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Can noise make nonbursting chaotic systems more regular?

Liqiang Zhu,[1,2] Ying-Cheng Lai,[1,2,3] Zonghua Liu,[2,3] and Arvind Raghu[1]
[1]*Department of Electrical Engineering, Arizona State University, Tempe, Arizona 85287*
[2]*Center for Systems Science and Engineering Research, Arizona State University, Tempe, Arizona 85287*
[3]*Department of Mathematics and Statistics, Arizona State University, Tempe, Arizona 85287*

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It has been known that noise can enhance the temporal regularity of dynamical systems that exhibit a *bursting* behavior—the phenomenon of coherence resonance. But can the phenomenon be expected for nonbursting chaotic systems? We present a theoretical argument based on the idea of time-scale matching and provide experimental evidence with a chaotic electronic circuit for coherence resonance in nonbursting chaotic systems.

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The interplay between noise and nonlinearity in dynamical systems can lead to a variety of interesting phenomena, such as stochastic resonance where the detectability of a signal can be enhanced by noise [1]. The phenomenon of noise-induced enhancement of the temporal regularity in nonlinear dynamical systems has been known since the work by Sigeti and Horsthemke [2]. They discovered that for a system near a saddle-node bifurcation, noise can induce a pronounced peak in the power spectrum of dynamical variables. Sigeti and Horsthemke named the phenomenon *noise-induced frequency*. A closely related phenomenon, i.e., noise-induced collective oscillation, or stochastic resonance in the absence of an external periodic signal in excitable dynamical systems, was discovered and analyzed by Hu *et al.* [3], who also introduced a quantitative measure β (to be described later) to characterize the degree of enhancement of the temporal regularity by noise. Recently, the phenomenon was analyzed utilizing the FitzHugh-Nagumo equations [5] by Pikovsky and Kurths who renamed it as *coherence resonance* [4].

Most existing theoretical [6] and experimental [7,8] works on coherence resonance address excitable dynamical systems that typically generate bursting time series. In such a system, there is usually a reference or a ''silent'' state, e.g., a fixed point, near which a trajectory can spend long stretches of time. The trajectory can also leave the reference state and return to its neighborhood in a relatively short time, giving rise to a ''burst.'' The bursts can be due to the inherent dynamics of the system itself, or they can be excited by external perturbations or noise through a threshold mechanism, as the firing behavior of many types of neurons in biological systems. The bursts can occur at either relatively regular or random time intervals, for which the corresponding Fourier spectrum either contains a pronounced peak or has a broadband feature. Coherence resonance thus means that noise can actually be utilized either to improve the sharpness of the existing spectral peak, as in the former case, or to induce a pronounced spectral peak and enhance it, as in the latter case. More recently, the phenomenon was extended to coupled chaotic oscillators exhibiting on-off intermittency [9,11]. In an applied sense, coherence resonance may be a useful mechanism for signal processing [10].

While many nonlinear dynamical systems, nonchaotic or chaotic, can indeed exhibit bursting behaviors, many others do not. A question of interest is then whether coherence resonance can occur in *nonbursting* dynamical systems. Our interest here is in chaotic systems [8]. Suppose there is a chaotic system that generates irregular but nonbursting signals, and suppose in a specific application the temporal regularity of a signal is of interest. Would external noise help improve the temporal regularity of this signal? The purpose of this paper is to provide an affirmative answer to this question by presenting both theoretical and experimental evidence. In particular, we argue theoretically that for a typical nonbursting chaotic system with many possible intrinsic time scales, noise can introduce a new time scale, or the external time scale. When the noise amplitude reaches a proper value, a resonant state can be reached in the sense that the external time scale matches one of the dominant internal time scales, leading possibly to coherence resonance. To verify our theory, we present experimental evidence with a chaotic electronic circuit, the Chua's circuit [12]. The implication of our work is that noise can generally be beneficial, not only for bursting chaotic systems but also for nonbursting ones, so coherence resonance is expected to be ubiquitous in chaotic systems in general.

Our argument for coherence resonance in nonbursting chaotic systems goes as follows. Note that in order for a resonance to occur, it is necessary to have two independent and competing time scales. At least one time scale should depend on noise. To gain insight we consider a chaotic system with a simple rotational structure so that there is a well-defined internal time scale τ_{int}. This time scale is thus deterministic and it does not change with noise. Noise, however, can induce another time scale. This can be seen by realizing that for a chaotic attractor, which is bounded in the phase space, in general there exists a *reference* state, such as that due to the harmonic oscillator embedded in the differential equations in the Rössler system [13]. Noise can cause a trajectory initiated in the reference state to wander away from it. Since the system is bounded, at a later time the trajectory will come back to the reference state. On average, this process defines a time scale, which is the stochastic first-passage time with respect to the reference state. This is an external time scale induced by noise, so it depends on the noise amplitude *D*. We write it as $\tau_{ext}(D)$. As the noise is strengthened, we expect to see a resonance at the optimal noise level D^*, where

