Perturbed on-off intermittency

Daniel Marthaler,[1] Dieter Armbruster,[1] Ying-Cheng Lai,[1,2] and Eric J. Kostelich[1]
[1]*Department of Mathematics, Arizona State University, Tempe, Arizona 85287*
[2]*Department of Electrical Engineering and Center for Systems Science and Engineering Research, Arizona State University, Tempe, Arizona 85287*
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A basic requirement for on-off intermittency to occur is that the system possesses an invariant subspace. We address how on-off intermittency manifests itself when a perturbation destroys the invariant subspace. In particular, we distinguish between situations where the threshold for measuring the on-off intermittency in numerical or physical experiments is much larger than or is comparable to the size of the perturbation. Our principal result is that, as the perturbation parameter increases from zero, a metamorphosis in on-off intermittency occurs in the sense that scaling laws associated with physically measurable quantities change abruptly. A geometric analysis, a random-walk model, and numerical computations support the result.

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I. INTRODUCTION

The phenomenon of on-off intermittency in nonlinear systems has been an area of continuing interest [1–4]. Phenomenologically, a system in on-off intermittency exhibits two distinct states in the course of time evolution. One is the ''off'' state, where the dynamical variables remain approximately constant in various time intervals. The other is the ''on'' state, which corresponds to intermittent ''bursts'' of the dynamical variables away from their approximately constant values in the ''off'' state. The characterization of on-off intermittency and the dynamical mechanisms [5] responsible for it have been investigated actively [1–4], due in part to their relevance to chaos synchronization [6,7].

A basic dynamical requirement for on-off intermittency to occur is that the underlying system possess an invariant subspace $\mathcal{M}$. Consider the situation where there is a chaotic set in the invariant subspace. A typical trajectory on the chaotic set is unstable in $\mathcal{M}$, but in the subspace $\mathcal{T}$ transverse to $\mathcal{M}$, the trajectory can be either stable or unstable. If there is no other attracting set in the phase space, on-off intermittency can occur when the trajectory is slightly unstable in $\mathcal{T}$ [1–4]. The time Θ that a trajectory spends in the ''off'' state, also called the *laminar phase*, has been shown to obey the following algebraic probability distribution at the onset of on-off intermittency [3]: $\Phi(\Theta) \sim \Theta^{-3/2}$. Subsequently, it has been shown [4] that at the onset, the time trace of an on-off intermittent variable is a fractal time series with box-counting dimension $D_0 = 1/2$. We note that the existence of an invariant subspace, which is usually due to a simple symmetry of the system, appears essential for on-off intermittency to occur.

In this paper we investigate how on-off intermittency is affected when there is a perturbation so that the invariant subspace no longer exists. If the unperturbed system possesses a symmetry, then the perturbation is equivalent to a symmetry breaking. We expect such perturbations to be inevitable, say, in laboratory experiments. Symmetry breaking also arises naturally in the context of synchronization between nonindentical chaotic oscillators [8]. When such a perturbation is present, we find in numerical experiments that behavior akin to on-off intermittency still can be observed readily.

There are two important scales in the problem [9]. The first scale is the threshold y_{th} of a variable y transverse to the invariant manifold. The system is said to be in the *off state* when $y < y_{\mathrm{th}}$. The second scale is the perturbation parameter η that characterizes the extent of the symmetry breaking. Loosely speaking, y_{th} represents a numerical accuracy or an experimental measurement scale. There are then two different regimes of dynamical interest: one for which $y_{\mathrm{th}} \approx \eta$ and another one for which $y_{\mathrm{th}} \gg \eta$. These regimes correspond, respectively, to a situation where the symmetry breaking is readily discernible and to a situation where it is not. We find that a qualitative change in the characteristics of on-off intermittency occurs immediately as η is increased from zero in both cases.

Our principal results are the following. (1) There is a natural way to define a laminar phase when $y_{\mathrm{th}} \approx \eta > 0$. (2) There is a crossover in the probability distribution of the laminar phases from algebraic to exponential as η increases from 0 in both cases, no matter how small η may be. (3) If $y_{\mathrm{th}} \gg \eta$ then the fractal dimension of the on-off intermittent time series changes discontinuously from 1/2 to 1 as η increases from zero. (4) The mean length of the laminar phase becomes very short for $y_{\mathrm{th}} \approx \eta$.

An implication of our results is that while on-off intermittency can indeed be observed easily in many practical situations, care should be exercised when interpreting the statistical properties of the on-off intermittent time series. For instance, it may be natural for an experimentalist to report the observation of on-off intermittency, together with an approximate power-law probability distribution of the laminar phase. But if there is a small amount of symmetry breaking, then such a distribution may be better described by an exponential distribution. A measurement of the fractal dimension of the on-off intermittent time series may also be a good indicator of whether there is a symmetry breaking in the system.

