Characterization of noise-induced strange nonchaotic attractors

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Strange nonchaotic attractors (SNAs) were previously thought to arise exclusively in quasiperiodic dynamical systems. A recent study has revealed, however, that such attractors can be induced by noise in nonquasiperiodic discrete-time maps or in periodically driven flows. In particular, in a periodic window of such a system where a periodic attractor coexists with a chaotic saddle (nonattracting chaotic invariant set), none of the Lyapunov exponents of the asymptotic attractor is positive. Small random noise is incapable of causing characteristic changes in the Lyapunov spectrum, but it can make the attractor geometrically strange by dynamically connecting the original periodic attractor with the chaotic saddle. Here we present a detailed study of noise-induced SNAs and the characterization of their properties. Numerical calculations reveal that the fractal dimensions of noise-induced SNAs typically assume fractional values, in contrast to SNAs in quasiperiodically driven systems whose dimensions are integers. An interesting finding is that the fluctuations of the finite-time Lyapunov exponents away from their asymptotic values obey an exponential distribution, the generality of which we are able to establish by a theoretical analysis using random matrices. We suggest a possible experimental test. We expect noise-induced SNAs to be general.

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I. INTRODUCTION

Strange nonchaotic attractors (SNAs), which are attractors that possess a strange geometry but exhibit no sensitive dependence on initial conditions, have attracted continuous interest in the study of nonlinear dynamical systems [1–5,7–13]. SNAs are typical in quasiperiodic systems in that they can occur in parameter regimes of positive Lebesgue measure [1]. Mathematically, if an SNA can persist under small perturbations, it is said to be robust [11]. While the issue of robustness is formally addressed in Ref. [11], examples of robust SNAs appeared earlier in the literature, e.g., in Refs. [14–16]. So far, four different routes to SNAs have been reported: (i) collision between a period-doubled torus and its unstable parent [2], (ii) fractalization of a torus [4], (iii) blowout bifurcation [5,6], and (iv) intermittency [7]. Experimental observations of SNAs have been reported in a quasiperiodically driven magnetic-ribbon system [8], in electronic circuits [17], in a plasma system [18], and in a system near the torus-doubling terminal critical point [19]. It has also been found that SNAs can play an intermediate role in the transition from regular motion to chaos [10,12,20–22]. Physically, SNAs are relevant to situations such as the localization of quantum particles in spatially quasiperiodic potential systems [14]. These exotic attractors may also be important for biological systems [9,23], and they may be useful for nonlinear dynamics based communication as well [24,25]. Mathematical issues concerning the generations and properties of SNAs have been addressed [11,26–28]. Studies of the fractal dimension spectrum of SNAs in quasiperiodically driven systems have revealed an interesting result: both the box-counting dimension and the information dimension are integers [11,16].

So far, SNAs have been identified and studied almost exclusively in quasiperiodically driven dynamical systems. Since many physical, biological, and engineering systems do not fall in this category, it is natural to ask whether SNAs can arise in more general situations [29], for instance situations in which the underlying system is autonomous or periodically driven. Identification of possible SNAs there would be of great interest because such systems are common in many applications. There have been debates in this regard [30–33], which makes the problem even more intriguing and appealing.

We have recently reported robust noise-induced SNAs in both autonomous and periodically driven systems [13]. Our idea is motivated by the well-documented fact that random dynamical systems permit robust chaotic attractors with well-defined fractal characteristics [34–37]. An intuition is then that robust SNAs can also arise in random systems, where a fundamental role of noise is to dynamically connect distinct, coexisting invariant sets in the phase space [37,38]. In particular, imagine the situation in which there is a periodic attractor coexisting with a nonattracting chaotic set (chaotic saddle), as can occur in any periodic window of a dynamical system. In the absence of noise, almost all initial conditions lead to trajectories that approach the periodic attractor. The chaotic saddle, however, provides a platform for trajectories to exhibit transient chaotic behavior before approaching the periodic attractor. Once a trajectory falls in the neighborhood of the periodic attractor, it can no longer visit the chaotic saddle. It is in this sense that the two invariant sets can be regarded as dynamically isolated. The presence of noise changes this picture in that it enables trajectories near the periodic attractor to revisit the chaotic saddle, generating an intermittent behavior. The final attractor created this way comprises two dynamically connected components and, under fairly general conditions, the attractor can be geometrically strange but it is possible that none of its Lyapunov exponents become positive. These are the fundamental ingredients for noise-induced SNAs. More specifically, the strangeness comes from the original chaotic invariant set that has a fractal structure and, because the original periodic component has a negative largest Lyapunov exponent, the largest exponent of the final attractor can remain negative. The final attractor will then be geometrically strange but not chaotic. This reasoning suggests further that in any finite-time interval, the Lyapunov exponent can be temporally positive due to intermittent visits to the original chaotic set and the Fourier spectrum of the trajectory will contain both a continuous and a discrete component, where the former comes from the chaotic set and the latter is a result of the periodic orbit. Thus the spectrum of trajectories on the final attractor should exhibit characteristics of being singular-continuous, typically observed for SNAs in quasiperiodically driven systems [39].

It is evident that noise-induced SNAs cannot arise in dynamical systems described by autonomous differential equations, because the largest Lyapunov exponent immediately becomes positive from zero when noise begins to connect the two invariant sets. Thus noise-induced SNAs can occur only in discrete-time maps or in periodically driven systems for which the largest nontrivial exponent can be nonpositive but the final attractor is strange in open parameter regions. The surprising consequence is that noise-induced SNAs can be generated without requiring quasiperiodic driving. Such attractors are expected to be general in physical systems where noise is inevitable.

While we have published a brief account of the dynamical mechanism for noise-induced SNAs in nonquasiperiodically driven systems [13], an important issue that needs to be addressed is the characterization of this class of attractors. Our concern is that one might naively think that noise-induced SNAs cannot exist in such systems because noise can smear out any strange geometry of the underlying attractor. Indeed, under noise, for a single trajectory any fractal structure of an attractor, chaotic or strange nonchaotic, can be resolved only for scales larger than one determined by the noise amplitude. However, it was pointed out by Romeiras, Grebogi, and Ott [40] that the fundamental fractal structure of a chaotic attractor *can* be resolved even under noise, if one examines the *snapshot* attractors formed by a large number of trajectories. In particular, one can use a large number of systems, started with random initial conditions and subjected to identical noise at any instant of time, evolve them under the dynamics, and plot the locations of the resulting trajectories at "frozen" times. The snapshot attractors so obtained are generally fractals. At different times, since the realization of noise is different, the details of the snapshot attractors are different but their fractal dimensions tend to a common value in the limit of an infinite number of initial conditions. This approach to studying the fractal structure of chaotic attractors in random dynamical systems has been explored extensively (see, for example, Refs. [36,41]) and has even been extended to quasiperiodically driven systems for investigating the transition between strange nonchaotic and chaotic attractors [42]. It is thus reasonable that noise-induced SNAs can be studied using snapshot attractors. Considering that the requirement of quasiperiodicity for SNAs has been a controversial topic in nonlinear dynamics [30–33], a systematic study of the dynamical and geometric properties of noise-induced SNAs is desirable.