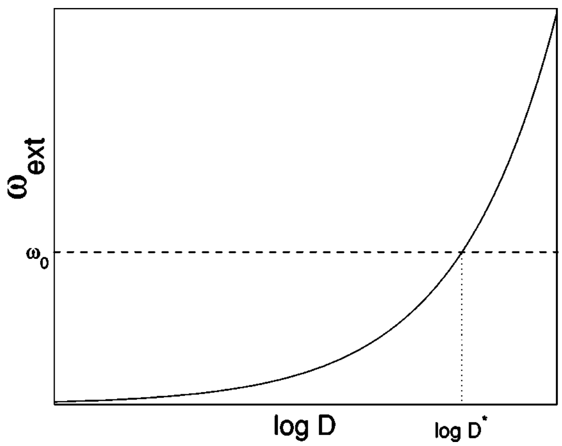

FIG. 1. Mechanism for coherence resonance in nonbursting chaotic systems: the dashed horizontal line denotes the deterministic frequency and the solid curve indicates the general behavior of the first-passage frequency of the underlying stochastic process. Coherence resonance occurs when there is a match between the two independent frequencies at some optimal noise level D^*.

$$\tau_{ext}(D^*) = \tau_{int}. \tag{1}$$

The above heuristic argument can in fact be made more quantitative. In particular, the existence of the external stochastic time scale τ_{ext} and how it varies with noise can be studied by considering the following simple one-dimensional model with a reference state, under the influence of noise:

$$\frac{dx}{dt} = [-\lambda + h(t)]x + D\xi(t), \tag{2}$$

where e^{λ} is the largest eigenvalue of the the reference state $x=0$, $h(t)$ is a zero-mean process (either random or chaotic) that models the finite-time fluctuations in the stability of the reference state, and $D\xi(t)$ is the external noise. The time series $x(t)$ is therefore a realization of a stochastic process $X(t)$, and its probability distribution function $P(x,t)$ obeys the Fokker-Planck equation,

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\left[\left(-\lambda x + \frac{1}{2}\eta x\right)P\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[(\eta x^2 + D)P], \tag{3}$$

where η is the amplitude of $h(t)$. To compute the first passage time, imagine there is an absorbing boundary at $x=a$. The boundedness of the system implies that there must be a reflecting boundary at $x=b$. With these boundary conditions, the Fokker-Planck equation can be solved to yield the following expression for the first-passage time [14]:

$$\langle T_{fp}\rangle = 2\int_{x_0}^{a} dy(\eta y^2 + D)^{\lambda/\eta - 1/2} \times \int_{b}^{y} (\eta z^2 + D)^{-1/2-\lambda/\eta} dz, \tag{4}$$

where x_0 is the initial value of $x(t)$. Figure 1 shows a typical behavior of the first-passage frequency $\omega_{ext}(D) \equiv 1/\langle T_{fp}\rangle(D)$ versus the noise amplitude D, which is obtained utilizing an arbitrary value of b and an arbitrary initial condition v_0. The general feature is that the frequency increases with noise. Since the internal frequency ω_0 is approximately constant, generically the $\omega_{ext}(D)$ curve can intersect ω_0 at some optimal noise amplitude D^*, leading to the time-scale match required for coherence resonance. The optimal noise level D^* depends on the details of the system and cannot be predicted by our simple theory.

For a more general chaotic system, the internal time scale τ_{int} can be regarded as arising from the recurrence of the flow. For instance, one can imagine a Poincaré surface of section and observe the average time interval between successive piercing through the section. A coherence resonance can occur when this deterministic time matches the first-passage time induced by noise.

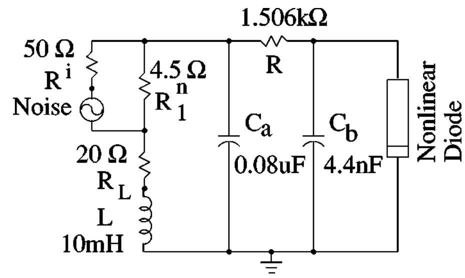

FIG. 2. The Chua's circuit used in our experiment.

Our experimental system is the Chua's chaotic circuit [12], as shown in Fig. 2. The differential equations that describe the circuit in a noiseless situation are $dx/dt = (1/C_b)[(y-x)/R - f(x)]$, $dy/dt = (1/C_a)[(x-y)/R + z]$, and $dz/dt = -(1/L)(y + R^n z)$, where x, y, and z are proportional to the voltages across the capacitors C_b and C_a, and the current through the inductor L, respectively. The nonlinear diode has the following piecewise linear, current-voltage relation: $f(x) = G_b x + (G_a - G_b)(|x+E| - |x-E|)/2$. The circuit is assembled on a high quality printed-circuit board and enclosed in a electromagnetic sheilding box to avoid the influence of external disturbances. The circuit is powered by a low ripple, low noise power supply (HPE3631A, HP). External Gaussian white noise is introduced in the circuit by using a synthesized function generator (DS345, SRS) in which the noise amplitude can be controlled digitally. The circuit operates in the audio-frequency range and the signals are measured using a 12-bit data acquisition board (KPCI3110, Keithley) with sampling frequency at least one order of magnitude higher than the Nyquist rate. The parameters of the circuit are tuned so that it generates a chaotic attractor.