The rest of the paper is organized as follows. Section II analyzes the case $y_{\mathrm{th}} \approx \eta$ by geometric arguments. Section III discusses the case where $y_{\mathrm{th}} \gg \eta$ by analyzing a biased ran-

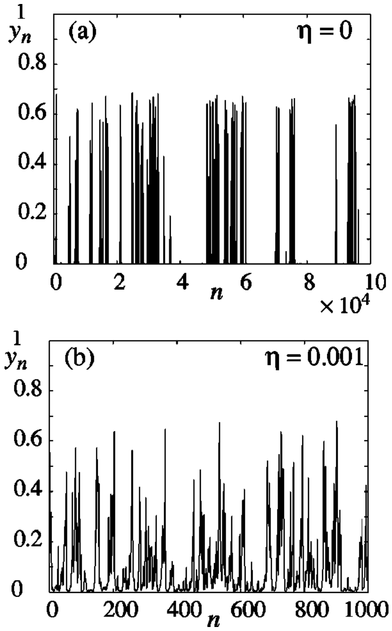

FIG. 1. Intermittent time series for y_n from Eq. (1) with $a=2.8$ for (a) $\eta=0$; (b) $\eta=10^{-3}$.

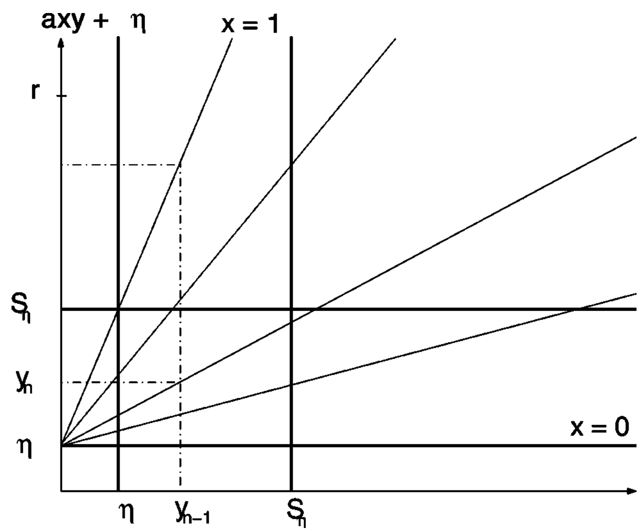

FIG. 2. A schematic illustration of the dynamics of Eq. (2) near the origin. The curves starting at $(0,\eta)$ are representative graphs of Eq. (2) for different choices of *x*.

dom walk model. A summary discussion is presented in Sec. IV.

II. MEASURABLE SYMMETRY BREAKING

We motivate our analysis by studying a class of two-dimensional maps introduced by Heagy *et al.* [3], with modifications to include the effect of a perturbation that destroys the invariant subspace. We confine our attention to maps of the form

$$x_{n+1}=f(x_n), \tag{1}$$

$$y_{n+1}=ax_ny_n(1-y_n)+\eta,$$

where f yields a chaotic process on the unit interval, $x\in[0,1]$ is a driving variable, a is a parameter, and η is the perturbation parameter. In this family of maps, y corresponds to an on-off intermittent variable when $\eta=0$, because there is an invariant subspace, namely the line $y=0$. The onset of on-off intermittency is determined by the condition: $\overline{\ln ax}\equiv\int[\ln(ax)]\rho(x)\,dx=0$, where $\rho(x)$ is the probability density function of the chaotic variable *x*. In the simplest case where ρ is the uniform density (e.g., when *f* is the tent map), on-off intermittency occurs when $a\geqslant a_c=e$. For $a\gtrsim a_c$, the probability distribution of the laminar phases follows the universal algebraic scaling law with exponent $-3/2$, which has been supported by numerical experiments using a variety of chaotic driving dynamics [10].

A. Perturbed ''off'' state

Figure 1 shows two representative time series generated by Eq. (1) for $a=2.8$. Figure 1(a) corresponds to a perfectly symmetric system ($\eta=0$) and Fig. 1(b) corresponds to a system with symmetry breaking ($\eta=10^{-3}$), where *f* produces a uniform distribution for $x\in[0,1]$. The initial conditions are the same for each case. Using visual accuracy as the threshold, Fig. 1(b) represents the case where $y_{\rm th}\approx\eta$. The time series in Fig. 1(a) displays the hallmark of on-off intermittent behavior: long periods of nearly constant signal (laminar phases) interrupted by short-lived, large-amplitude bursts; the plot shows 10^4 iterations. Many laminar phases last for several hundred iterations. The time series in Fig. 1(b) is qualitatively similar, even though the symmetry has been broken, but it appears more noisy and seems to have shorter laminar phases (only 10^3 iterations are shown). Our goal is to determine to what extent the time series in Fig. 1(b) can still be characterized as on-off intermittent.

On-off intermittency, as illustrated in Fig. 1(a), consists of laminar phases that begin when an orbit is reinjected into a small neighborhood of the invariant manifold. The bursts occur when the orbit escapes from the small neighborhood. We will show that there is a natural extension of this idea to the case where a symmetry breaking destroys the invariant manifold. We focus our analysis on the family of maps in Eq. (1).

