# Detecting and characterizing phase synchronization in nonstationary dynamical systems

Ying-Cheng Lai,[1] Mark G. Frei,[2] and Ivan Osorio[2,3]
[1]*Department of Electrical Engineering, Arizona State University, Tempe, Arizona 85287, USA*
[2]*Flint Hills Scientific L.L.C., 5020 Bob Billings Parkway, Suite A, Lawrence, Kansas 66049, USA*
[3]*Department of Neurology, University of Kansas Medical Center, 3901 Rainbow Boulevard, Kansas City, Kansas 66160, USA*
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We propose a general framework for detecting and characterizing phase synchronization from noisy, nonstationary time series. For detection, we propose to use the average phase-synchronization time and show that it is extremely sensitive to parameter changes near the onset of phase synchronization. To characterize the degree of temporal phase synchronization, we suggest to monitor the evolution of phase diffusion from a moving time window and argue that this measure is practically useful as it can be enhanced by increasing the size of the window. While desynchronization events can be caused by either a lack of sufficient deterministic coupling or noise, we demonstrate that the time scales associated with the two mechanisms are quite different. In particular, noise-induced desynchronization events tend to occur on much shorter time scales. This allows for the effect of noise on phase synchronization to be corrected in a practically doable manner. We perform a control study to substantiate these findings by constructing and investigating a prototype model of nonstationary dynamical system that consists of coupled chaotic oscillators with time-varying coupling parameter.

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## I. INTRODUCTION

In this paper, we present a general methodology to analyze time series from nonstationary dynamical systems. Here by nonstationarity we mean that the rules governing the system's evolution are not fixed but they can vary with time. For instance, for a system described by a set of differential equations, its being nonstationary means that the mathematical form of the equations can vary with time. A common situation is where some key parameters of the system equations vary slowly with time. This is so because in many realistic applications parameter changes may be assumed to be much slower than the change of the dynamical (state) variables. Because of the various bifurcations that can potentially occur as a system parameter changes, at certain times the system can undergo characteristic changes, examples of which include sudden change in the attractors (crises), emergence of collective behavior such as synchronization, occurrence of intermittency, etc. In terms of the system performance, such changes may generate undesirable or even catastrophic behavior that should be avoided. A central goal in nonstationary time series analysis is to detect or possibly to predict characteristic changes in the system as it evolves with time.

Our approach is based on detecting and characterizing phase synchronization from multivariate nonstationary time series. The basic assumption is that the underlying dynamical system can be regarded as a system of weakly coupled oscillators, such as a network of coupled neurons. Our interest in the weakly coupling regime stems from the observation that if the coupling is strong, coherence among the dynamics of the oscillators will also be strong, which may lead to an unrealistic reduction in the degrees of freedom of the system and consequently to relatively less complex dynamics. However, in real physical or biological situations, the dynamics can be extraordinarily complex despite the coherence among the coupled elements. In such a case, a description based on the weak-coupling hypothesis seems more reasonable. In the past decade, phase synchronization has been recognized as a general phenomenon in weakly coupled dynamical systems [1–5], where the phase differences of the oscillators are bounded but their amplitudes may remain uncorrelated. Phase synchronization in this sense is *nontrivial*, versus the situation where the dynamical variables of (slightly different) oscillators are synchronized so that the phases are trivially synchronized, as can usually occur in strongly coupled systems.

Our interest is in real-time detection and/or prediction of characteristic changes in the system from time series, so naturally we will use the moving time-window technique. That is, at any given time, calculations and analyses are done using the finite data set contained in a window immediately preceding that time. The key is thus to derive useful quantities that are sensitive to changes in the underlying coupled system and are, at the same time, robust against noise. A natural way to overcome noise is to increase the size of the moving window. Thus by robustness we mean that the quantity can be enhanced significantly as we increase the window size. In terms of phase synchronization, a previously proposed measure that has been used widely in biomedical time-series analysis [6] is the Shannon entropy, but we find that this measure is sensitive to noise and therefore may not be desirable in dealing with short and noisy moving-window data.

For a deterministic, stationary system of coupled oscillators, phase synchronization refers to the situation where the differences among the phase variables from the oscillators stay within $2\pi$. Due to noise and nonstationarity, such an ideal situation cannot be maintained forever and, in fact, there can typically be $2\pi$ changes in the phase differences. As a result, phase synchronization can be observed only in finite time intervals. This defines an *average phase-synchronization time*. We shall argue that this time typically shows an extremely rapid increase as the underlying deterministic system evolves into the phase-synchronized state and, hence, the time can be effective for detecting phase synchronization. To characterize the degree of temporal phase synchronization as the system evolves, we propose to use the variance of phase fluctuations, or phase diffusion, and show that it is particularly suitable for noisy, moving-window data from multivariate nonstationary time series. We will show that the phase diffusion is sensitive to characteristic changes in the system and it is robust against noise in the sense that it can be enhanced proportionally as we increase the moving-window size.

While both the average phase-synchronization time and phase diffusion can work well for monitoring phase synchronization for relatively small-amplitude noise, large noise can be quite detrimental. Quite interestingly, we have observed that desynchronization events induced by noise tend to occur on much shorter time scales than those of deterministic origin. Taking advantage of this observation, we propose a simple and practical method to minimize the effect of noise on phase synchronization. We will present examples to demonstrate the power of this noise-reduction method.

In Sec. II, we introduce the average phase-synchronization time and the phase-diffusion measure and argue that they can be superior to the existing entropy measure in dealing with noisy, moving-window data from nonstationary time series. In Sec. III, we introduce our control model, describe our strategy to overcome noise, and present evidence that the measures are sensitive to system change but robust against noise. Conclusions and discussions are presented in Sec. IV.

## II. MEASURES FOR DETECTING AND CHARACTERIZING PHASE SYNCHRONIZATION

### A. Phase synchronization

Consider a system of coupled nonlinear oscillators. The term "synchronization" in a conventional sense means that all oscillators evolve identically in time. In reality, parameter mismatch among the oscillators and the presence of noise render perfect synchronization impossible. Yet, due to coupling, a certain degree of coherence among the oscillators can occur. Let $x(t)$ and $y(t)$ be signals from two oscillators. If the overall temporal evolutions of $x(t)$ and $y(t)$ tend to follow each other but not their details, there is a phase synchronization between $x(t)$ and $y(t)$. For instance, in a given time interval both $x(t)$ and $y(t)$ may complete the same number of oscillations, but these need not match exactly with each other. Or more generally, in a time interval $x(t)$ may complete $n$ oscillations and $y(t)$ may go through $m$ oscillations, but insofar as the ratio $n/m$ remains constant in time, there is a phase coherence or synchronization between $x(t)$ and $y(t)$. Let $\phi_1(t)$ and $\phi_2(t)$ be some properly defined phase variables for the two oscillators, respectively. Phase synchronization is defined by [1,7]

$$\Delta\phi(t) \equiv |n\phi_1(t) - m\phi_2(t)| < 2\pi, \tag{1}$$

where $n$ and $m$ are integers. A *phase-desynchronization* event occurs when $\Delta\phi(t)$ changes by $2\pi$, which can be caused by either insufficient coupling in the deterministic system or noise. In this case, phase synchronization occurs only temporally in the time interval between two successive desynchronization events.