Our main results in this paper (beyond those in Ref. [13]) are (i) noise-induced SNAs possess a stronger fractal property in that the box-counting dimension and the information dimension are generally fractional values (versus SNAs in quasiperiodic systems where these dimensions are integers); (ii) the fractal structure of noise-induced SNAs can be revealed by snapshot attractors and the evolution of their sizes exhibits a highly intermittent behavior, particularly near the transitions from a periodic attractor to a noise-induced SNA and from it to a chaotic attractor; and (iii) the fluctuations of the finite-time Lyapunov exponents away from their asymptotic values are exponentially distributed, and the generality of this phenomenon can be established by an analytic treatment based on random matrices .

The dynamical mechanism for generating noise-induced SNAs is described in Sec. II. Numerical studies of a map model, the random Hénon map [43], are presented in Sec. III. A random-matrix theory is developed in Sec. IV to explain the numerically observed exponential distribution of the fluctuations of the finite-time Lyapunov exponents. Snapshot-attractor characterization of noise-induced SNAs is presented in Sec. V. A brief discussion is offered in Sec. VI.

II. DYNAMICAL MECHANISM FOR NOISE-INDUCED SNAs

We shall argue that for a system with a bifurcation parameter p (its variation can lead to the occurrences of various periodic windows) in the presence of noise of amplitude D, there exist open sets of finite areas in the two-dimensional parameter space (p,D) for which the asymptotic attractor is strange but nonchaotic. In particular, we consider general discrete-time maps

$$\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n, p) \tag{1}$$

and periodically driven systems described by differential equations of the following form:

$$d\mathbf{x}/dt = \mathbf{F}(\mathbf{x}, z, p) \quad \text{and} \quad dz/dt = \omega, \tag{2}$$

where $\mathbf{x} \in \mathcal{R}^d$. For the continuous-time system, z is a time variable and the velocity field $\mathbf{F}$ depends periodically on z. We choose p such that the system is in a periodic window of period m and the asymptotic attractor of the system is a periodic attractor of period $2^k m$, where $k=0,1,\ldots$, due to period-doubling bifurcations. Let p_m be the parameter value for the beginning of the window, which is triggered by a saddle-node bifurcation that creates a period-m stable orbit, and let p_{m*} be the parameter value for the end of the period-doubling cascade of the original stable period-m orbit. We focus on the parameter interval $p_m < p < p_{m*}$ in which the attractor is periodic. Note that the intervals $\{\Delta p_m \equiv |p_{m*} - p_m| > 0\}_m$ are open on the parameter axis [44]. In such a setting, for maps the largest Lyapunov exponent is negative, except for a set of parameter values of Lebesgue measure zero where the period-doubling bifurcations occur. For periodically driven systems, there is a null Lyapunov exponent generated by $dz/dt = \omega$, but in a periodic window the largest nontrivial exponent is negative. Now consider additive noise of amplitude D (for simplicity). Our goal is to show that for $p_m < p < p_{m*}$, there exists a range of the noise amplitude $\Delta D_m > 0$ for which the asymptotic attractor is nonchaotic but strange, and robust with respect to small perturbations.

In a periodic window, a periodic attractor and a chaotic saddle coexist. A trajectory from a random initial condition typically moves toward the chaotic saddle along its stable manifold, stays near the saddle for a finite time, and leaves along its unstable manifold before finally approaching the periodic attractor. There is thus transient chaos for $p_m < p < p_{m*}$. In the absence of noise, the asymptotic attractor is periodic, despite transient chaos. If noise is not strong enough to kick a trajectory on the attractor to a nearby region where the stable manifold of the chaotic saddle lies, the final attractor is still approximately periodic with a negative largest Lyapunov exponent. Only when the noise amplitude D exceeds a critical value D_m is the probability *physically appreciable* for a trajectory on the original periodic attractor to be perturbed to the vicinity of the stable manifold of the chaotic saddle and to move toward the chaotic saddle. Because the saddle is nonattracting, the trajectory can spend only a finite amount of time near it before approaching the original periodic attractor again, and so on. For $D \gtrsim D_m$, a trajectory switches intermittently between the original periodic attractor and the chaotic saddle. There is then a sudden change in the structure of the asymptotic attractor at D_m: for $D \gtrsim D_m$, the attractor contains both periodic and chaotic components.

For discrete maps, the largest Lyapunov exponent of the periodic attractor is $\lambda_1^P < 0$. As a trajectory begins to visit the chaotic saddle for $D \gtrsim D_m$, the largest exponent λ_1 of the new attractor starts to increase from λ_1^P. It has been shown [37] that the increase in λ_1 obeys the following universal algebraic scaling law:

$$\lambda_1 - \lambda_1^P \sim (D - D_m)^{\alpha}, \tag{3}$$

where the scaling exponent $\alpha > 0$ depends on the phase-space dimension of the system and the dynamical invariants of the chaotic saddle such as its average lifetime and the Lyapunov spectrum [37]. We see that λ_1 can remain negative for a range of the noise amplitude above D_m: $D_m < D < D_{*m}$, where D_{*m} is the noise amplitude for which $\lambda_1 = 0$. We thus have

$$\Delta D_m \equiv D_{*m} - D_m \sim |\lambda_1^P|^{1/\alpha} > 0.$$

In this noise range, the attractor of the system is geometrically complicated but its largest Lyapunov exponent remains negative. The same consideration [45] applies to periodically driven systems for which a null Lyapunov exponent always exists but the largest nontrivial Lyapunov exponent of the attractor remains negative for $D < D_{*m}$. Thus, for $D_m < D < D_{*m}$, the asymptotic attractor of the system can have a strange geometry because it contains a chaotic component, yet the largest Lyapunov exponent is nonpositive.