When $\eta>0$, there is no longer an invariant manifold at $y=0$ for Eq. (1), and so there is no notion of a transverse Lyapunov exponent. We focus on the dynamics of the *y* variable, given by

$$y_{n+1}=ax_ny_n(1-y_n)+\eta. \tag{2}$$

Figure 2 shows a schematic illustration of a region near the origin. The straight lines extending from $(0,\eta)$ are the graphs of Eq. (2) for some representative values of *x*, where the topmost line corresponds to $x=1$, the horizontal line corresponds to $x=0$, and other lines represent other values of $x\in[0,1]$. Note that $y_n\geqslant\eta>0$ for all *n*. Now consider an iterate y_n that is slightly larger than η. Given any $y_{n-1}\in[0,1]$, there exists exactly one *x* in Eq. (2) such that y_{n-1} is the preimage of y_n. If y_n is sufficiently large, then the corresponding point in the phase space has no preimages. For example, $y_n = r$ can never be reached by the map (2) with the given y_{n-1} for $x \in [0,1]$. The largest value $y_n = s_\eta$ for which there exists an x such that every point y_{n-1} in (0,1) can be a preimage of s_η is $s_\eta = a\eta(1-\eta)+\eta$. Let S denote the interval $[\eta, s_\eta]$, let $I=[z, z+\delta z]$ be a subinterval of S, and let $P(z)\delta z$ denote the probability that $z \in I$. Then

$$P(z)\delta z = \int_\eta^1 P(y)P(z|y)dy,$$

where $P(z|y)$ is the transition probability that a given point y in the unit interval maps into I. For a fixed $y \in [\eta, 1]$, this probability is the same as the probability that

$$x_n \in \left[\frac{z-\eta}{ay(1-y)}, \frac{z+\delta z-\eta}{ay(1-y)}\right].$$

Since x is assumed to be uniformly distributed in (0,1), this probability is just the length of the interval: $\delta z/ay(1-y)$. Thus, if $y \in [\eta, 1]$, then the probability that y maps into I is

$$P(y_{n+1}\in I|y_n \in [\eta,1]) = \frac{1}{m(S)}\int_\eta^1 \frac{\delta z}{ay(1-y)}dy, \qquad (3)$$

where $m(S)=a\eta(1-\eta)$ denotes the length of the interval S. This analysis does not depend on any specific choice of z or δz, as long as the resulting interval I is contained in S. Hence, the probability density of the y component of the orbits of Eq. (2) is constant on S.

This result is illustrated by the numerical simulations shown in Fig. 3. Figure 3(a) shows the probability density $p(y)$ of the y components of orbits of Eq. (1) for $a=1.5$ and three different choices of the perturbation parameter η. The unit interval has been divided into 10^5 equal subintervals, and Eq. (1) has been iterated 10^7 times. The plot shows a histogram of the fraction of the y components that is contained in each subinterval. (The inset shows similar data, but the horizontal axis has been magnified to make the interval S more visible.) The probability density of orbits that lie to the right of S appears to drop off exponentially. Figure 3(b) shows analogous plots for $a=2.8$.

Additional analysis illustrates the impact of the symmetry breaking on the probability distributions of the orbits. Suppose that $a<e$. As $\eta\to 0$, the interval S shrinks and moves towards 0. At the same time, the probability that a given iterate y lies in S increases, and the limiting distribution is a δ function at $y=0$. Next suppose that $a \geq e$. Even as $\eta \to 0$, there is a positive probability that a given iterate y lies to the right of S, because the origin is not a sink for Eq. (2) when $a>e$.

Figure 4 shows some numerical computations of the probability that a given point y lies to the right of S as a function of the perturbation parameter η for three choices of a. If $a<e$, then the probability that a given iterate lies to the right of S tends to 0 as $\eta\to 0$, but if $a\geq e$, then the probability that a given iterate lies to the right of S tends to a positive constant as $\eta \to 0$.

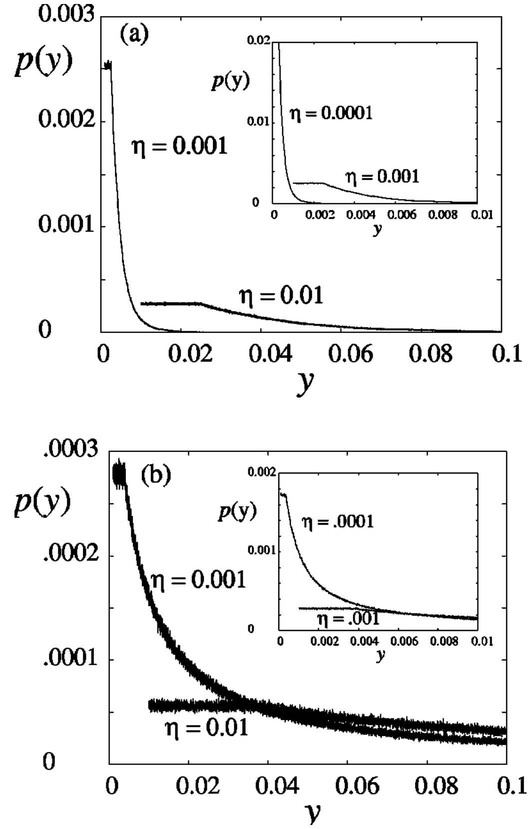

FIG. 3. The numerically computed probability density $p(y)$ of the y component of orbits of Eq. (1) for (a) $a=1.5$ and (b) $a=2.8$. Each inset shows an enlargement of the $\eta=0.001$ curve for $0 \leq y \leq 0.01$. In addition, each inset includes a new curve $p(y)$ for $\eta=0.0001$. All together, for both (a) and (b), three different cases, corresponding to three different values of η, are shown.

