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Universal and nonuniversal features in shadowing dynamics of nonhyperbolic chaotic systems with unstable-dimension variability

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An important quantity characterizing the shadowability of computer-generated trajectories in nonhyperbolic chaotic system is the shadowing time, which measures for how long a numerical trajectory remains valid. This time depends sensitively on an initial condition. Here, we show that for nonhyperbolic systems with unstable-dimension variability, the probability distribution of the shadowing time contains two distinct scaling behaviors: an algebraic scaling for short times and an exponential scaling for long times. The exponential behavior depends on the system details but the small-time algebraic behavior appears to be universal.

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The validity of numerical trajectories is a fundamental problem in chaotic dynamics. Given a chaotic system, one can compute a numerical trajectory, starting from a random initial condition, and ask whether there is a true trajectory of the system dynamics from a slighly different initial condition, which stays in a small neighborhood of the numerical one. This is the problem of shadowing of numerical trajectories.

An understanding of the shadowing dynamics relies on the mathematical notion of hyperbolicity. Roughly, the dynamics is hyperbolic on a chaotic set if at each point of the trajectory, the tangent space can be split into expanding and contracting subspaces and the angle between them is bounded away from zero. Furthermore, the expanding subspace evolves into the expanding one along the trajectory and the same holds for the contracting subspace. Otherwise, the set is nonhyperbolic. The following results have been established.

(1) Hyperbolic chaotic systems permit infinite shadowing of numerical trajectories [1,2].

(2) For nonhyperbolic chaotic systems with tangencies (i.e., points at which the expanding and contracting directions coincide), shadowing can be expected for a finite amount of time that depends on the computer roundoff error [3,4].

(3) If the dimensions of the expanding and contracting subspaces are not constant on different parts of the invariant set, i.e., if there is unstable dimension variability, then shadowing of numerical trajectories for relatively long time is impossible [5–9]. The severe obstruction to shadowing in the presence of unstable-dimension variability appears to be common in high-dimensional chaotic systems, i.e., those with multiple positive Lyapunov exponents [6–9].

A key quantity to characterize shadowing is the *shadowing time*, which measures for how long a numerical trajectory remains valid in the sense that it stays close to a true trajectory. Due to chaos, this time depends sensitively on initial conditions. It is thus natural to speak about the *average shadowing time* [3,8] and the *probability distribution* of shadowing time [8]. For a random initial condition, the shadowing time can be measured by examining the evolution of the *pointwise shadowing distance*, the local phase-space distance between the resulting numerical trajectory and a true one. For chaotic systems with tangencies, the issue of shadowing time is relatively well settled. In particular, rigorous works indicate that the average shadowing time is inversely proportional to the square root of the computer roundoff [3]. The situation is more complicated for nonhyperbolic systems with unstable-dimension variability. The following is where current understanding stands [8]. For such a system, the shadowing distance typically increases exponentially after encountering a glitch point, where a change in the unstable dimension occurs, then decreases exponentially in hyperbolic regions, and so on, with a lower bound determined by the computer roundoff. The switches between the expanding and contracting behaviors occur randomly in time, suggesting that the behavior of the logarithm Z of the pointwise shadowing distance mimics that of a random walker. A calculation of the corresponding first-passage time gives the average shadowing time, which depends on the system details in the following manner: $\langle\tau\rangle \sim -2m/\sigma^2$, where $m>0$ and σ are the mean and standard deviation of the finite-time Lyapunov exponent that is closest to zero.

In this paper, we examine the probability distribution $\Phi(\tau)$ of the shadowing time and show that there are universal and nonuniversal scaling features. We find that for small τ values the distribution exhibits a universal algebraic scaling, while the distribution is exponential for large values of τ. The exponential distribution depends on system details. That is,

$$\Phi(\tau) \sim \begin{cases} \tau^{-3/2} & \text{for small } \tau, \\ \exp(-a\tau) & \text{for large } \tau, \end{cases} \tag{1}$$

where the constant a is system dependent. The scaling relation (1) means that for nonhyperbolic systems with unstable-dimension variability, shadowing of numerical trajectories can be expected only in short time because longer shadowing time is exponentially improbable.