In the recent experimental work in Ref. [8], the Chua's circuit is tuned to generate a double-scroll chaotic attractor and the bistable behavior is extracted for testing coherence resonance. In particular, the dynamical variable $x(t)$, which exhibits chaotic switchings between the two components of the attractor, is digitized to yield a signal $u(t)$ that assumes only two distinct values, say ± 1. This is done by assigning $u(t)=1$ when the trajectory moves on one scroll and $u(t) = -1$ when it moves on another. The effect of noise on the

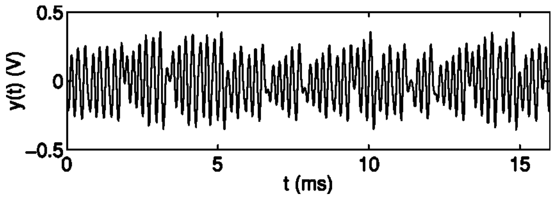

FIG. 3. Chaotic signal $y(t)$ from the Chua's circuit that we use to study coherence resonance.

temporal regularity of the bistable signal $u(t)$ is then investigated and coherence resonance is found. Our aim here is on coherence resonance for more general *continuous-time* chaotic signals, so we focus on a dynamical variable that apparently does not exhibit a bursting behavior. The signal $y(t)$ from the Chua's circuit satisfies this requirement, as shown in Fig. 3.

For the parameter setting in Fig. 2, under the influence of noise, the Fourier spectrum of $y(t)$ exhibits a peak centered at $f_0 \approx 4.5$ kHz, as shown in Figs. 4(a)–4(c) for noise voltage at 1.0 V, 2.5 V, and 8.0 V, respectively. The dominant peak in the power spectrum at the intermediate noise level (2.5 V) is apparently sharper than those at the (relatively) small and large noise levels, indicating a higher degree of temporal regularity at noise levels near $D = 2.5$ V. This is clearly the sign of noise-induced enhancement of a frequency component [2] or coherence resonance [4]. To quantify it, we utilize the following quantity first proposed by Hu *et al.* [3]:

$$\beta \equiv H f_p / \Delta f, \tag{5}$$

where H is the height of the spectral peak, f_p is the location of the peak in the spectrum, and Δf is the half width of the peak. Thus, a sharper and higher spectral peak yields a higher value of β, indicating a higher degree of temporal regularity. Figure 4(d) shows the coherence-resonance measure β, defined with respect to the spectral peak at f_0, versus the noise voltage D. We see that β is small at low noise levels, increases as the noise is increased, reaches a maximum at an optimal noise level, and decreases as the noise is

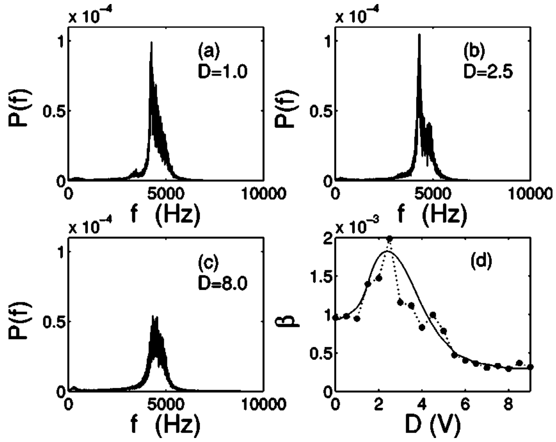

FIG. 4. From the Chua's chaotic circuit in Fig. 2: (a)–(c) Fourier spectra at three different noise voltages; (d) the coherence-resonance measure β_1 versus the noise voltage.

increased further. These are features of coherence resonance, which appear to be quite general in wide parameter regimes of the circuit. Figures 4(a)–4(d) thus represent a direct experimental support that coherence resonance can occur in nonbursting chaotic systems.

In summary, we have presented theoretical arguments and experimental evidence for the existence of coherence resonance in chaotic systems. Our emphasis is on chaotic signals that do not exhibit any excitable feature such as bistability. We show that in a general sense, coherence resonance is the result of the match between two time scales, one deterministic and the other stochastic. While the deterministic time scale can be readily identified for excitable systems, we argue that any chaotic system, excitable or not, naturally possesses such a time scale due to recurrence. Our results suggest that coherence resonance is a very common phenomenon in chaotic dynamical systems.

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