This analysis suggests that it is reasonable to regard the dynamics of Eq. (1) as on-off intermittent even in the presence of symmetry breaking. We regard the dynamics as being in an ''off'' state or laminar phase whenever the orbit lies in $S=[\eta, a\eta(1-\eta)+\eta]$. The dynamics ''burst'' or fall into an ''on'' state whenever the orbit leaves the interval S. This definition has the added property that the laminar phase is featureless with respect to the probability measure of the orbit, i.e., the probability density of orbits in every subinterval of S is uniform.

B. Laminar phases

We continue the geometric analysis illustrated in Fig. 2. An iterate y such that $y<s_\eta = a\eta(1-\eta)+\eta$ is in an ''off'' state (laminar phase), and an iterate $y>s_\eta$ is in an ''on'' state (burst). As above, we let $S=[\eta, s_\eta]$.

Suppose that $y_{n-1}\in S$. We define $x_{\text{crit}}(y_{n-1})$ to be the largest value of x such that $y_n \in S$; in general, $x_{\text{crit}}(y)=\eta(1-\eta)/y(1-y)$. Since x is uniformly distributed in [0,1], the probability that x is larger than $x_{\text{crit}}(y)$ is $1-x_{\text{crit}}(y)$. An analysis similar to that leading to Eq. (3) implies that

$$P(y_{n+1}\notin S|y\in S) = 1-x_{\text{crit}}(y)$$
$$= \frac{1}{m(S)}\int_\eta^{s_\eta}\left(1-\frac{\eta(1-\eta)}{y(1-y)}\right)dy,$$

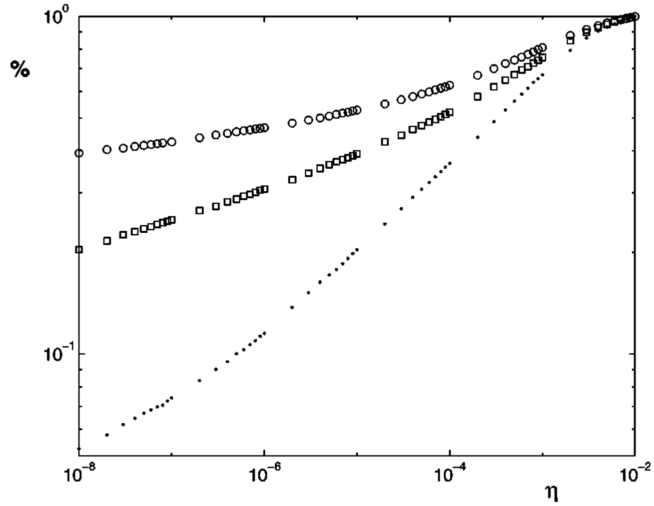

FIG. 4. The fraction of time that orbits of Eq. (1) spend outside the interval S as a function of the perturbation η. The circles give numerically computed results for the case where $a>e$; the squares, $a=e$; and the dots, $a<e$.

where $m(S)=a\eta(1-\eta)$ denotes the length of the interval S. Let P_S denote the probability that $y_{n+1}\in S$ when $y_n\in S$. Then

$$P_S=\frac{1}{m(S)}\int_{\eta}^{s_\eta}\frac{\eta(1-\eta)}{z(1-z)}dz=\frac{1}{a}\ln\left[\frac{a(1-\eta)+1}{1-a\eta}\right]. \quad (4)$$

The probability of having a laminar phase of length n is simply the probability that the first n iterates lie in S and the $(n+1)$st leaves S, i.e., $P_S^n(1-P_S)$. Thus, the lengths of the laminar phases are exponentially distributed. Exponential distributions of laminar phases have also been seen in maps of the form $y_{n+1}=z_n f(y_n)+\delta_n 10^{-\nu}$, where δ_n is a bounded noise process and $\nu>0$ scales the noise amplitude [11].

Numerical simulations confirm that there are laminar phases whose length is of order 10. It is possible to compute the probability distribution of the laminar phases numerically for values of n up to several dozen, but long laminar phases are rare events. As a result, the mean length M of the laminar phases is short,

$$M=(1-P_S)\sum_{n=1}^{\infty} nP_S^n=\frac{P_S}{1-P_S}, \quad (5)$$

where P_S, which depends on η, is given by Eq. (4).

Figure 5 shows a plot (solid line) of Eq. (5) as a function of η when $a=2.75$. The boxes are numerical computations of M for selected values of η, using Eq. (1). Notice that M is roughly constant for $\eta<10^{-2}$. Because the value of M is close to 1 for small values of η, the system remains in the interval S for only two iterations on the average.