We now argue that the attractors created for $D_m < D < D_{*m}$ are strange but not chaotic. We first consider the finite-time behavior of the largest Lyapunov exponent. It is known [5,10,46,47] that an SNA, while having a nonpositive largest Lyapunov exponent, possesses regions in the phase space in which infinitesimal vectors in fact grow in length. That is, in any finite-time interval, there is a finite probability that the largest exponent is temporally positive. The asymptotic exponent can be regarded as the weighted sum of the temporally positive exponent when the trajectory visits the expanding regions, and the temporally negative exponent when the trajectory is in regions in which the tangent vectors contract [5,10,47]. The asymptotic exponent is negative when the negative component weighs more than the positive one [48]. Here, the existence of the two sets with distinct behaviors for the evolution of infinitesimal vectors is apparent: for $D > D_m$, a trajectory visits both the original periodic attractor for which the tangent vectors contract and the chaotic saddle for which the vectors expand. The largest Lyapunov exponent can then be approximately written as

$$\lambda_1 \approx f_P(D)\lambda_1^P + f_S(D)\lambda_1^S, \tag{4}$$

where $\lambda_1^S > 0$ is the largest Lyapunov exponent of the chaotic saddle, and $f_P(D)$ and $f_S(D)$ are the frequencies of visit to the original periodic attractor and to the saddle, respectively. For $D_m < D < D_{*m}$, the first term weighs more than the second term in Eq. (4), potentially giving rise to a noise-induced SNA. For $D > D_{*m}$, the second term dominates, leading to a chaotic attractor. It is useful to note that, since both the periodic attractor and the chaotic saddle are dynamically invariant in the noiseless situation, the exponents λ_1^P and λ_1^S are well defined with respect to their respective invariant measures [49].

Another key characteristic of noise-induced SNAs lies in the Fourier spectrum. It has been recognized that SNAs usually possess a singular-continuous spectrum that contains both discrete and continuous components [39]. For the attractors for $D_m < D < D_{*m}$, their spectra naturally contain these distinct components for an apparent reason: such an attractor consists of a periodic component for which the spectrum is discrete and a chaotic component for which the spectrum is broad. Thus we expect the attractors for $D_m < D < D_{*m}$ to have singular-continuous Fourier spectra.

Our analysis thus suggests that, in the parameter plane (p, D), there are open areas of the various sizes $(\Delta p_m, \Delta D_m)$, where m denotes the period of every possible periodic window, in which there are noise-induced SNAs. They are typical in the parameter space. In addition, since the nonpositivity of the Lyapunov exponent for $D_m < D < D_{*m}$ and the strangeness of the noise-induced attractors, as characterized by fluctuations of the finite-time Lyapunov exponent into the positive side and a singular-continuous spectrum, are statistical properties of the attractors under random perturbation, they are robust. Noise-induced SNAs are thus physically observable.

An issue of practical interest concerns the meaning and definition of the critical noise threshold (e.g., D_m and D_{*m} in the preceding discussion). Note that the basic dynamical mechanism responsible for noise-induced SNAs is the noise-activated connection between the periodic attractor and the chaotic saddle, which is possible only for noise level above the threshold. For Gaussian noise, if one is allowed an infinite amount of computational or experimental time, the two sets will connect for arbitrarily weak noise. (This is similar to the situation in which, under Gaussian noise in the infinite-time limit, no attractor in the finite phase-space region and its basin of attraction can be defined.) However, for finite time, such a threshold can be defined in an *ad hoc* but physically meaningful manner. See Refs. [50–54] for details.

III. EXAMPLES OF NOISE-INDUCED SNAS

In Ref. [13], examples of noise-induced SNAs are presented for the logistic map and for the periodically driven Duffing oscillator. Here we shall present a different example: the noisy Hénon map given by [55]

$$x_{n+1} = a - x_n^2 + by_n + D\xi(n), \quad y_{n+1} = x_n, \tag{5}$$

where a and b are parameters, $\xi(n)$ is a discrete random variable of uniform probability distribution in $[-1,1]$, and D is the noise amplitude. For $b=0.4$, the map exhibits a period-8 window for $a \in (1.1435, 1.145)$. For periodic attractors in the window, the largest Lyapunov exponent is $\lambda_p \approx -0.02$. To demonstrate noise-induced SNAs, we arbitrarily fix the parameters in the window at $a=1.14358$.

A. Noise-induced SNAs

For small noise $(D<D_8)$, the period-8 attractor and the chaotic saddle are not dynamically connected. In this case, the attractor remains approximately periodic, despite noise. As D passes through the critical value D_8, the two sets are dynamically connected in the sense that a trajectory will visit both sets in time. For $D \gtrsim D_8$, the trajectory spends most time near the original periodic attractor but with intermittent visits to the chaotic saddle. The largest Lyapunov exponent of this "dynamic" attractor, which contains both the original periodic attractor and the chaotic saddle, increases with D. Since the chaotic saddle has become part of the attractor, its geometry is necessarily strange and, in fact, is fractal. As D is increased further, λ_1 keeps increasing from λ_1^P but it still remains to be negative until D exceeds D_{8^*}, after which the exponent becomes positive so that the attractor is chaotic. We thus expect to observe noise-induced SNAs for $D_8 < D < D_{8^*}$. Since λ_1 changes continuously with D and since λ_1^P is finitely negative, the parameter range $\Delta D = D_{8^*} - D_8$ is finite. In terms of the bifurcation parameter a, noise-induced SNAs can occur for any value of a in the periodic window for which the deterministic attractor is periodic. The range of such values of a is also finite. Thus in the two-dimensional parameter space (a, D), we expect to have an open area in which there are noise-induced SNAs, indicating the typicality of such attractors. Because the attractors are noise-induced, they are naturally robust. Numerically we find $D_8 \approx 10^{-4.6}$ and $D_{8^*} \approx 10^{-4.4}$. All these features are illustrated in Fig. 1.

B. Intermittency

For $D \gtrsim D_8$, a trajectory exhibits an intermittent behavior: it spends most time near the original periodic attractor with occasional visits to the chaotic saddle, as can be seen from the time series x_n in Fig. 2(a). The intermittency can be regarded as "on-off," where the "on" and the "off" states correspond to dwelling in the neighborhood of the original periodic attractor and away from it, respectively. To better observe the on-off behavior, we calculate the distance between the trajectory and its nearest component of the periodic attractor. In particular, let $(x^{(m)}, y^{(m)})$ $(m=1,\ldots,8)$ denote the locations of all components of the period-8 attractor. For a given trajectory point (x_n, y_n), the distance is

$$\delta_n = \min_m \sqrt{(x_n - x^{(m)})^2 + (y_n - y^{(m)})^2}. \tag{6}$$

Figure 2(b) shows the evolution of δ_n for the attractor in Fig. 1(b), which apparently exhibits an on-off intermittent behavior. Figure 2(c) shows the laminar-phase (i.e., the time τ between two successive bursts from the "off" state) distribution, which can be fit by an exponential-decay law. Exponential laminar-phase distribution is a typical feature of noise-induced on-off intermittency [36,56].