To calculate a phase variable, the standard Hilbert-transform approach is often used. Given $x(t)$, its Hilbert transform is

$$H[x(t)] = \mathrm{PV}\left[\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(t')}{t-t'}dt'\right], \tag{2}$$

where "PV" stands for the Cauchy principal value of the integral. From $H[x(t)]$, the following complex analytic signal can be constructed:

$$\psi(t) = x(t) + iH[x(t)] \equiv A(t)\exp[i\phi(t)], \tag{3}$$

which defines the phase variable $\phi(t)$. The Hilbert transform and the analytic signal are meaningful only when $x(t)$ possesses a proper structure of rotation. This can be seen by referring to the rotational motion of a mechanical particle on a circle centered at the origin in the plane. The coordinates $x(t)$ and $y(t)$ are sinusoidal signals with the property that in a time interval containing many periods, the number of maximum and minimum points is the same as the number of zeros, which can be taken as the defining characteristic of a proper rotation. In realistic applications this condition may not be met. In such a case the procedure developed by Huang *et al.* [8] or that in [9] can be used to decompose $x(t)$ into a small number of modes of proper rotation and an analytic signal can be obtained for each of the modes to allow for phase variables to be calculated. An alternative procedure based on filtering is proposed in Ref. [5] and has been applied to data from laser experiments.

### B. Shannon entropy

A method to characterize the degree of phase synchronization, which has been applied to detecting synchronization among biomedical signals, was proposed by Tass *et al.* [6]. Given two phase variables $\phi_1(t)$ and $\phi_2(t)$, the phase difference $\Delta\phi(t)$ can be normalized to the $2\pi$ interval—say, $[-\pi,\pi]$. For chaotic and/or stochastic signals, $\Delta\phi(t)$ can be regarded as a stochastic process. The probability distribution $P(\Delta\phi)$ can then be calculated. In the lack of synchronization, all phase differences in the $2\pi$ interval are possible, leading to nearly uniform distribution of $\Delta\phi(t)$. When synchronization occurs, the distribution $P(\Delta\phi)$ becomes nonuniform and typically tends to concentrate on a narrow range in $\Delta\phi(t)$. A standard approach to quantify the degree of uniformity of a probability distribution is the following Shannon entropy [10]:

$$S = -\int_{-\pi}^{\pi} P(\Delta\phi)\ln P(\Delta\phi)d\Delta\phi. \tag{4}$$

Let $S_{max}=\ln(2\pi)$ be the Shannon entropy for a uniform distribution. The ratio

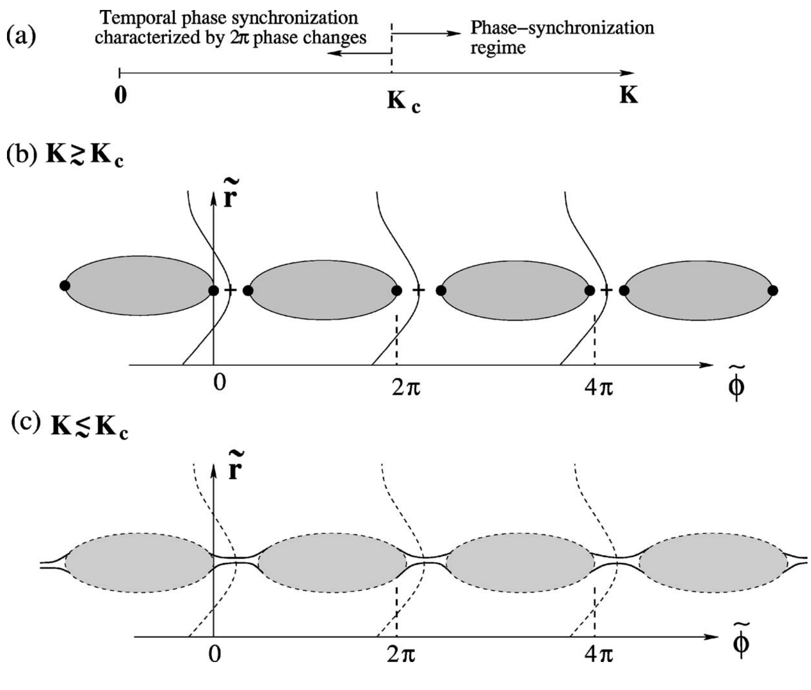


FIG. 1. For the system of a pair of coupled chaotic oscillators, (a) two distinct parameter regimes of interest, (b) for $K \gtrsim K_c$ (phase-synchronized regime), phase space with a "lifted" phase variable, and (c) phase-space structure for $K \lesssim K_c$.

$$\rho = \frac{S}{S_{max}} \tag{5}$$

can be used to characterize the degree of phase synchronization, where $\rho=0$ indicates perfect synchronization and $\rho=1$ signals complete lack of synchronization.

The Shannon entropy is a statistical measure, and it ignores temporal features such as $2\pi$ phase jumps associated with desynchronization events. It can be useful for distinguishing phase-coherent from phase-incoherent states for signals of long duration so that sufficient statistics can be accumulated. In the type of real-time applications that we are interested in here, the goal is to be able to tell, as soon as possible, when a characteristic change in the system occurs. This stipulates the use of short signals, typically from a moving time window. In this case, the entropy can exhibit large statistical fluctuations from window to window, which is worsened by the presence of noise. This presents a difficulty to detect or predict system changes.

### C. Average phase-synchronization time

To present our idea, we consider a prototype model of two weakly coupled chaotic oscillators. The chaotic attractor is assumed to be phase coherent so that a proper phase variable can be defined. The average frequencies of the oscillators, denoted by $\omega_1$ and $\omega_2$, are assumed to be slightly different: $\Delta\omega \equiv |\omega_1 - \omega_2| \ll \omega_{1,2}$. Let $K$ be the coupling parameter and assume that phase synchronization occurs for $K > K_c > 0$. In the absence of coupling ($K=0$), the two oscillators evolve independently so that the ensemble-averaged phase difference increases linearly with time,

$$\langle \Delta\phi(t) \rangle \approx (\Delta\omega)t. \tag{6}$$

The average time for a $2\pi$ change in $\Delta\phi(t)$ is thus

$$\tau(K=0) \approx 2\pi/(\Delta\omega) = \text{const.} \tag{7}$$

As the coupling parameter $K$ is increased from zero, $\langle \Delta\phi(t) \rangle$ still increases approximately linearly with time but at a smaller rate than $\Delta\omega$, leading to an increase in the average $2\pi$-phase-change time. For $K < K_c$, $\Delta\phi(t)$ exhibits persistent $2\pi$ phase changes so that $\tau$ is finite. As $K$ is increased through $K_c$, the phase difference $\Delta\phi(t)$ becomes confined within $2\pi$ so that $\tau=\infty$. These two parameter regimes are shown schematically in Fig. 1(a).