III. SMALL SYMMETRY BREAKING

We now consider the case where the threshold y_{th} for measuring on-off intermittency is much greater than η, the symmetry-breaking parameter. We will show, by both numerical evidence and a random walk model, that the following two properties hold when $\eta\neq 0$ and the line $y=0$ is no longer invariant. (1) The distribution of the laminar phases becomes exponential, no matter how small η may be. (2) The fractal dimension of the set of intersecting points of the on-off intermittent time series at y_{th} changes abruptly from 1/2 to 1 as η is increased from zero.

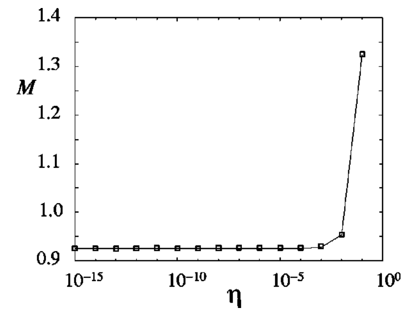

FIG. 5. The mean laminar phase length M of orbits of Eq. (1) when $a=2.75$. The solid line is a graph of the theoretically predicted length, Eq. (5), as a function of η, and the boxes are numerically computed values of M.

A. Numerical evidence

As a numerical experiment, we choose $f(x)$ to be the tent map, set $a=2.75$, and iterate Eq. (1) until we accumulate 10^7 laminar phases [12]. For this purpose, we regard an iterate y_n as being in a laminar phase if $y_n<y_{th}=10^{-2}$. Figure 6(a) shows a plot of the fraction $\Phi(\Theta)$ of laminar phases as a function of their length Θ for two different values of the symmetry breaking parameter η. The plot is semilogarithmic and suggests that the lengths of the laminar phases are exponentially distributed, i.e., $\Phi(\Theta)\sim\exp(b\Theta)$ for some constant b that depends on η. The exponential behavior of $\Phi(\Theta)$ in this case is evident even for small values of Θ. Figure 6(b) shows the contrasting situation where $\eta=0$ and the line $y=0$ is an invariant subspace. Here the distribution $\Phi(\Theta)$ appears to be algebraic, i.e., $\Phi(\Theta)\sim\Theta^{\beta}$ for a constant β; in fact, $\beta=-3/2$ [3].

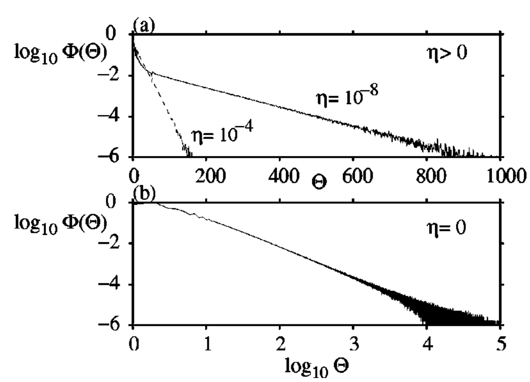

FIG. 6. The distribution $\Phi(\Theta)$ of laminar phase lengths Θ of time series from Eq. (1) with $a=2.75$. (a) The symmetry-breaking case with selected positive values of the perturbation parameter η. (b) The symmetric case, $\eta=0$.

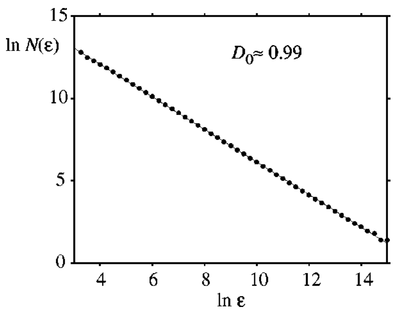

FIG. 7. The filled circles indicate the numerically computed values of the number of intervals $N(\epsilon)$ of length ϵ needed to cover the set of points formed by the intersection of the graph of the time series of y values generated by Eq. (1) with the line $y=y_{\rm th}$.

Our confidence in the exponential distribution of the laminar phase lengths in the symmetry-breaking case is bolstered by a computation of the fractal dimension of the on-off intermittent time series. The fractal dimension is computed as follows. Set a threshold $y_{\rm th}>\eta$ and consider the set of time intervals whose endpoints are determined by the intersection of the graph of y_n with the horizontal line $y=y_{\rm th}$. Let $N(\epsilon)$ be the number of intervals of length ϵ required to cover the set of intersecting points of the on-off intermittent time series with $y_{\rm th}$. In the range of time intervals Θ for which the exponential distribution $\Phi(\Theta)$ is valid, we expect that $N(\epsilon)\sim\epsilon^{-D_0}$, where D_0 is the fractal dimension of the on-off intermittent time series.

Venkataramani *et al.* [4] have argued that an algebraic distribution of laminar phase lengths Θ is equivalent to a fractal dimension $D_0=1/2$ of the on-off intermittent time series. However, if the distribution of the laminar phase lengths Θ is exponential, then, as we show below, the fractal dimension of the on-off intermittent time series is $D_0=1$. Figure 7 shows a plot of numerically computed values of $\ln N(\epsilon)$ versus $\ln\epsilon$ for a time series of Eq. (1) with $a=2.75$ and $\eta=10^{-4}$. The regression function is indicated by the solid line, which suggests that D_0 is close to 1.