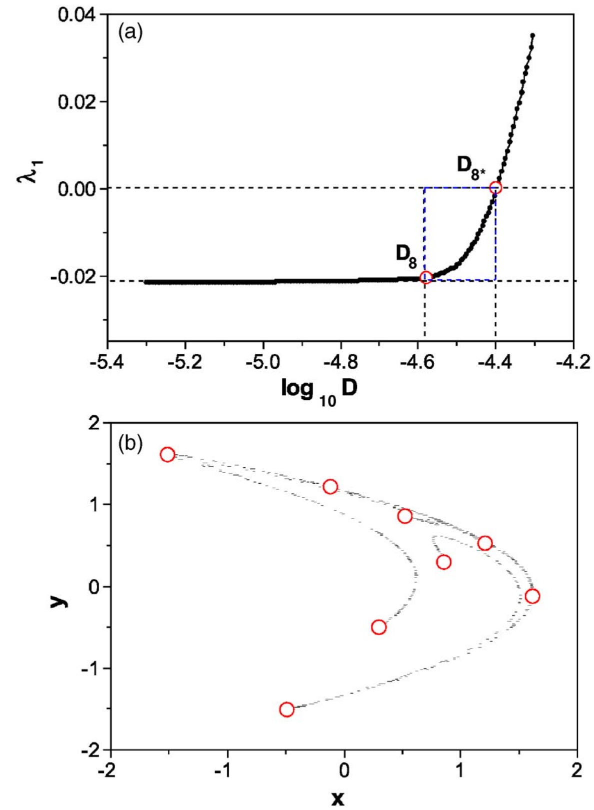

FIG. 1. (Color online) For the Henon map of parameters $b=0.4$ and $a=1.14358$, (a) increase of the largest Lyapunov exponent λ_1 with the noise amplitude D. Noise-induced SNAs occur for $10^{-4.6} \approx D_8 < D < D_{8^*} \approx 10^{-4.4}$, where λ_1 is negative but the geometry of the attractor is strange. (b) A noise-induced SNA for $D=10^{-4.43}$ with the largest exponent $\lambda_1 \approx -7.5 \times 10^{-3} < 0$. The locations of the original period-8 attractor are indicated by the open circles.

C. Finite-time Lyapunov exponent

A characteristic indicator of SNA in quasiperiodic systems is that, in any finite-time interval, the largest Lyapunov

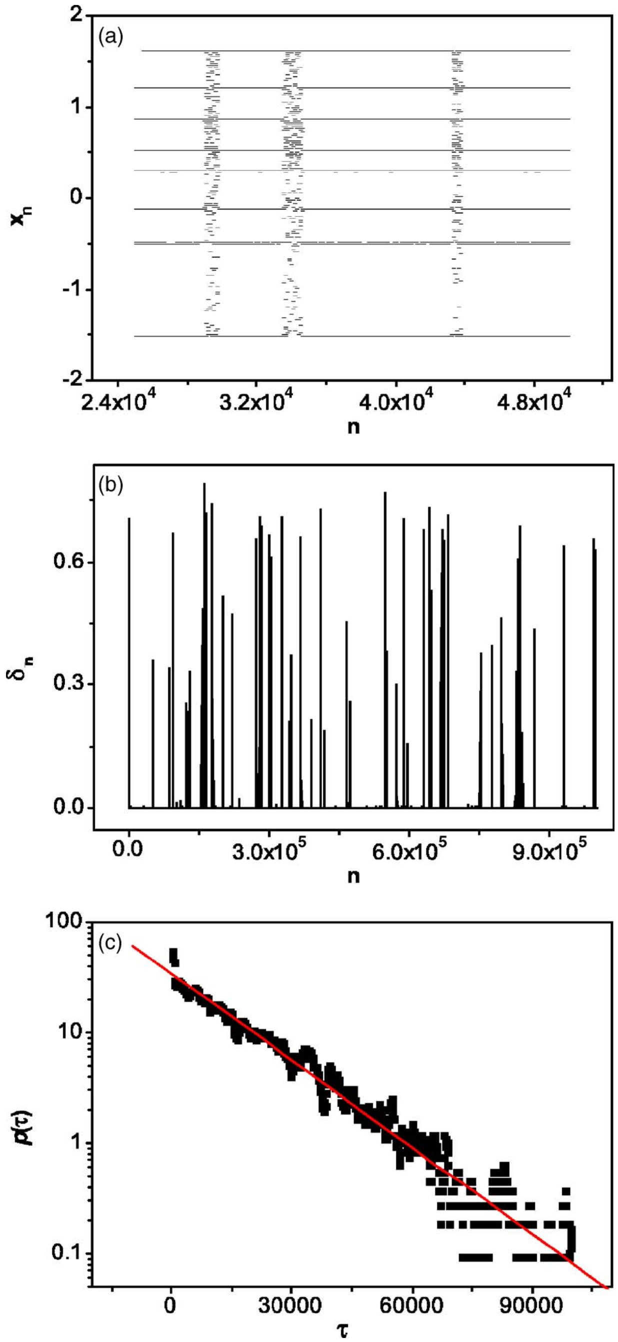

FIG. 2. (Color online) For the parameter setting in Fig. 1(b), (a) an intermittent time series x_n, (b) on-off intermittency in the evolution of the distance as defined in Eq. (6), and (c) the exponential laminar-phase distribution.

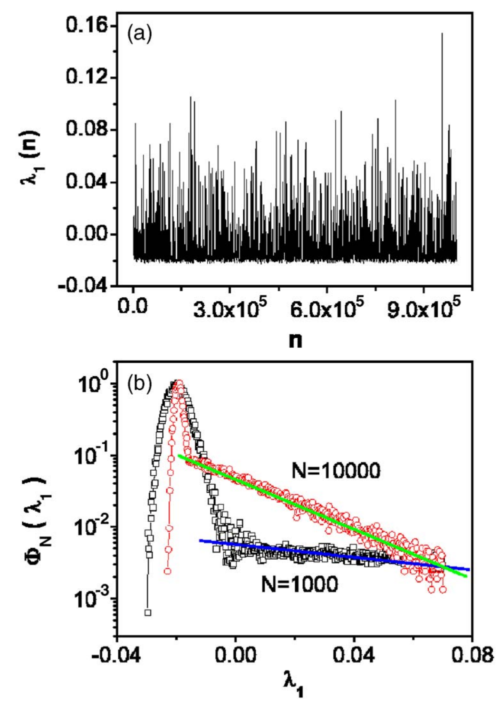

FIG. 3. (Color online) For the same parameter setting as in Fig. 2, (a) temporal evolution of the finite-time Lyapunov exponent calculated using $N=1000$ trajectory points, and (b) two distributions of the exponent for $N=1000$ and 10 000.

exponent can be positive while having nonpositive values asymptotically [1,3,2,4,5,7–13]. This is in fact a robust feature for noise-induced SNAs, which, by mechanism, contain a chaotic component. The intermittent visit to the chaotic component from a typical trajectory stipulates that it be temporally chaotic, giving rise to a temporally positive Lyapunov exponent. This feature is demonstrated in Fig. 3(a), where the temporal evolution of the exponent evaluated from a trajectory segment of length $N=1000$ is plotted. We observe that the exponent is mostly negative but there are intermittent bursts into the positive side. Figure 3(b) shows the histogram of the exponent evaluated using two time intervals: $N=1000$ and 10 000. In both cases we observe that the distributions have a peak about the asymptotic (negative) value of the exponent, but they extend into the positive side and are apparently *exponential*. The distribution corresponding to the longer time interval is narrower.