The above discussion is for the case of deterministic, stationary dynamical systems. Since our goal is to use $\tau$ to detect and characterize phase synchronization in noisy, nonstationary dynamical systems, it is important to understand how $\tau$ changes with $K$. The generic behavior as $K$ is increased from well below $K_c$ is that $\tau$ increases slowly until, as $K$ approaches $K_c$, $\tau$ starts to increase extremely rapidly. This can be understood heuristically by referring to the general dynamical mechanism for a transition to phase synchronization, unstable-unstable pair bifurcation [11]. In particular, let $\tilde{\phi}$ be a "lifted" phase variable corresponding to $\Delta\phi(t)$, where the $2\pi$-periodic behavior in $\Delta\phi(t)$ is unwrapped and successive $2\pi$ intervals are considered as distinct phase-space regions, as shown schematically in Figs. 1(b) and 1(c), where $\tilde{r}$ is an arbitrary amplitude variable. In the phase-synchronized regime, $\Delta\phi(t)$ is confined within $2\pi$, so there is an attractor in the phase space $(\tilde{r}, \Delta\phi)$, where $|\Delta\phi| < 2\pi$. In the lifted phase space, there are then an infinite number of identical attractors, each being confined within a $2\pi$ interval corresponding to its basin of attraction, as in Fig. 1(b). Because of the synchronization, the basins of attraction are not dynamically connected. Desynchronization in phase can be regarded as being caused by the dynamical connection among the basins. Rosa *et al.* [11] argued and presented numerical evidence that the connection is typically a result of unstable-unstable pair bifurcations. In particular, for $K \gtrsim K_c$,

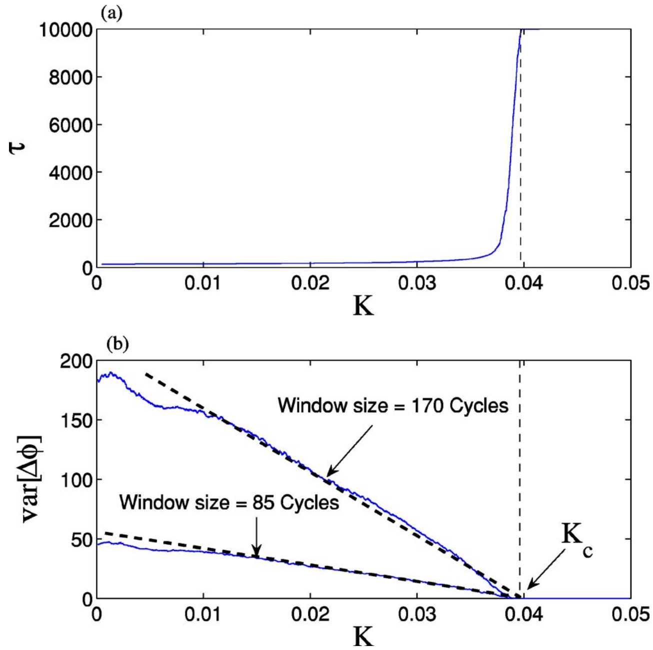


FIG. 2. (Color online) For the model system, Eq. (9), (a) the average $2\pi$-phase-change time (or the average phase-synchronization time) $\tau$ versus the coupling parameter $K$. We observe an extremely fast increasing behavior in $\tau$ as $K$ approaches the critical value $K_c$, suggesting that $\tau$ can be effective to distinguish between phase-synchronized and phase-desynchronized regimes. (b) Approximately linear variation with $K$ of the phase diffusion for two time intervals of computation (two window sizes), indicating that the measure can be used to characterize the degree of temporal phase synchronization.

there are unstable periodic orbits on the attractors and on the basin boundaries [denoted by closed circles and pluses in Fig. 1(b), respectively]. The two classes of orbits typically have distinct unstable dimensions. As $K$ is decreased toward $K_c$, the two classes of unstable periodic orbits approach each other and coalesce at $K_c$, leading to the opening of an infinite set of narrow "channels" that dynamically connect adjacent basins of attraction for $K$ below $K_c$. Tunneling between adjacent basins of attraction gives rise to $2\pi$-phase jumps observed for $\Delta\phi(t)$, as shown schematically in Fig. 1(c). For $K \lesssim K_c$, a trajectory typically spends an extremely long time in a basin before tunneling occurs. It is known that chaotic transients triggered by unstable-unstable pair bifurcations are superpersistent [12–14] which, in the context of phase synchronization, means the following scaling relation [15] in the average $2\pi$-phase-change time for $K < K_c$:

$$\tau \sim \exp[C(K_c - K)^{-\gamma}], \qquad (8)$$

where $C > 0$ and $\gamma > 0$ are constants. We see that as $K$ approaches $K_c$ from below, $\tau$ increases in the manner roughly described by $e^{\infty}$ (herein the term "superpersistent"). For $K$ below this superpersistent regime, different scaling laws were reported [16], but they all indicate much smaller values for $\tau$.

An example of the extremely fast increasing behavior in $\tau$ as $K$ approaches $K_c$ is shown in Fig. 2(a), where the model is the following system of two coupled chaotic Rössler oscillators:

$$dx_{1,2}/dt = -\omega_{1,2}y_{1,2} - z_{1,2} + K(x_{2,1} - x_{1,2}),$$

$$dy_{1,2}/dt = \omega_{1,2}x_{1,2} + 0.165y_{1,2},$$

$$dz_{1,2}/dt = 0.2 + (x_{1,2} - 10.0)z_{1,2}, \qquad (9)$$

and $\omega_1 = 0.98$ and $\omega_2 = 1.02$ are the frequencies of the oscillators when decoupled. Here we assume all parameters are constant so the system is stationary. (In Sec. III we shall study the nonstationary version of this system by assuming that the coupling parameter $K$ depends on time.) Each Rössler oscillator, when uncoupled, exhibits a chaotic attractor with well-defined rotation. The phase, or the angle of rotation, can be conveniently calculated from

$$\tan \phi_{1,2}(t) = \frac{y_{1,2}(t)}{x_{1,2}(t)}. \qquad (10)$$

Phase synchronization occurs [1] for $K > K_c \sim \omega_2 - \omega_1 \approx 0.04$. To generate Fig. 2(a), a large number of $K$ values are chosen and, for each $K$, 100 trajectories of length $T = 10\,000$ are used to obtain the average $2\pi$-phase-change time. The maximum computable value of $\tau$ is thus $T$. (The average period of oscillation for the Rössler attractor is about 5.9, so the time interval $T = 10\,000$ contains approximately 1700 cycles of oscillations.) We observe that, in its range of variation, $\tau$ increases relatively slowly for $K < K_c$ but shows a sharp increase as $K$ approaches $K_c$.