B. Random walk model

Qualitatively, the crossover from an algebraic to an exponential distribution of the laminar phase lengths as η increases from 0 can be understood as follows. Rewrite Eq. (1) as

$$y_{n+1}=ax_ny_n[(1-y_n)+\eta/ax_ny_n].$$

Consider the case where $1\gg\eta>0$ and y_n is in the ''off'' state. Then $y_n\approx\eta$, so $y_{n+1}\approx ax_ny_n(1+\delta_n)$, where $\delta_n=\eta/ax_ny_n$. The probability that x_n is close to 0 is small. Therefore, most of the time, δ_n is on the order of 1. Letting

$$Y_n=-\ln y_n, \tag{6}$$

we obtain

$$Y_{n+1}=Y_n+v_n, \tag{7}$$

where $v_n=-\ln(ax_n)-\ln(1+\delta_n)$ is a random variable. In this way, we obtain a random walk in Y_n. If $\eta=0$ and $a\gtrsim a_c$, then $\overline{\ln ax_n}\approx 0$ and $\delta_n=0$ so that $\overline{v_n}=0$, which implies that the random walk is *unbiased*. But when $\eta\neq 0$, then depending on the value of y_n, the term δ_n can be either large or small. In particular, if y_n is close to η (in the ''off'' state), then δ_n is on the order of unity. As y_n moves away from the ''off'' state, the value of δ_n becomes negligible. Because there is a high probability that the trajectory remains in the ''off'' state, we have $\overline{v_n}<0$. Thus, the random walk in Y_n is *biased* when $\eta\neq 0$. Therefore, we expect a sudden, qualitative change in the characteristics of on-off intermittency immediately after η is increased from zero.

We note that, even in the case of $\eta=0$ (no symmetry breaking), the random walk so obtained is also biased [3] when the parameter a is increased from a_c, the blowout bifurcation point, because $\overline{\ln ax_n}\gtrsim 0$ for $a\gtrsim a_c$. In this case, one also expects an exponential behavior in the distribution of the laminar phase lengths Θ for sufficiently large values of Θ. In numerically or physically reasonable time scales, one still observes an algebraic distribution [3]. When the symmetry is broken, however, the switch from an algebraic to an exponential distribution of laminar phase lengths is metamorphic, in the sense that the onset of the exponential behavior is almost immediate at small time scales even when η is many orders of magnitude smaller than typical scales of the measurable physical quantities of the system.

We are thus led to the analysis of the biased random walk model (7), where we assume that v_n is a random variable with a probability distribution $F(v)$ [13]. The drift of v_n is given by

$$\bar{v}=\int vF(v)dv.$$

The Fokker-Planck equation for $P(Y,t)$, the probability distribution of finding the walker at location Y at time t, is given by [14]

$$\frac{\partial P}{\partial t}+\bar{v}\frac{\partial P}{\partial Y}=D\frac{\partial^2 P}{\partial Y^2}, \tag{8}$$

where $D=\int(v-\bar{v})^2F(v)dv$ is the diffusion coefficient. The range of the variable Y is $(0,\infty)$. In the phase space y, the drift is towards $y=1$ if $\eta\gtrsim 0$. In the walker's space, the drift is towards $Y=0$, i.e., $\bar{v}<0$. We write $\bar{v}=-h$, where $h>0$.

We analyze Eq. (8) in a manner that mimics what is typically done in numerical experiments for computing the laminar phases. We set a threshold, say at $y=1$ $(Y=0)$, start from the steady-state distribution (which is equivalent to eliminating transients), and examine the probability that the value of y exceeds the threshold at some time t. Thus, we have an absorbing boundary at $Y=0$,

$$P(0,t)=0. \tag{9}$$

For $Y>0$, the initial condition is

$$P(Y,0)=P_s(Y)=\left(\frac{h}{D}\right)\exp\left(-\frac{hY}{D}\right), \tag{10}$$

where $P_s(Y)$ is the steady-state solution of the Fokker-Planck equation. Since y can never reach zero, we have another boundary condition

$$P(\infty,t)=0. \tag{11}$$

Let $\mathcal{P}(Y,s)$ denote the Laplace transform of $P(Y,t)$, defined by

$$\mathcal{P}(Y,s)=\int_0^\infty P(Y,t)e^{-st}dt, \tag{12}$$

where $\Re s>0$. The initial condition in Eq. (10) yields the following equation for $\mathcal{P}(Y,s)$:

$$D\frac{d^2\mathcal{P}}{dY^2}+h\frac{d\mathcal{P}}{dY}-s\mathcal{P}=-P_s(Y). \tag{13}$$

The boundary conditions in Eqs. (9) and (11) lead to the solution of Eq. (13),

$$\mathcal{P}(Y,s)=\frac{h}{Ds}\left\{\exp\left(-\frac{hY}{D}\right)-\exp[\lambda(s)Y]\right\}, \tag{14}$$

where $\lambda(s)=(-h-\sqrt{h^2+4Ds})/(2D)$, which satisfies the relation $\mathrm{Re}\lambda<0$.

