

A few remarks are in order. (1) Distributing coupling strength according not only to the degree of the node itself but also to the degrees of its neighbors is one of the key features that distinguishes our scheme from previous ones. (2) The parameter β not only determines the direction of the gradient field, but also controls its weight. (3) For a given network, there usually exists a small set of low-degree nodes with some hub nodes as their neighbors. Such nodes may play an important role in promoting synchronization as they provide "bridges" between the hubs. Our scheme emphasizes the role of these small-degree nodes [Eq. (2)]. This is the main reason that our scheme can lead to highly synchronizable networks.

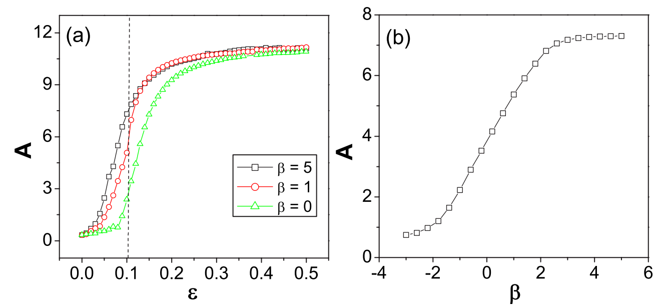

FIG. 3. (Color online) Ensemble of Rössler oscillator networks constructed using our coupling scheme Eq. (5): (a) The amplitude A of the mean field versus the coupling parameter ε and (b) A versus the control parameter β in Eq. (5). Network parameters are $N = 1024$ and $\langle k \rangle = 10$, and each data point is the result of averaging over 100 network realizations.

In summary, we have argued that the topology of gradient networks can be expected naturally in any weighted, asymmetrical network, and this can be used to devise effective coupling schemes for designing complex networks with enhanced synchronizability. We have presented a general coupling scheme and demonstrated that scale-free networks so constructed can be more synchronizable than homogeneous networks of the same system size and total number of links. The present scheme also possesses a higher synchronizability than the previous ones, and actually reaches the "optimal" configuration that has been proposed more recently [15,16]. The importance of gradient networks has been recognized only recently, with particular focus on the problem of traffic jamming [13,14]. Here we have shown that they can also be useful for studying other types of dynamics on complex networks, such as synchronization. Many interesting issues concerning gradient networks can arise, such as the detection of any possible gradient structure for a given complex network based on experimental measurements. It seems that gradient networks represent an interesting topic of study in network science.

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[19] In the limiting case of $\beta \rightarrow \infty$, it is possible that the network will be disconnected. This probability, however, is very small and can usually be neglected, especially for the densely connected scale-free networks of small values of γ. A detailed analysis about network breaking will be presented elsewhere (e-print arXiv:nlin.CD/0608035).