In applied situations where both noise and nonstationarity may be present, the phase-synchronization time is finite even for $K > K_c$. The rapid increasing behavior shown in Fig. 2(a) suggests that this time can be used to reveal whether the underlying deterministic system is in a phase-synchronized or in a phase-desynchronized regime. However, in the desynchronized regime where phase synchronization is typically of short duration, the time may not be effective for differentiating the degree of the temporal phase synchronization.

### D. Phase diffusion

We desire a measure that (1) characterizes the degree of temporal phase synchronization and (2) can be improved by increasing the time interval of computation (or the window size). The latter point is particularly important because, in the presence of strong noise, a viable approach is to increase the size of the moving window. Here we propose a measure based on the phase diffusion and argue that it satisfies the two criteria. Still consider the prototype system of two coupled chaotic oscillators. For $0 < K < K_c$, the average phase difference increases with time but with a slope smaller than $\Delta\omega$. In general, we can write

$$\Delta\phi(t) = \Delta\phi(0) + \alpha(K)t + \theta(t), \qquad (11)$$

where $\alpha(K)$ is a small constant that depends on the coupling strength $K$ and $\theta(t)$ denotes the random process of phase fluctuations. In an infinitesimal time interval $\Delta t$, the change in $\theta(t)$ satisfies $-\pi \leqslant \Delta\theta(t) < \pi$. Thus the average value of

$\Delta\theta(t)$ is zero: $\langle\Delta\theta(t)\rangle=0$. However, much like a random-walk process, the root-mean-square value of $\theta(t)$ typically increases with time as $\sqrt{t}$. This can be justified by considering the differential equation governing the evolution of $\Delta\phi(t)$,

$$\frac{d\Delta\phi(t)}{dt} = \alpha(K) + \xi(t), \tag{12}$$

where $\xi(t)$ is a stationary random process. We thus have

$$\theta(t) = \int_0^t \xi(t')dt'. \tag{13}$$

In time scales larger than the correlation time of the underlying chaotic or stochastic process $x(t)$, $\xi(t)$ can be regarded as being independent of $\Delta\phi(t)$. Under the assumption that the average of $\xi(t)$ vanishes, the ensemble-averaged value of the phase difference $\Delta\phi(t)$ at time $t$ is $\Delta\phi(0)+\alpha(K)t$. For large times, the variance of $\Delta\phi(t)$ obeys the Green-Kubo relation for diffusion processes [17]:

$$\langle[\Delta\phi(t) - \Delta\phi(0) - \alpha(K)t]^2\rangle = \langle\theta^2(t)\rangle \approx Dt, \tag{14}$$

where $D$ is the diffusion coefficient defined by

$$D = \int_{-\infty}^{\infty} \langle\xi(t')\xi(t'+t)\rangle dt. \tag{15}$$

Now consider a moving window of length $\Delta T$ and assume that it is large so that the diffusion approximation, Eq. (14), holds. Within the window, we have

$$\langle\Delta\phi(t)\rangle \approx \alpha(K)\Delta T/2. \tag{16}$$

The variance of $\Delta\phi(t)$, or the phase diffusion, is

$$\mathrm{var}[\Delta\phi] \equiv \langle[\Delta\phi(t) - \langle\Delta\phi(t)\rangle]^2\rangle \approx \frac{\alpha^2(\Delta T)^2}{12} + D\Delta T. \tag{17}$$

For $K<K_c$, there is no phase synchronization and, hence, we have $\alpha\neq 0$. Thus for $\Delta T$ large we have $\mathrm{var}(\Delta\phi)\sim(\Delta T)^2$. In the phase-synchronization regime ($K>K_c$), we have $\alpha=0$ so that $\mathrm{var}(\Delta\phi)\sim\Delta T$. Thus, by increasing the window length, the phase diffusion can exhibit relatively more significant changes as $K$ approaches $K_c$, as shown in Fig. 2(b). This can be particularly useful for real-time detection of system changes, and the effect of noise can be overcome by increasing the moving-window length.

To verify these behaviors numerically, we plot in Fig. 3 the phase evolution $\Delta\phi(t)$ for $K=0$, $K=0.02<K_c$, $K=0.03<K_c$, and $K=0.04\gtrsim K_c$. The linear growth is seen for the first three cases but the growth rate decreases as the coupling parameter $K$ is increased. These validate the linear-growth assumption in Eq. (11). To see the randomly fluctuating behavior in the phase difference associated with phase synchronization, we show in Fig. 4, for $K=0.04$, $\Delta\phi(t)$ on the scale of $2\pi$. We see that in large time, $\Delta\phi(t)$ can be regarded effectively as a random process in the phase-synchronization regime. This provides justification for treating $\theta(t)$ in Eq. (11), the phase variation on top of the linear growth, as a random process.

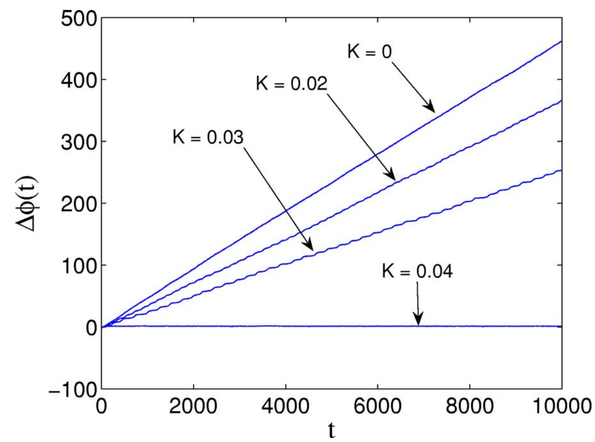


FIG. 3. (Color online) For the prototype model, Eq. (9), approximately linear behavior in $\Delta\phi(t)$ for $K=0$, $K=0.02<K_c$, $K=0.03<K_c$, and $K=0.04\gtrsim K_c$. These results validate the linear-growth assumption in Eq. (11).

## III. NUMERICAL TEST AND NOISE REDUCTION

Here we present a prototype model of nonstationary dynamical systems for which the degree of phase synchronization can be varied with time in a controlled manner and test the power of the average synchronization time and the phase diffusion for detecting and characterizing phase synchronization from measured time series.