To compute the shadowing time, it is necessary to monitor the evolution of the shadowing distance, which can be defined as follows. Consider a D-dimensional map of the form $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n, p)$, where $\mathbf{x} \in \mathcal{R}^D$ and p is a parameter. The map can be regarded, for example, as arising from a Poincaré surface of section of a $(D+1)$-dimensional flow, or it may represent the time-T map obtained at times nT ($n=0,1,\ldots$) by numerically solving a set of ordinary differential equations. Consider a numerical (pseudo-) trajectory of length $(N+1)$: $\{\mathbf{p}_n\}_{n=0}^{N}$. Due to the computer roundoff error, typically there is a small difference between $\mathbf{p}_{n+1}$ and $\mathbf{f}(\mathbf{p}_n)$ for $n=0,1,\ldots,N-1$, where $\mathbf{f}(\mathbf{p}_n)$ is the image of $\mathbf{p}_n$ under the true dynamics. Let δ be an upper bound of all these errors along the pseudotrajectory, i.e., $|\delta_n| \equiv |\mathbf{p}_{n+1}-\mathbf{f}(\mathbf{p}_n)| < \delta$, for $n=0,\ldots,N-1$, where δ is on the order of computer roundoff ϵ_c. A true trajectory $\{\mathbf{x}_n\}_{n=0}^{N}$, on the other hand, satisfies $\mathbf{f}(\mathbf{x}_n)=\mathbf{x}_{n+1}$, for $n=0,\ldots,N-1$. The true trajectory ϵ *shadows* the pseudotrajectory if there exists an ϵ such that $|\mathbf{x}_n-\mathbf{p}_n|<\epsilon$ for $n=0,\ldots,N$. The quantity $|\mathbf{x}_n-\mathbf{p}_n|$ is the pointwise shadowing distance [3,6–8]. For hyperbolic attractors, given a δ pseudo-orbit, it is always possible to find a true trajectory whose pointwise shadowing distance with respect to the pseudo-orbit is of order δ. For nonhyperbolic systems, however, the pointwise shadowing distance can reach the size of the entire attractor, for instance, at glitch points where shadowing breaks down [3]. The *shadowing distance* is the maximum distance deformed from any point of a pseudotrajectory to the corresponding point of a true trajectory as the computing error and roundoff go to zero.

The above definition of the pointwise shadowing distance requires knowledge about the true trajectory, which is generally not available. It is thus necessary to obtain an approximation to the true trajectory. The following procedure has been proposed [3]. Given a pseudotrajectory $\{\mathbf{p}_n\}_{n=0}^{N}$, a Newton-Raphson root-finding procedure is used to find a correction $\mathbf{c}_n$ to each point of the pseudotrajectory, which yields a less noisy trajectory, say $\{\mathbf{y}_n\}_{n=0}^{N}$. Let $\mathbf{y}_n=\mathbf{p}_n+\mathbf{c}_n$. The corrections $\mathbf{c}_n$ are determined by decomposing them into components along the local stable and unstable subspaces as $\mathbf{c}_n=\mathbf{s}_n+\mathbf{u}_n$. Reference [3] gives the details of an iterative refinement scheme that computes, for each n, corrections in the stable and unstable directions as $\mathbf{s}_{n+1}=\mathcal{S}_p[\mathbf{Df}(\mathbf{p}_n)\cdot(\mathbf{s}_n+\delta_n)]$ and $\mathbf{u}_n=\mathcal{U}_p[\mathbf{Df}^{-1}(\mathbf{p}_{n+1})\cdot(\mathbf{u}_{n+1}-\delta_{n+1})]$ using the boundary conditions $\mathbf{s}_0=\mathbf{0}$ and $\mathbf{u}_N=\mathbf{0}$. Here, $\mathbf{Df}(\mathbf{p}_n)$ is the Jacobian matrix of $\mathbf{f}$ at $\mathbf{p}_n$, and $\mathcal{S}_p$ and $\mathcal{U}_p$ are the projection operators into the stable and unstable subspaces, respectively. After a refined trajectory is computed, the pointwise shadowing distance can be computed by using this trajectory and the original one.