Let

$$W(t)=1-\int_0^\infty P(Y,t)dY \tag{15}$$

be the probability that Y is absorbed at $Y=0$ (which is the probability that y bursts out of $y=1$). The Laplace transform $\mathcal{W}(s)$ of $W(t)$ is

$$\mathcal{W}(s)=\frac{1}{s}-\int_0^\infty \mathcal{P}(Y,s)dY=\frac{2}{s(1+\sqrt{1+s\tau})}, \tag{16}$$

where $\tau\equiv 4D/h^2$ defines the characteristic time scale of the random walk model.

We now consider the inverse Laplace transform of $\mathcal{W}(s)$ [15], given by

$$W(t)=\frac{1}{2\pi i}\int_{-i\infty+\sigma}^{i\infty+\sigma}\mathcal{W}(s)e^{st}ds. \tag{17}$$

There is a pole at $s=0$ and a branch singularity at $s^*=-1/\tau$. The pole corresponds to $t\to\infty$ in the time domain and thus makes no contribution to $W(t)$ in finite time. We choose a branch cut from $s=-\infty$ to $s=s^*<0$ as shown in Fig. 8. The integration over the infinitesimal circular path about s^* vanishes. The contribution to $W(t)$ comes from the two integrals, one above the branch cut and another below, along the negative real axis. The integration yields

$$W(t)=g(t)e^{-t/\tau}, \tag{18}$$

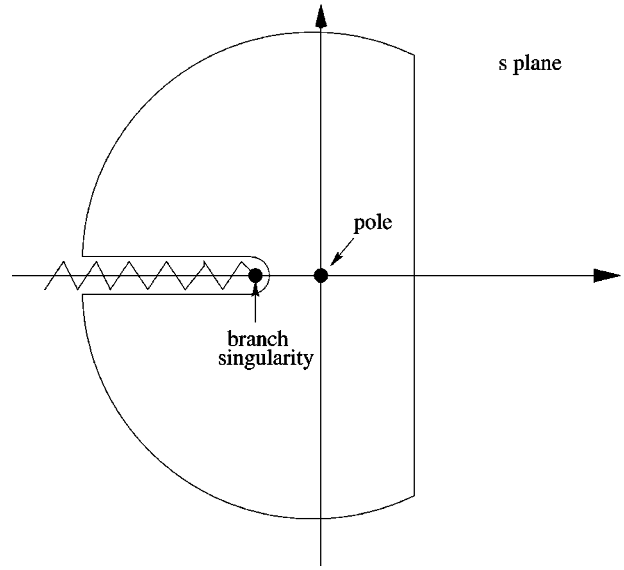

FIG. 8. The integration path and branch cut for obtaining $W(t)$, Eq. (17).

where

$$g(t)=\frac{2}{\pi\sqrt{\tau}}\int_0^\infty \frac{\sqrt{x}\,e^{-tx}}{(x-s^*)^2}dx, \tag{19}$$

which can be evaluated by using the standard saddle-point method [16]. We obtain $g(t)\sim t^{-3/2}$ and consequently, the following scaling relation for $W(t)$:

$$W(t)\sim t^{-3/2}e^{-t/\tau}. \tag{20}$$

We can now compute the fractal dimension of the intermittent time series and the laminar-phase distribution based on Eq. (20). Let $y(t)$ be the time series and choose a uniform set of time intervals of length ϵ to cover the set of points at which bursts occur. The fractal dimension D_0 is given by $N(\epsilon)\sim\epsilon^{-D_0}$, where $N(\epsilon)$ is the number of ϵ intervals that are required to cover the set. Now $W(\epsilon)$ is the probability that a randomly chosen interval of length ϵ is a part of the cover. Suppose that the time series has length T. The total number of ϵ intervals contained in $[0,T]$ is $T\epsilon^{-1}$. Thus,

$$N(\epsilon)=(T\epsilon^{-1})W(\epsilon)\sim \epsilon^{-5/2}e^{-\epsilon/\tau}, \tag{21}$$

which yields $D_0=1$ [17].

The distribution of laminar phase lengths can be computed as follows. Let $R(\epsilon)$ be the probability of having a laminar phase whose length is at least ϵ. Let $r(\epsilon)$ be the conditional probability that there is a laminar phase of length between ϵ and 2ϵ given that there is a laminar phase of length ϵ. Then

$$r(\epsilon)=[R(\epsilon)-R(2\epsilon)]/R(\epsilon)=1-R(2\epsilon)/R(\epsilon).$$

Note that $r(\epsilon)$ is the conditional probability that there is a burst in the time interval $[(n_0+1)\epsilon,(n_0+2)\epsilon]$ given that there is a burst in the time interval $[n_0\epsilon,(n_0+1)\epsilon]$, where $n_0\epsilon$ is an arbitrary reference initial time. It follows that