### A. Model description

The model consists of a pair of bidirectionally coupled chaotic Rössler oscillators under noise as follows:

$$dx_{1,2}/dt = -\omega_{1,2}y_{1,2} - z_{1,2} + K(t)(x_{2,1} - x_{1,2}),$$

$$dy_{1,2}/dt = \omega_{1,2}x_{1,2} + 0.165y_{1,2} + \epsilon\xi_{1,2}(t),$$

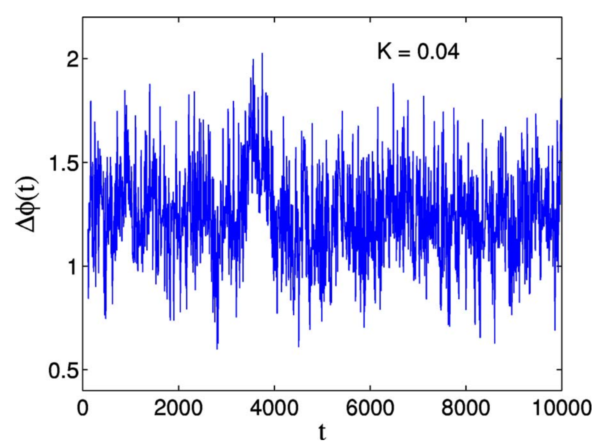


FIG. 4. (Color online) For the prototype model, Eq. (9), random behavior in $\Delta\phi(t)$ for $K=0.04\gtrsim K_c$. This provides the justification for regarding $\theta(t)$ in Eq. (11) as a random process.

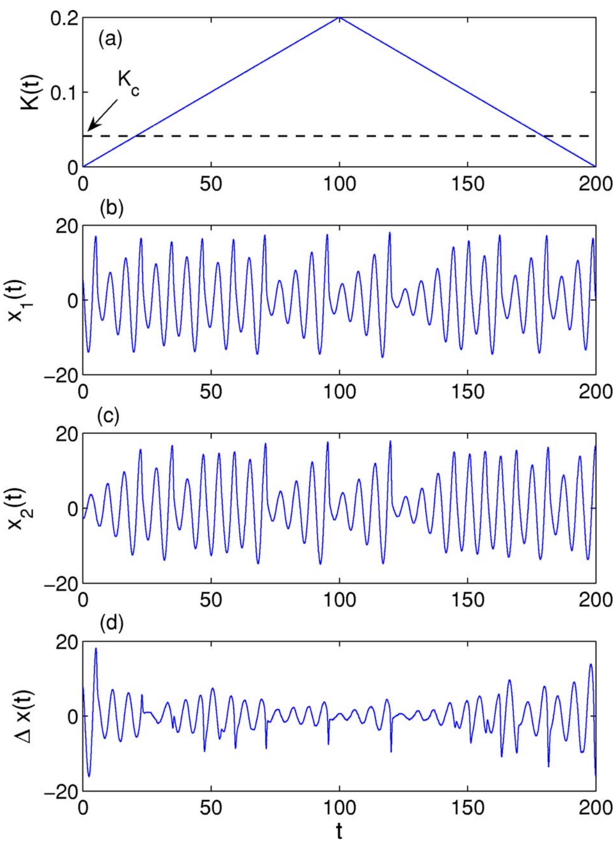


FIG. 5. (Color online) For the prototype model, Eq. (18), (a) variation of the coupling parameter, (b)–(d) for $\epsilon=0$, time series $x_1(t)$, $x_2(t)$, and $\Delta x(t)=x_1(t)-x_2(t)$, respectively.

$$dz_{1,2}/dt = 0.2 + (x_{1,2} - 10.0)z_{1,2}, \tag{18}$$

where the coupling parameter $K(t)$ is time dependent so that the system is nonstationary and $\epsilon\xi_{1,2}(t)$ are independent additive noise [$\xi_{1,2}(t)$ are Gaussian random processes of zero mean and unit variance]. The noise terms are added to the $y$ equation because it is similar to the "force" equation in a physical situation. The stochastic differential equation (18) is integrated using a standard second-order routine [18].

Given a long experimental time interval $T$, we allow the coupling parameter $K(t)$ to vary in the range $[0,K_m]$, where $K_m > K_c$, according to the following simple rule:

$$K(t) = \begin{cases} 2K_m t/T, & 0 \leq t < T/2, \\ 2K_m(1 - t/T), & T/2 \leq t < T. \end{cases} \tag{19}$$

Figure 5(a) shows, for $T=200$, $\epsilon=0$, and $K_m=0.2$, the variation of $K(t)$ over time. The resulting time series $x_1(t)$, $x_2(t)$, and $\Delta x(t)=x_1(t)-x_2(t)$ are shown in Figs. 5(b)–5(d), respectively. The average periods of oscillation for both oscillators are $T_0 \approx 5.9$. This defines a basic time unit in that time can be conveniently referred to in terms of the number of cycles of oscillation. From Fig. 5(d), we observe that in the time interval where $K(t)$ is large, there is a tendency for $x_1(t)$ and $x_2(t)$ to stay close, indicating synchronization in a general sense.

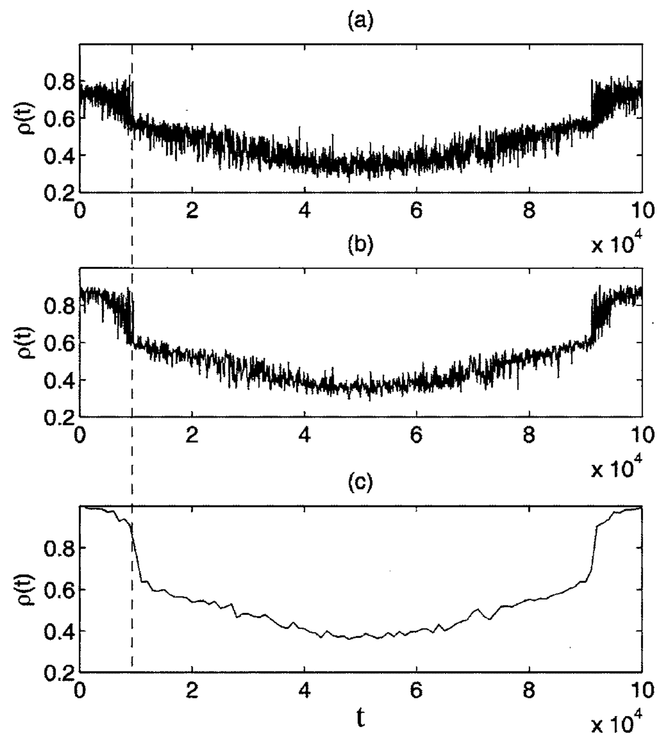


FIG. 6. (a)–(c) For $\epsilon=0.2$, evolution of the Shannon-entropy measure $\rho(t)$ from a moving window of size $\Delta T \approx 8.5$ cycles, $\Delta T \approx 17$ cycles, and $\Delta T=170$ cycles, respectively. Time is counted at the end of the window.