In our numerical computation, we use the kicked double rotor, which has been a paradigmatic model for studying high-dimensional chaotic phenomena [10], particularly, the shadowing problem [6,7]. We use the parameters as given in Ref. [6] and choose the periodic forcing strength ρ as the control parameter. At $\rho\approx 8.0$, the second largest Lyapunov exponent becomes positive, leading to a high-dimensional chaotic attractor with two positive exponents for $\rho \gtrsim 8.0$. Severe unstable-dimension variability occurs for $\rho\approx 8.0$ [6]. In numerical simulation, the number N_u of unstable directions is chosen as the number N_+ of asymptotically positive Lyapunov exponents. For the special case where one asymptotic exponent cannot be distinguished from zero, we find that choosing either $N_u=N_+$ or $N_u=N_+ +1$ has no influence on our results, which are all statistical in nature. This means that, for instance, for the double-rotor map for $\rho\equiv\rho_0=8.0$, N_u can be chosen to be either 1 or 2. Figures 1(a)–(c) show, on a logarithmic scale, the evolution of the pointwise shadowing distance for $\rho=8.0$, 8.5, and 9.0, respectively. Due to the severe unstable-dimension variability for $\rho=\rho_0$, numerical trajectories cannot be shadowed for appreciable lengths of time. This is reflected by the wide variations of the pointwise shadowing distance over many orders of magnitude. The large values arise from sudden and frequent changes in the dimensions of the stable and unstable subspaces along the trajectory that stymie the refinement procedure. As ρ is increased from ρ_0, the degree of unstable-dimension variability is reduced, causing a progressive improvement in shadowing. For instance, for $\rho=8.5$ [Fig. 1(b)] and $\rho=9.0$ [Fig. 1(c)], the pointwise shadowing distance appears to stay below unity most of the time. In the time interval of 10^4 iterations, there are two events for $\rho=8.5$, in which the pointwise shadowing distance exceeds unity, while there is none for $\rho=9.0$.

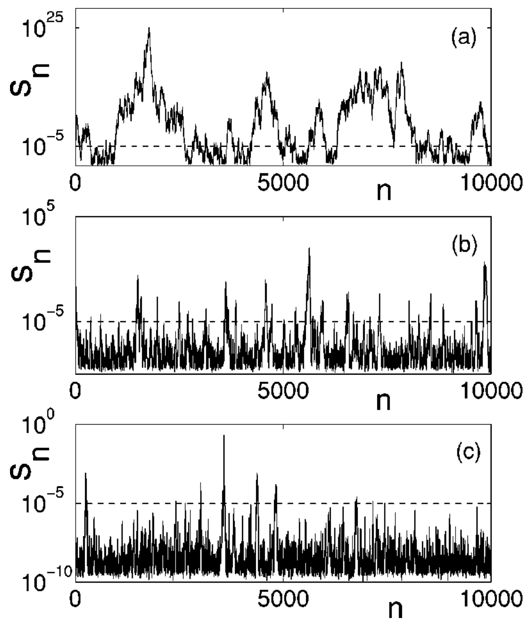

FIG. 1. (a)–(c) Realization of the evalution of the pointwise shadowing distance for the double-rotor map for $\rho\equiv\rho_0=8.0$, $\rho=8.5$, and $\rho=9.0$, respectively.

The shadowing time can be conveniently defined as the time interval during which the pointwise shadowing distance stays less than $\epsilon \ll 1$. With the seemingly random variations in the pointwise shadowing distance, the shadowing time can be regarded as a random variable. The dynamics governing the evolution of the pointwise shadowing distance can naturally be modeled as a stochastic process, in the sense that for a fixed parameter, a different initial conditioin gives a different realization of the process [such as Fig. 1(a)]. To obtain the probability distribution of the shadowing time, we construct a histogram of the values of time intervals τ, during which the shadowing distance is less than the threshold ϵ. Figure 2(a) shows, on a logarithmic scale, for $\epsilon=10^{-5}$ the

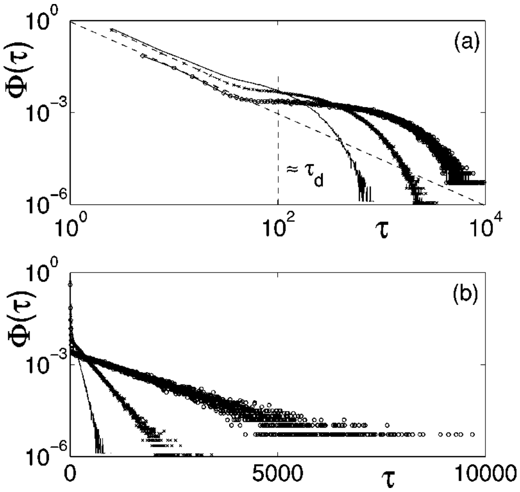

FIG. 2. Probability distributions of the shadowing time τ for $\rho \equiv \rho_0 = 8.0$ (thin solid line), $\rho = 8.5$ (crosses + dashed line), and $\rho = 9.0$ (circles + thin solid line). In (a) the distributions are shown on a logarithmic scale, indicating a universal algebraic scaling behavior with exponent $-3/2$ for small values of τ. In (b) the distributions are plotted on a semilogarithmic scale, indicating an exponential decaying behavior that depends apparently on the system details.