D. Fractal dimensions

For SNAs in quasiperiodic systems, both the capacity and the information dimensions are integers [11,16]. For instance, SNAs in the quasiperiodically driven circle map have capacity dimension 2 and information dimension 1 [16]. The intuition for the integer information dimension is that, due to the quasiperiodicity, the largest Lyapunov exponent is zero and the second largest exponent is negative. Application of the Kaplan-Yorke formula [57] then gives an integer. For noise-induced SNAs, the largest Lyapunov exponent is negative, rendering inapplicable the Kaplan-Yorke formula. We shall demonstrate that, due to the chaotic component, noise-induced SNAs are typically fractal sets with fractional dimension values. (Note that, under small noise, the fine structure of an attractor is smeared out. Thus the fractal dimensions are meaningful only for scales larger than that determined by the noise amplitude. However, we shall show later that the fine fractal structure of noise-induced SNAs can be revealed by using snapshot attractors.)

We calculate the capacity dimension and the information dimension of noise-induced SNAs by using the standard box-counting procedure: $d_0=\lim_{\epsilon\to 0} \ln N(\epsilon)/|\ln \epsilon|$ and

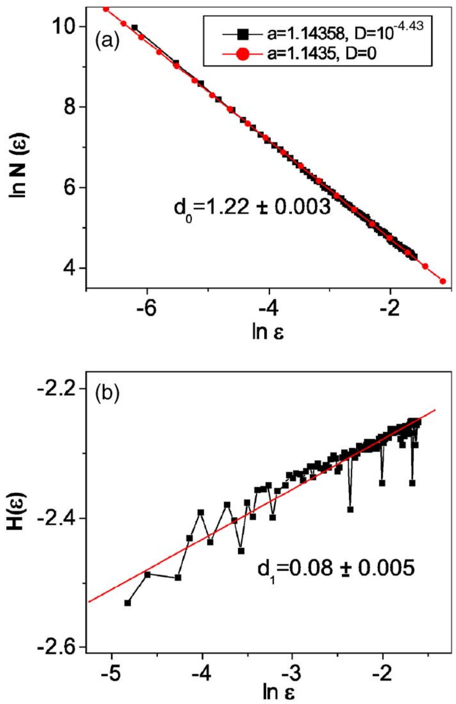

FIG. 4. (Color online) For the noise-induced SNA in Fig. 1(b), (a) $\ln N(\epsilon)$ and (b) $H(\epsilon)$ vs $\ln \epsilon$, respectively. The absolute value of the slope from the linear fit in (a) gives $d_0 \approx 1.22$, which is essentially the same as the capacity dimension of the chaotic attractor immediately preceding the periodic window in the corresponding deterministic system ($a=1.1435$). The slope from a linear fit in (b) gives $d_1 \approx 0.08$.

$d_1 = \lim_{\epsilon \to 0} H(\epsilon)/\ln \epsilon$, where $N(\epsilon)$ is the number of ϵ-size boxes required to cover the whole attractor, $H(\epsilon) = \Sigma_{i=1}^{N(\epsilon)} \mu_i(\epsilon) \ln \mu_i(\epsilon)$, and $\mu_i(\epsilon)$ is the frequency of visit of a typical trajectory to the ith box. Since the original periodic attractor consists of a finite number of points, the main contribution to $N(\epsilon)$ comes from the number of boxes needed to cover the original chaotic saddle. We thus expect the capacity dimension of a noise-induced SNA to have approximately the same value as that of the chaotic saddle, which is approximately equal to the capacity dimension of the chaotic attractor near the periodic window in the underlying deterministic system. The information dimension of the noise-induced SNA, however, will be smaller than that of the chaotic saddle, as the probability of a visit to the original saddle is smaller compared with that to the original periodic attractor. Figures 4(a) and 4(b) illustrate these results, where $\ln N(\epsilon)$ and $H(\epsilon)$ are plotted versus $\ln \epsilon$, respectively. We obtain $d_0 \approx 1.22$ and $d_1 \approx 0.08 \ll d_0$.

IV. DISTRIBUTION OF FINITE-TIME LYAPUNOV EXPONENTS

A feature associated with noise-induced SNAs, which seems to be independent of the system details, is the exponential distribution of the finite-time Lyapunov exponents observed for both the Hénon and the IHJM map. To establish the generality of this phenomenon, here we provide a heuristic theory. Consider a d-dimensional map system, as in Eq. (1). The starting point of analysis of Lyapunov exponents is the following matrix product:

$$\mathbf{Q}_N = \prod_{i=0}^{N-1} \mathbf{DF}(\mathbf{x}_i), \tag{7}$$

where $\{\mathbf{x}_i\}_{i=0}^{N-1}$ is a trajectory of length N on the attractor and $\mathbf{DF}(\mathbf{x}_i)$ is the Jacobian matrix associated with trajectory point $\mathbf{x}_i$. To obtain the spectrum of d Lyapunov exponents [58], it is necessary to calculate the largest Lyapunov exponent λ_1, which is defined as

$$\lambda_1 = \lim_{N\to\infty} \frac{1}{N} \ln|\mathbf{Q}_N \cdot \mathbf{u}_0^{(1)}|, \tag{8}$$

where $\{\mathbf{u}_0^{(1)}\}$ is a random unit vector in the tangent space of the initial trajectory point. Let $\{\mathbf{u}_i^{(1)}\}_{i=0}^{N-1}$ be the set of tangent vectors along the trajectory $\{\mathbf{x}_i\}_{i=0}^{N-1}$, where $\mathbf{u}_i^{(1)} \equiv \mathbf{Q}_i \cdot \mathbf{u}_0^{(1)}$. The second exponent can be expressed as

$$\lambda_2 = \lim_{N\to\infty} \frac{1}{N} \ln|\mathbf{u}_{N-1}^{(2)}|, \tag{9}$$

where $\{\mathbf{u}_i^{(2)}\}_{i=0}^{N-1}$ is a set of unit tangent vectors that are orthogonal to $\{\mathbf{u}_i^{(1)}\}_{i=0}^{N-1}$. Continuing this way, the lth Lyapunov exponent is

$$\lambda_l = \lim_{N\to\infty} \frac{1}{N} \ln|\mathbf{u}_{N-1}^{(l)}|, \tag{10}$$

where the unit tangent vectors $\{\mathbf{u}_i^{(l)}\}_{i=0}^{N-1}$ are orthogonal to $\{\mathbf{u}_i^{(j)}\}_{i=0}^{N-1}$ for $j=1,\ldots,l-1$.