### B. Shannon entropy

To emulate an experimental situation we set a long time interval $T=10^5$ (corresponding to about 17 000 cycles of oscillation) and collect time series $x_{1,2}(t)$ and $y_{1,2}(t)$ from non-overlapping, moving time windows of duration $\Delta T \ll T$. The coupling parameter varies in $[0,T]$ according to Eq. (19). Figures 6(a)–6(c) show, for $\epsilon=0.2$, the time-varying Shannon entropy $\rho(t)$ for $\Delta T=50$ (corresponding to approximately 8.5 cycles of oscillation), $\Delta T=100$ ($\approx 17$ cycles), and $\Delta T=1000$ ($\approx 170$ cycles), respectively, where time $t$ is counted at the end of window. It can be seen that phase synchronization can be detected and the detectability can be improved by increasing the length of the moving window. We have observed that, however, larger noise tends to deteriorate the detectability.

A drawback of the Shannon entropy $\rho(t)$ for detecting phase synchronization appears to be its weak ability to distinguish between a phase-incoherent and a phase-synchronized state in the presence of noise. In particular, from Fig. 6, we see that the maximum value of $\rho(t)$ is about unity, which indicates complete lack of phase synchronization. Its minimum value is about 0.5, which signifies a certain degree of phase synchronization. To quantify the effectiveness of the entropy measure, we define the following *contrast* measure [19]:

$$C_\rho \equiv \frac{\rho_{max} - \rho_{min}}{\rho_{max} + \rho_{min}}. \tag{20}$$

We desire the contrast to be as close to unity as possible in order to detect system changes [19]. For the Shannon en-

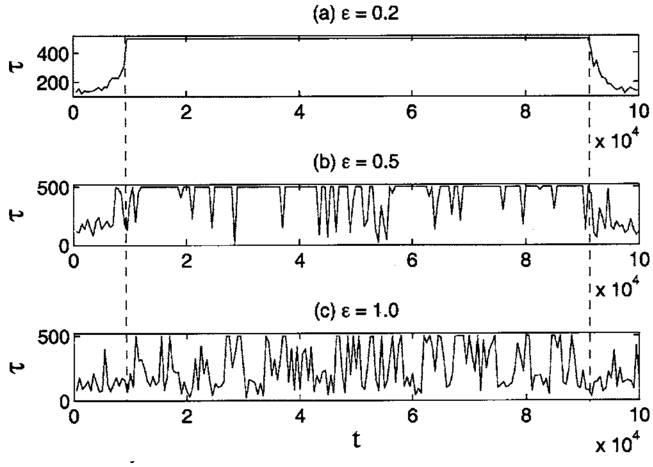


FIG. 7. For the model of Eq. (18), (a), (b), and (c) $\epsilon$=0.2, 0.5, and 1.0, respectively, evolution of the average phase-synchronization time from moving window of size $\Delta T \approx 85$ cycles.

tropy we have $C_\rho \approx 1/3$. In practical applications where strong noise may be present, the contrast value can be even smaller. It may then be difficult to use the Shannon-entropy measure $\rho(t)$ for detecting phase synchronization.

### C. Phase-synchronization time

To calculate the average phase-synchronization time, we use a moving window of size $\Delta T$=500 (corresponding to about 85 cycles of oscillation) and determine the various time intervals within $\Delta T$ during which a $2\pi$-phase change in $\Delta\phi$ occurs. The maximum synchronization time is thus $\Delta T$, which occurs for $K > K_c$. Figures 7(a)–7(c) show, for $\epsilon$=0.2, 0.5, and 1.0, respectively, the evolution of $\tau$ with time. We observe that for a low-noise level ($\epsilon$=0.2), $\tau$ remains at relatively low values for $K < K_c$ but increases rapidly as $K$ approaches $K_c$ and remains at the maximum allowable time for $K > K_c$, suggesting that $\tau$ can be used to detect phase synchronization. Even better, the fast-rising behavior may be useful for short-term prediction of the onset of phase synchronization. The performance of $\tau$ apparently deteriorates as the noise level is increased. For instance, there are irregular dips in $\tau$ even in the phase-synchronized regime [Fig. 7(b)]. For relatively large noise ($\epsilon$=1.0), the detective or short-term predictive power of $\tau$ for phase synchronization is lost almost completely [Fig. 7(c)]. Using alternative statistical quantities such as the median in the distribution of $\tau$ yields essentially the same result for this system.

We thus seek for practical ways to reduce the effect of noise. An interesting observation is that $2\pi$-phase jumps induced by noise tend to occur much more rapidly than those of deterministic origin (e.g., due to insufficient coupling). Figure 8(a) shows a typical $2\pi$-phase jump in $\Delta\phi$ for the deterministic case ($\epsilon$=0) for $K$=0.037$\lesssim K_c$. The instantaneous time derivative of $\Delta\phi$ is shown in Fig. 8(b), where its magnitude is of the order of unity. The temporal evolution of $\Delta\phi$ under noise of amplitude $\epsilon$=1.0 is shown in Fig. 8(c), where we observe random, sharp phase jumps. The sharpness of the phase jumps can also be seen in $d\Delta\phi/dt$, as shown in

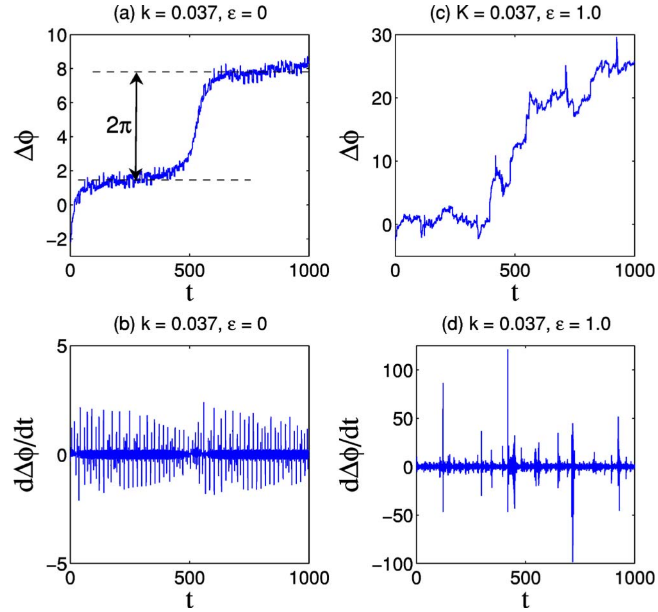


FIG. 8. (Color online) (a) A $2\pi$-phase jump in the noiseless case for $K$=0.037$\lesssim K_c$, (b) the corresponding instantaneous time derivative $d\Delta\phi/dt$, (c) phase changes under the influence of noise of amplitude $\epsilon$=1.0, and (d) the corresponding noisy time derivative. Phase changes induced by noise apparently occur much more quickly than those of deterministic origin.

Fig. 8(d), where its values can reach the order of $10^2$. These behaviors also occur in phase-synchronized regimes, as shown in Figs. 9(a)–9(d) for $K$=0.05$> K_c$.

The results in Figs. 8 and 9 suggest a practical strategy to reduce or even to eliminate noise-induced $2\pi$-phase jumps.