histograms for $\rho \equiv \rho_0$ (thin solid line), $\rho = 8.5$ (crosses + dashed line), and $\rho = 9.0$ (circles + thin solid line), respectively. We observe that for $\tau < \tau_d \approx 10^2$, the distributions $\Phi(\tau)$ appear to be algebraic, while for $\tau > \tau_d$, $\Phi(\tau)$'s decrease rapidly with τ. In fact, the decaying behavior of $\Phi(\tau)$ for $\tau > \tau_d$ appears to be exponential, as shown on a semilogarithmic scale in Fig. 2(b). The exponential decay is system dependent in the sense that its rate depends on the parameter ρ. In particular, the rate is large for $\rho = \rho_0$, indicating that it is highly improbable to have a long shadowing time due to the severe unstable-dimension variability at this parameter value. As ρ is increased from ρ_0 to 9.0, the degree of unstable-dimension variability is reduced so that the exponential decay in $\Phi(\tau)$ becomes slower. The remarkable feature is that the algebraic decay for small τ appears to be *universal* with the scaling exponent $-3/2$, which holds for many other values of ρ in the interval [7.8,10] that we have examined. This universal feature, which governs the shadowing dynamics in short time scale for dynamical systems with unstable-dimension variability, has not been noticed previously.

To explain the universal and nonuniversal features in shadowing, as exemplified by Fig. 2, we consider a random-walk model. A trajectory encounters both approximately hyperbolic regions and regions with glitches. The shadowing dynamics in the hyperbolic regions is equivalent to a random walk toward the reflecting barrier determined by the computer roundoff because, in this case, shadowing theory guarantees the existence of a nearby true trajectory [1]. An approximation of the true trajectory can be found with a refinement technique [3] that adjusts the points on the trajectory in a consistent manner along the stable and unstable directions. As a result, insofar as the trajectory is in a hyperbolic region, on an average, the pointwise shadowing distance decreases exponentially with time toward the lower bound δ. When a glitch occurs, the consistency in the trajectory adjustments, which can be achieved in hyperbolic regions, is immediately destroyed causing the pointwise shadowing distance to increase in an exponential manner. In the walker *i* space, it is equivalent to an excursion away from the reflecting barrier.

We are thus led to consider the following model: $s_{n+1} = w_n s_n$, where s_n stands for the shadowing distance at time *n*, and w_n is a random variable that describes the expansion or contraction of the local shadowing distance at time *n*. Introducing a new variable $y_n = \log_{10} s_n$, we obtain $y_{n+1} = y_n + \nu + z_n$, where $\nu \equiv \langle \log_{10} w_n \rangle$ is the drift of the random walk and $z_n = \log_{10} w_n - \langle \log_{10} w_n \rangle$ is a zero mean random variable. Approximately [11], we can write down the Fokker-Planck equation, $\partial P/\partial t = -\nu \partial P/\partial y + (D/2)\partial^2 P/\partial y^2$, where $P(t,y)$ is the probability distribution for having the walk at distance y at time *t*, and the diffusion coefficient is given by $D = \langle z_n^2 \rangle$. For computing the probability distribution of the shadowing time, the maximum relevant pointwise shadowing distance is $y_{th} = \log_{10}\epsilon$, the threshold distance below which shadowing is considered to hold. There is thus an absorbing boundary condition at y_{th}: $P(t, y_{th}) = 0$. The shadowing distance cannot be smaller than the computer roundoff ϵ_c, which stipulates a reflecting boundary condition at $\log_{10}\epsilon_c$: $[J(t,y) \equiv -\nu P + (D/2) dP/dy]|_{y=\log_{10}\epsilon_c} = 0$. Assuming the walker starts at an arbitrary place $\log_{10}\epsilon_c < y_0 < y_{th}$ at $t=0$, we have the initial condition $P(0,y) = \delta(y - y_0)$. Under these boundary and initial conditions, the Fokker-Planck equation can be solved [12], which gives the following probability distribution for the first-passage time of the walk across y_{th} (the shadowing time):