For notational convenience, here we focus on the statistical fluctuations of the largest Lyapunov exponent, simply denoted by λ, noting that the same derivation applies to all other exponents as defined above. The finite-time exponent is

$$\lambda(N) = \frac{1}{N} \ln|\mathbf{Q}_N \cdot \mathbf{u}_0|. \tag{11}$$

The statistical behavior of the finite-time exponent is determined completely by the matrix product $\mathbf{Q}_N$. While it is generally difficult to analyze the product, for noise-induced SNAs, if we focus on the fluctuations of the finite-time exponent far away from their asymptotic values, it is possible to draw conclusions concerning the statistics of the fluctuations. The basic fact is the intermittent visits of a trajectory to sets that are dynamically invariant in the underlying deterministic system. In particular, as we have explained, in a periodic window, a noise-induced SNA contains the original periodic attractor and the chaotic saddle. There are thus two characteristically different types of matrices in the product: one associated with the periodic attractor and another with the chaotic saddle, denoted by $\mathbf{A}$ and $\mathbf{B}$, respectively. The matrix product $\mathbf{Q}_N$ can be effectively regarded as a product of *random matrices*, where each entry in the product can randomly select $\mathbf{A}$ or $\mathbf{B}$ with certain probabilities. This is basically a Bernoulli process. For fixed time N, the time average in Eq. (11) can thus be represented by an ensemble average in terms of the contributions from matrices of type $\mathbf{A}$

and of type **B**. Since the total number of matrices is fixed (i.e., N), the ensemble is microcanonical-like [59]. We can thus write, symbolically,

$$\lambda(N) = \langle \ln|\mathbf{Q}_N \cdot \mathbf{u}_0| \rangle_m, \tag{12}$$

where $\langle \cdot \rangle_m$ denotes a microcanonical-like ensemble average.

Let $\lambda^P < 0$ and $\lambda^S > 0$ be the largest exponent of the periodic attractor and of the chaotic saddle, respectively, in the deterministic system. Alternatively, we can interpret λ^P and λ^S as the Lyapunov exponents determined by matrices of type **A** and of type **B**, respectively. In the presence of noise, let p and q be the probabilities of a visit to the original periodic attractor and to the chaotic saddle, respectively. Use of the ensemble average in Eq. (12) allows us to write the largest exponent of the noise-induced SNA as

$$\lambda \approx p\lambda^P + q\lambda^S. \tag{13}$$

In the microcanonical ensemble, the finite-time fluctuations of the exponent are due entirely to the fluctuations in the numbers of times that **A** and **B** appear in the product $\mathbf{Q}_N$ [59]. That is, the probabilities of choosing **A** and **B** are fixed constants and we write them as $\tilde{p}$ and $\tilde{q}$, respectively. The distribution of the finite-time exponent is given by the binomial distribution

$$\Phi_N(\lambda) = \binom{N}{k} \tilde{p}^k \tilde{q}^{N-k}, \tag{14}$$

where k and $N-k$ are the numbers of times that **A** and **B** appear in $\mathbf{Q}_N$, respectively. Focusing on the extreme fluctuations of the exponent away from its asymptotic (negative) value, say, on the positive side, we have $0 < \lambda \lesssim \lambda^S$. Equation (13) then implies

$$p \to 0 \quad \text{and} \quad q \lesssim 1. \tag{15}$$

The binomial distribution $\Phi_N(\lambda)$ can be simplified, as follows:

$$\Phi_N(\lambda) = \frac{N(N-1)\cdots(N-k+1)}{N^k} \frac{(N\tilde{p})^k}{k!} \left(1 - \frac{N\tilde{p}}{N}\right)^{N-k}$$
$$= \left(\prod_{m=0}^{k-1} \frac{N-m}{N-\alpha}\right) \frac{\alpha^k}{k!} \left(1 - \frac{\alpha}{N}\right)^N,$$

where $\alpha \equiv N\tilde{p}$=const for N large and $\tilde{p}$ small. We thus have $(1-\alpha/N)^N \approx e^{-\alpha}$ and

$$\left(\prod_{m=0}^{k-1} \frac{N-m}{N-\alpha}\right) = \prod_{m=0}^{k-1} \left(1 + \frac{\alpha - m}{N-\alpha}\right) \approx 1.$$

Under these approximations, the distribution becomes

$$\Phi_N(\lambda) \approx \frac{\alpha^k}{k!} e^{-\alpha}. \tag{16}$$

Since the fluctuation of the finite-time exponent is determined by the quantity k in the microcanonical ensemble, equivalently we can regard k as being dependent upon λ and write $k(\lambda)$. Using Eq. (13), we have

$$k(\lambda) = Np = N\frac{\lambda^S - \lambda}{\lambda^S - \lambda^P}. \tag{17}$$

To determine the main dependence of $\Phi_N(\lambda)$ on λ, we consider two values of λ: $\lambda^{(1)} \gtrsim \lambda^{(2)} \approx \lambda^S$, which gives

$$k(\lambda^{(1)}) \gtrsim k(\lambda^{(2)}) \approx 0.$$

Consequently, both $k(\lambda^{(1)})!$ and $k(\lambda^{(2)})!$ are small, and we have

$$\frac{\Phi_N(\lambda^{(1)})}{\Phi_N(\lambda^{(2)})} = \frac{\alpha^{k(\lambda^{(1)})}}{\alpha^{k(\lambda^{(2)})}} \frac{k(\lambda^{(2)})}{k(\lambda^{(1)})} \approx \alpha^{k(\lambda^{(1)})-k(\lambda^{(2)})} = \frac{e^{-\beta_N \lambda^{(1)}}}{e^{-\beta_N \lambda^{(2)}}}, \tag{18}$$

where $\beta_N \equiv N \ln \alpha / (\lambda^S - \lambda^P)$. We thus see that the main dependence of $\Phi_N(\lambda)$ on λ is exponential and conclude that the fluctuations of the finite-time Lyapunov exponent obey the following exponential decay law:

$$\Phi_N(\lambda) \sim e^{-\beta_N \lambda}, \tag{19}$$

where the exponential rate constant β_N is proportional to N. That is, for larger time, the decay rate in the exponential distribution is larger, making smaller its spread. These features have indeed been observed in numerical experiments [e.g., Fig. 3(b)].