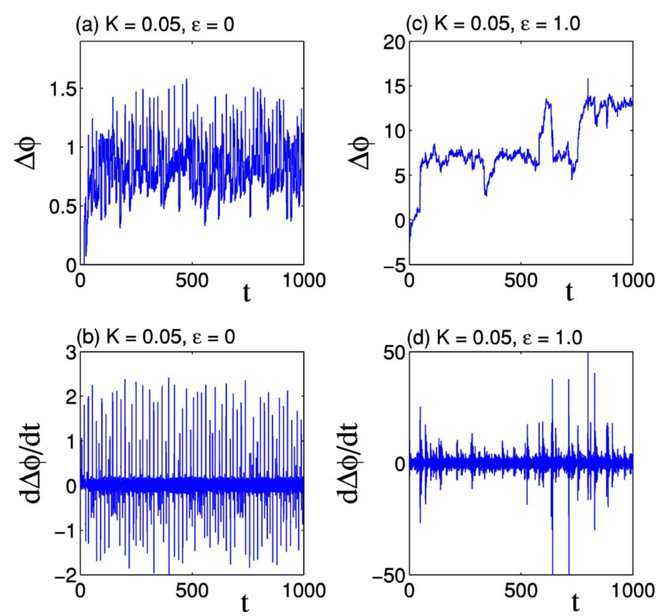


FIG. 9. (Color online) (a) Evolution of phase difference in the noiseless case for $K$=0.05$> K_c$, (b) the corresponding instantaneous time derivative $d\Delta\phi/dt$, (c) phase changes under the influence of noise of amplitude $\epsilon$=1.0, and (d) the corresponding noisy time derivative.

In particular, since such $2\pi$-phase jumps tend to occur rather suddenly, we can set up a threshold [denoted by $(d\Delta\phi/dt)_{th}>0$] for the instantaneous derivative. Phase changes with $|d\Delta\phi/dt|>(d\Delta\phi/dt)_{th}$ are regarded as being induced by noise. Given a time series $\Delta\phi(t_n)$ uniformly sampled at time interval $\Delta t$, we approximate the instantaneous derivative by

$$\frac{d\Delta\phi}{dt} \approx \frac{\Delta\phi(t_{n+1}) - \Delta\phi(t_n)}{\Delta t}.$$

Suppose at time $t_j$ the derivative exceeds the threshold so that the phase change at $t_j$ is regarded as being noise induced. We can remove this sudden change by substracting $\Delta\phi(t_{j+1})-\Delta\phi(t_j)$ from the time series $\Delta\phi(t_n)$ for all $n>j$. The process is repeated for all time instants at which noise-induced phase jumps are considered to have occurred.

An important issue is the choice of the derivative threshold $(d\Delta\phi/dt)_{th}$. If it is too small, phase jumps intrinsic to the system dynamics may be eliminated incorrectly. If it is too large, some noise-induced phase jumps will remain. For a model system such as Eq. (18), we can simply examine some noiseless time series and determine the maximum value for the magnitude of the derivative $d\Delta\phi/dt$. The threshold can be set at a value slightly above the maximum. For applied situations where inevitable noise makes the deterministic behavior of the system inaccessible, one can calculate the instantaneous phase derivatives from time series and construct a histogram of these derivatives. Phase jumps of deterministic origin are likely to be associated with peaks in the histogram at relatively small derivatives, making identification of the threshold feasible. In cases where the histogram does not exhibit such apparent peaks, estimates of the largest possible derivatives by using physical mechanisms specific to the underlying system may be performed to help determine the threshold. For example, the procedure can utilize any useful underlying structure in the histogram that may be representative of a portion of the underlying system dynamics. While there is no general guarantee that a threshold value chosen this way would be accurate, it is hoped that utilizing a sensible estimate can help reduce the effect of noise on phase dynamics.

Figures 10(a) and 10(c) show, for $K=0.037\lesssim K_c$ and $K=0.05>K_c$, respectively, the corrected time series $\Delta\phi(t)$ for $\epsilon=1.0$, where we set $(d\Delta\phi/dt)_{th}=3.0$. The corresponding instantaneous derivatives are shown in Figs. 10(b) and 10(d), respectively. Relatively sudden phase jumps have apparently been removed. Figure 11 shows the corrected behaviors of the phase-synchronization time as compared with the original behaviors in Fig. 7. Improvement is apparent.

### D. Phase diffusion

Figures 12(a)–12(c) show the evolution of the phase diffusion calculated from moving window of size $\tau\approx 8.5$ cycles, $\tau\approx 17$ cycles, and $\tau=170$ cycles, respectively, for $\epsilon=0.2$. We see that in the phase-incoherent regime [$K(t)<K_c$], large fluctuations in the phase diffusion exist for small window size. The fluctuations, however, are significantly reduced for large window size. In all cases, in the phase-synchronized regime the amount of phase diffusion is close to zero as compared with that in the unsynchronized regime, giving rise to a close-to-unity contrast value. This suggests that the phase-diffusion measure can be quite sensitive to characteristic changes in the system. As noise becomes larger, the phase diffusion can exhibit larger fluctuations in both the phase-incoherent and phase-synchronized regimes for small window size. In this case, it is necessary to use a

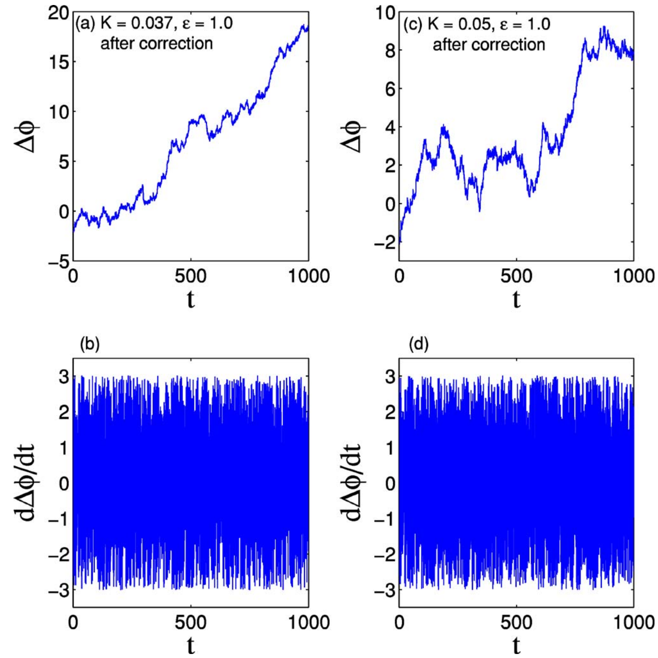


FIG. 10. (Color online) (a) Phase evolution $\Delta\phi(t)$ after the noise-reduction procedure for $K=0.037<K_c$, (b) the corresponding instantaneous time derivative, (c) $\Delta\phi(t)$ after the noise-reduction procedure for $K=0.05>K_c$, and (d) the time derivative.