$$\Phi(\tau) \sim \frac{\tau^{-3/2}}{\sqrt{2\pi D}} \exp\left(-\frac{\nu^2 \tau}{2D}\right), \qquad (2)$$

where the proportional constant depends on the choice of the initial condition y_0. For small values of τ, the dependence of $\Phi(\tau)$ on τ is mainly algebraic with the universal scaling exponent of $-3/2$. For large values of τ, the exponentially decaying behavior in $\Phi(\tau)$ dominates with the rate given by $a = \nu^2/(2D)$. These are the scaling results in Eq. (1). The dependence of the exponential rate on system details can be assessed by computing the dependence of the diffusion parameters ν and D on a system parameter. We find that, approximately, the average drift depends inversely on the parameter variation: $|\nu| \sim 1/(\rho - \rho_0)$, for $\rho > \rho_0$), and the diffusion coefficient D is relatively constant (data not shown).

A few remarks are in order. First, the average drift ν, which is a key parameter in the random-walk model, decreases as ρ is increased from ρ_0. In fact, the value of the average drift appears to be maximum when unstable-dimension variability is most severe. This is somewhat expected from Figs. 1(a–c), the plots of the logarithmic pointwise shadowing distance, or the displacement of the random walker for different values of ρ, where we see that the apparently random evolution of the distance indeed exhibits much larger drift for $\rho=\rho_0$, compared with other values of ρ. Dynamically, this happens due to the existence of the maximally possible number of glitch points on the attractor for $\rho=\rho_0$ when unstable-dimension variability is most severe. As a result, the pointwise shadowing distance suffers a relatively large number of expanding phases as compared to the number of contracting phases experienced during the hyperbolic phases leading to an appreciable amount of drift in the random walk model. This is somewhat different from the diffusion model used in Ref. [8]. Second, the solution to the Fokker-Planck equation, under the boundary and initial conditions, gives satisfactory explanations for our numerical results. The setting of the initial and boundary value problem is in fact quite standard [12], and it also appears in other contexts such as noisy on-off intermittency [13,14]. Our results are completely consistent with those in that context. Third, between the universal scaling in $\Phi(\tau)$ (algebraic) and the nonuniversal scaling (exponential) regimes, there is a crossover regime in τ, where both the algebraic and exponential contributions are important. This is the so-called "shoulder regime" in noisy on-off intermittency [13]. The crossover time is approximately given by $\tau_d \approx D^{-1}[\ln(\epsilon/\epsilon_c)]^2$, which defines the time scale of diffusion [14]. There is another time of interest, which is the drift time $|\nu|^{-1}\ln(\epsilon/\epsilon_c)$. These represent the typical times for the shadowing distance to reach the threshold from the level of computer roundoff due to diffusion and drift, respectively.

In summary, we have uncovered universal and nonuniversal features in the shadowing dynamics of nonhyperbolic chaotic systems with unstable-dimension variability [15]. Our results provide a more detailed understanding of the fundamental problem of shadowing in terms of statistical characterizations. Our theoretical treatment suggests that the shadowing problem shares the same dynamical mechanism as that for on-off intermittency under noise.

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[11] Strictly, the random-walk model can be solved by the Fokker-Planck equation when *Z* is a zero mean, Gaussian random variable. For our shadowing problem, numerically, we find the distribution of *Z* is approximately Gaussian (by definition *Z* has a zero mean).

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[15] The shadowing lemma of Anosov and Bowen [1], which holds for hyperbolic systems, has recently been extended to nonuniformly hyperbolic systems [2]. The nonhyperbolic systems studied here, i.e., dynamical systems with unstable-dimension variability, violate one of the essential conditions for hyperbolicity: the continuous splitting of the tangent space between the stable and unstable subspaces. Thus, the shadowing lemma in Ref. [2] does not hold for these severely nonhyperbolic systems. For them, shadowing of numerical trajectories, even of relatively short lengths, cannot be expected. Our method of the combination of algebraic (for short time) and exponential (for long time) behaviors in the statistical distribution of the shadowing time answers the question, "for how long a numerical trajectory can be expected to be valid?," in a quantitative way.