V. CHARACTERIZATION OF NOISE-INDUCED SNAs BY SNAPSHOT ATTRACTORS

Due to noise, a single trajectory (even from a chaotic attractor) will not exhibit fractal structure in the phase space because noise smears out fine phase-space details. However, the technique of snapshot attractors, which are attractors obtained from an ensemble of trajectories under identical noise at fixed instants of time, can be used to visualize and analyze the fractal structure of attractors in random systems [36,40]. An interesting result in Ref. [36] is that in the nonchaotic regime, snapshot attractors approach a finite set of points, but for chaotic attractors with a slightly positive largest Lyapunov exponent, the size of the snapshot attractors exhibits on-off intermittency.

For noise-induced SNAs, because there is no positive Lyapunov exponent, there can be time intervals during which trajectories on the snapshot attractors come to the small neighborhoods of a finite set of points. If the sizes of the neighborhoods become of the order of magnitude of the computer roundoff, numerically obtained snapshot attractors will consist of a finite set of points, which is artificial. To resolve the strange geometry of noise-induced SNAs by snapshot attractors, we can introduce a small amount of inhomogeneity in the noisy perturbations to different trajectories in the ensemble. That is, at a fixed time, we use an additional noise source of infinitesimal amplitude (numerically on the order of the computer roundoff) and apply different realizations of the noise to different trajectories. This idea is similar to that used to obtain chaotic numerical trajectories from, say, the tent map, where a small noise is

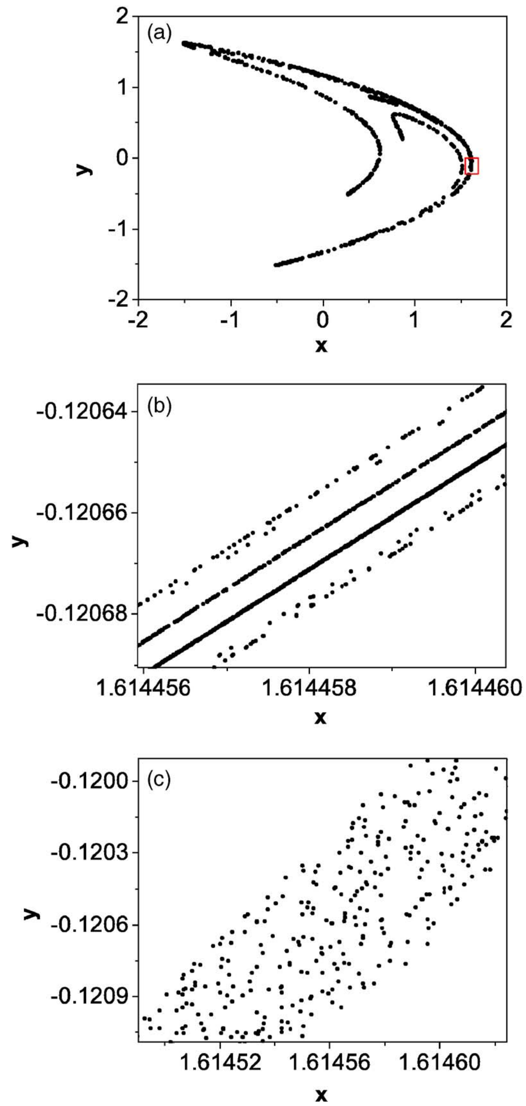

FIG. 5. (Color online) For the noisy Hénon map Eq. (5), (a) a snapshot attractor, (b) blowup of the small box in (a), which reveals the fractal structure of the noise-induced SNA, and (c) points from a single trajectory in the small box.

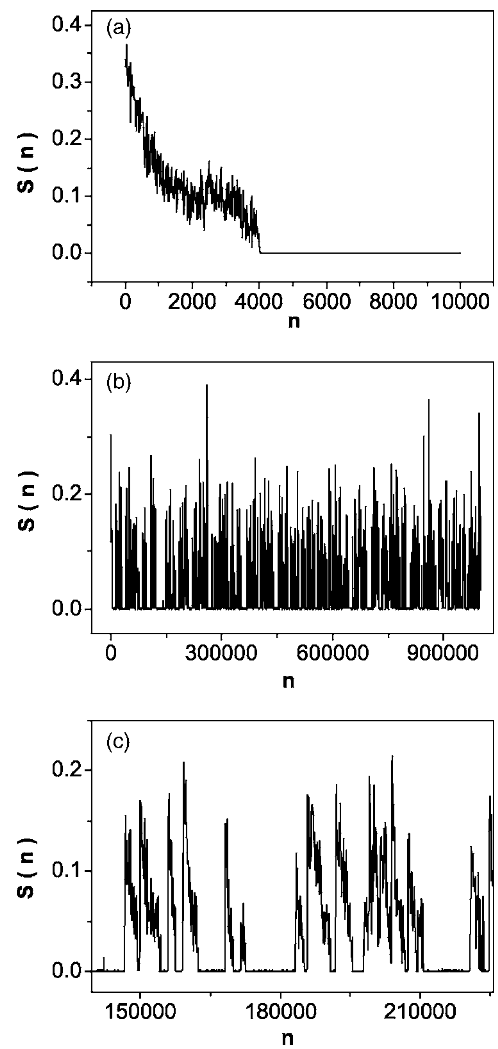

FIG. 6. For the noisy Hénon map Eq. (5), (a) temporal evolution of the size of the snapshot attractor under identical noise, which artificially approaches zero, (b) intermittency in the size of the snapshot attractor under small, inhomogeneous noise, and (c) blowup of part of (b).

necessary to avoid artificial convergence of computer trajectories to zero.

For the noisy Hénon map Eq. (5), an example of the snapshot attractor is shown in Fig. 5(a) [for the same parameter setting as in Fig. 1(b)]. The attractor is obtained by using a grid of 100×100 initial conditions uniformly distributed in the region $-1.0 \leqslant (x_0, y_0) \leqslant 1.0$, and the amplitude of the inhomogeneous noise is 10^{-10}. Figure 5(b) shows a blowup of the small box in Fig. 5(a), which apparently exhibits a fractal structure. In contrast, Fig. 5(c) shows the points from a single trajectory in the small box, which appears to be randomly distributed.