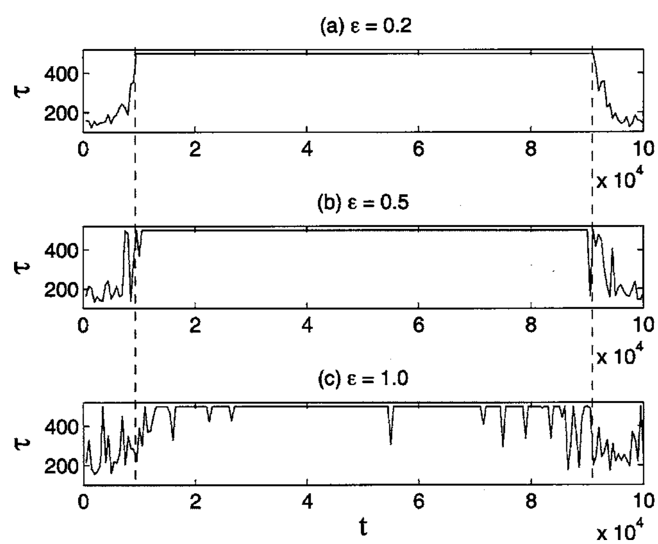


FIG. 11. For the model of Eq. (18), (a), (b), and (c) $\epsilon=0.2$, 0.5, and 1.0, respectively, evolution of the average phase-synchronization time from moving window of size $\Delta T\approx 85$ cycles after performing the noise-reduction procedure.

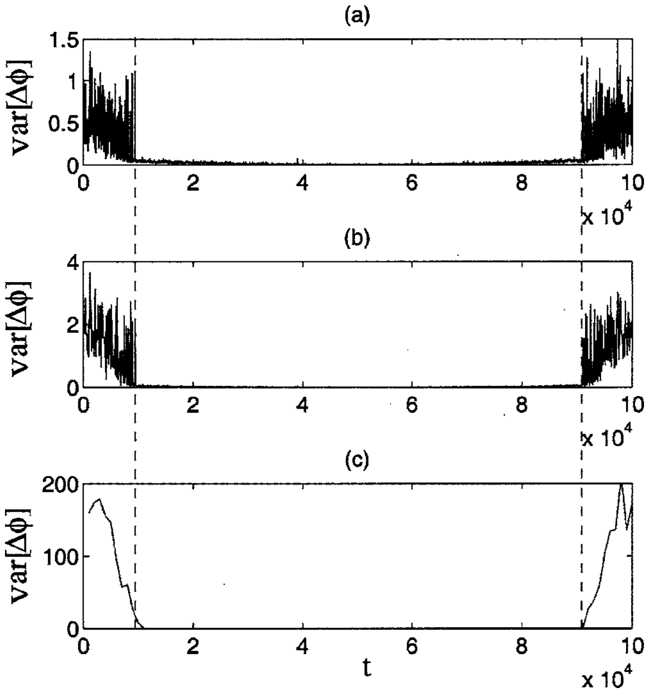


FIG. 12. (a), (b), and (c) For $\epsilon=0.2$, evolution of the phase diffusion calculated from moving window of size $\tau\approx 8.5$ cycles, $\tau\approx 17$ cycles, and $\tau=170$ cycles, respectively.

large window size for the detection of phase synchronization. Improvement can also be achieved by applying our procedure for identifying and eliminating noise-induced phase jumps (Sec. III C), as we have verified numerically.

The above results are obtained by taking advantage of the proper rotational structure of the chaotic Rössler oscillator so that Eq. (10) can be used for calculating the phase. Provided with only time series, it is necessary to use the Hilbert transform and analytic-signal method to calculate the phase. Qualitatively, this yields no difference in the detection of phase synchronization.

## IV. DISCUSSIONS

The problem that we address in this paper concerns nonstationary dynamical systems under temporal changes such as parameter drifts or perturbations from the surroundings. Such changes can lead to characteristic changes in the system dynamics or even to catastrophic events such as seizures in an epileptic brain. Because of the strong nonstationarity, detection and quantification of the system changes need to be done using moving time windows of relatively small size. The goal here is to detect and characterize phase synchronization based on time-series signals measured from multiple sensors. We have demonstrated that the average phase-synchronization time and the phase diffusion can be powerful tools for the task in the sense that they are generally highly sensitive to changes in the degree of phase synchronization. We have presented heuristic arguments and constructed a prototype model of weakly coupled chaotic oscillators to test the effectiveness of these tools.

A previously proposed measure for quantifying the degree of phase synchronization in chaotic or stochastic systems is the Shannon entropy [6]. For systems where long, relatively stationary time series are available, the entropy can be effective for detecting changes in the degree of phase synchronization and thus be quite useful in some applications [6]. We have demonstrated, however, that there can be difficulties associated with the Shannon entropy, particularly in highly nonstationary systems under the influence of strong noise. For our prototype nonstationary model, we defined the quantity contrast that somewhat characterizes the sensitivity of the entropy to changes in the degree of phase synchronization and demonstrated that the maximally achievable value of the contrast for the Shannon entropy is usually far less than the ideal value of unity. Our phase-diffusion measure, however, can easily yield values of contrast close to unity.

An area in which our method may be applied is epilepsy. Epileptic seizures affect about 1% of the population. Seizures are usually characterized by abnormal electrical activity in one or multiple brain regions and can be monitored by an electroencephalogram (EEG) recorded via electrodes attached to the scalp or by an electrocorticogram (ECoG) from electrodes in direct contact with the cortex. The underlying dynamical system responsible for seizures is most likely nonstationary, extremely high dimensional, and nonlinear, and one can intuitively associate the occurrence of seizures with, for example, some catastrophic bifurcations triggered by slow drifts in the system parameters. In epilepsy, since the dynamical system details are unknown, the most convenient means by which seizure dynamics can be studied is through the analysis of nonstationary time series produced by the system—i.e., EEG or ECoG. Since the subsystem responsible for generating seizures (in the case of localization-related epilepsies) interacts with (i.e., is coupled to) many other nonepileptogenic subsystems in a complicated way that is mostly unknown at the present, measures of synchronization (both in phase and in general) are worthwhile for improved understanding of the system dynamics. Another important aspect of the seizure problem, which is shared by many realistic applications of nonstationary time series analysis, is the presence of noise. Overcoming noise is thus a fundamental requirement that must be met by any algorithm if meaningful results quantifying phase synchronization are to be obtained for such systems.

## ACKNOWLEDGMENTS

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[15] This scaling law is a theoretical consequence of the work by Rosa *et al.* [11], which establishes unstable-unstable pair bifurcation as the dynamical mechanism for the phase synchronization-desynchronization transition, and of the previous works on superpersistent chaotic transients [12–14]. So far, to our knowledge, there is no numerical evidence for this scaling law. Possible difficulties are that (1) numerical verification of the scaling law requires an estimate of the critical coupling parameter $K_c$ with high precision and (2) the extremely long transient time may be computationally prohibitive.

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