A snapshot attractor can be characterized by its size in the phase space. For the class of noise-induced SNAs here, because the underlying deterministic system is in a periodic window, it is convenient to use the original periodic attractor to define the size. In particular, for a period-m window, we examine the m-times iterated map, for which the periodic attractor is simply a fixed point. For $D < D_m$, the size of the snapshot attractor remains small, as the trajectory stays in the vicinity of the periodic attractor. For $D > D_m$ when noise-induced SNAs occur, a typical trajectory deviates from the original periodic attractor intermittently, giving rise to snapshot attractors with larger size. Thus, as D is increased through D_m, we expect to see a sudden increase in the size of the snapshot attractor. As we have pointed out, due to the negativeness of the largest Lyapunov exponent for a noise-induced SNA, under identical noise the size of the corresponding snapshot attractor approaches asymptotically zero, as exemplified in Fig. 6(a) for the noisy Hénon map. The finite size of the snapshot attractor can be restored by using small inhomogeneous noise, as shown in Figs. 6(b) and 6(c) [a blowup of part of Fig. 6(b)], a variation of the size of the snapshot attractor with time.

Figures 6(b) and 6(c) reveal an interesting behavior: the size of the snapshot attractor exhibits on-off intermittency with time. The explanation again lies in the coexistence of two sets in the noiseless system: a periodic attractor and a chaotic saddle. Under noise, there can be time intervals where the ensemble of trajectories concentrates in the vicinity of the original periodic attractor, leading to extremely small size in the snapshot attractor. Because of noise, after a time the trajectories will be kicked out of the small neighborhoods to spread over the chaotic saddle, giving rise to a large size. The process of shrinking to periodic points and spreading over a chaotic set continues in an intermittent fashion. Analyses similar to those in Ref. [36] can be carried out to characterize the intermittency and to explore universal properties.

Noise-induced SNAs and fractal snapshot attractors can also occur in periodically driven systems. To give an example, we consider the following kicked Duffing's oscillator:

$$d^2x/dt^2 + 0.1dx/dt + (1.0 + 0.45 \cos t)x - x^3 + D\xi(t) = 0, \tag{20}$$

where $\xi(t)$ is a Gaussian process of zero mean and unit variance. For the chosen set of parameter values, for $D=0$ the system is in a period-4 window with the largest nontrivial Lyapunov exponent $\lambda_1^P \approx -0.047$. The range of noise amplitude for which noise-induced SNAs can possibly occur is $D_4(\approx 0.03) < D < D_{*4}(\approx 0.08)$. A demonstration that single trajectories do not reveal the fractal structure of noise-induced SNA, while snapshot attractors do, is given in Fig. 7.

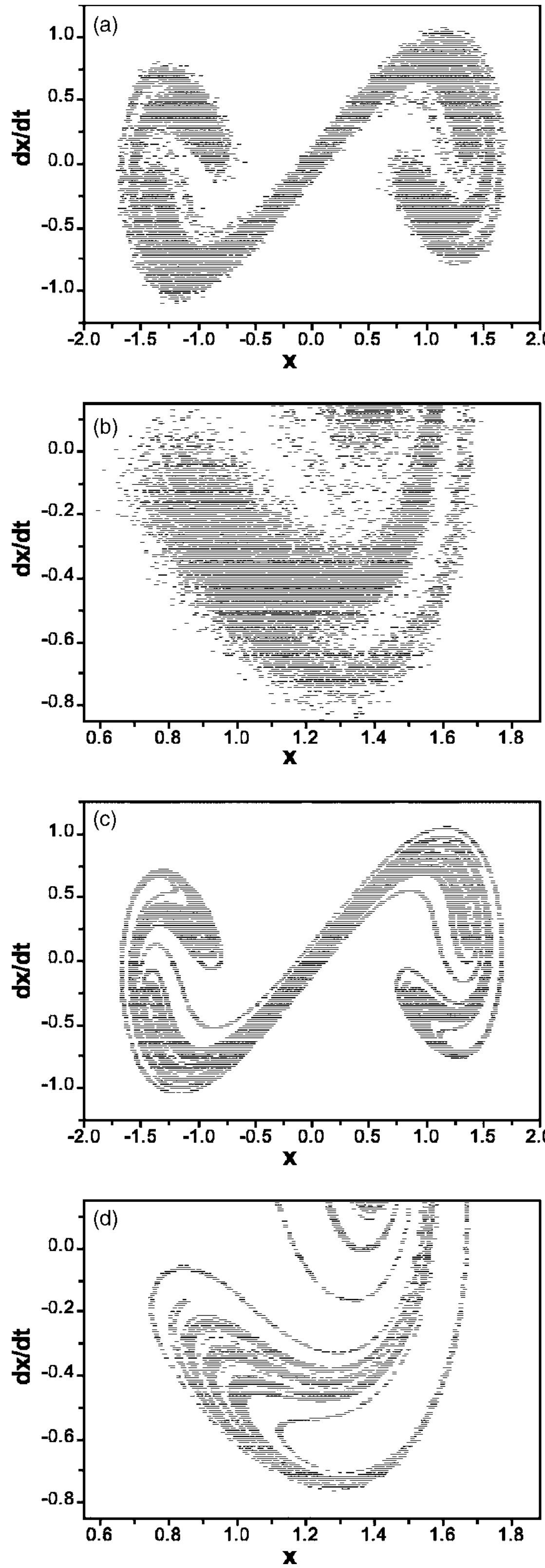

FIG. 7. On the stroboscopic section of the periodically driven Duffing oscillator under Gaussian noise of amplitude $D=0.06$, (a) a single trajectory, (b) blowup of part of (a), (c) a snapshot attractor from 40 000 trajectories, (d) blowup of part of (c). The snapshot attractor is apparently fractal.

VI. DISCUSSION

We have presented a detailed study of noise-induced SNAs in maps or in periodically driven systems. The results here and those from our previous short paper [13] represent evidence that noise-induced SNAs can occur in physical systems other than quasiperiodically driven. The findings of this paper (beyond those in Ref. [13]) are (i) the fractal dimensions of noise-induced SNAs typically assume fractional values, in contrast to SNAs in quasiperiodically driven systems, (ii) the fractal structures can be observed through the snapshot attractors, and (iii) while the asymptotic values of the largest Lyapunov exponent of noise-induced SNAs are negative, their finite-time fluctuations on the positive side are exponentially distributed. We have worked out a random-matrix theory to explain and to establish the generality of the exponential behavior.

Some findings reported here are accessible to experimental test. For instance, one can use electronic circuits to build up a periodically driven system, such as the kicked Duffing oscillator. The parameters of the system can be chosen so that it is in a periodic window. Application of noise can then induce SNAs. A possible technical issue is the determination of the asymptotic value of the largest Lyapunov exponent from measured time series. We suggest to generate the histogram of the finite-time exponent, which should exhibit the following features: (i) there is a peak at a negative value, and

(ii) the positive values are exponentially distributed. Another issue concerns the fractal dimension. A good candidate is the correlation dimension, which is experimentally accessible by calculating the correlation integral from measured time series [60]. As we have demonstrated, for noise-induced SNAs, the correlation dimensions typically assume fractional values.

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