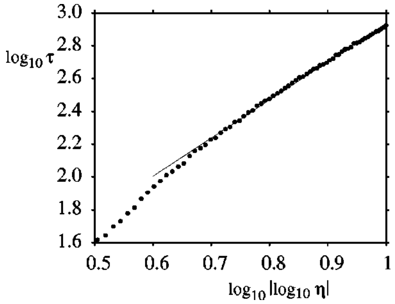

FIG. 9. Average length τ of the laminar phases of iterates of Eq. (1) as a function of the perturbation parameter η.

$$N(2\epsilon)=N(\epsilon)-N(\epsilon)r(\epsilon)=N(\epsilon)[1-r(\epsilon)].$$

Equation (21) implies that $r(\epsilon)=1-r_0e^{-\epsilon/\tau}$ (r_0 is a constant). Hence

$$R(\epsilon)\sim\epsilon^q e^{-\epsilon/\tau},$$

where $q=-5/2$. Now $R(\epsilon)$ is related to the probability distribution $\Phi(\Theta)$ of the laminar phase by

$$R(\epsilon)=\int_\epsilon^\infty\Phi(\Theta)d\Theta. \tag{22}$$

Therefore, the probability distribution of the laminar phases is exponential,

$$\Phi(\Theta)\sim\Theta^q e^{-\Theta/\tau}\sim e^{-\Theta/\tau}\quad\text{for}\quad\Theta\gg1. \tag{23}$$

One prediction of the biased random walk model is that the average length of the laminar phase can be related to the drift $-h$ and the diffusion coefficient D as $\tau=4D/h^2$. Heuristically, we expect the dependence of the drift and the diffusion coefficient on the perturbation parameter η to be logarithmic for the following two reasons. First, the random-walk picture is valid in the logarithmic space of the intermittent variable in the phase space, and second, as we have argued, the term that causes a bias in the random walk is almost independent of η. Thus, we expect the following scaling relation for the average length of the laminar phase:

$$\tau(\eta)\sim|\ln\eta|^\gamma, \tag{24}$$

where $\gamma>0$ is the scaling exponent. This scaling relation implies that the average length of the laminar phases increases only logarithmically as the amount of perturbation is decreased and, as a practical matter, many orders of decrease in η yield only an incremental increase in the average length of the laminar phases. For example, in the simple model Eq. (1), we find that, when η is decreased from 10^{-3} to 10^{-10}, the value of $\tau(\eta)$ increases only by a factor of 10. Figure 9 shows a plot of $\log_{10}\tau(\eta)$ versus $\log_{10}|\log_{10}\eta|$ for $a=2.75$. The approximate linear behavior for small values of η supports the scaling law in Eq. (24). We stress, however, the scaling law (24) is only meant to be speculative [20].

The predictions from the biased random walk model do not depend on any specific details of the underlying on-off intermittent system. Therefore, we expect these predictions to be general. In fact, for Eq. (1), we are able to obtain *rigorous* expressions for the probability distribution and the average length of the laminar phases that agree with those from the random walk model.

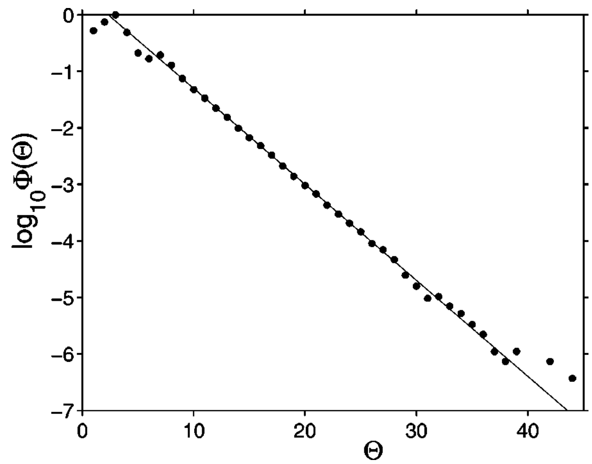

FIG. 10. Probability distribution $\Phi(\Theta)$ of the laminar phase lengths Θ of iterates of the model (25).

We have performed numerical tests on various models of on-off intermittency with perturbations and have found a robust exponential behavior in the distribution of the laminar phase lengths. Figure 10 shows, on a semilogarithmic scale, the probability distribution of the laminar phases obtained at $y_{\rm th}=10^{-2}$ for the model

$$x_{n+1}=T(x_n), \tag{25}$$

$$y_{n+1}=ax_ny_n(1-y_n)+\epsilon+\epsilon_0\cos(2\pi y_n),$$

where $T(x)$ is the tent map and $a=2.75$, $\epsilon=10^{-3}$, and $\epsilon_0=10^{-3}$. The exponential nature of the probability distribution of the laminar phase lengths is apparent.

IV. DISCUSSION

Perturbations of an on-off intermittent system that destroy the invariant manifold have a distinct impact on the properties of the dynamics. Most importantly, there is a metamorphosis of the scaling law for the probability distribution of the lengths of the laminar phases that is exponential, in contrast to the algebraic distribution in a system with an invariant manifold.

Moreover, we have shown that the relative amplitude of the symmetry breaking and the threshold amplitude at which one defines the ''off'' state are important. When the threshold amplitude is comparable to the size of the perturbation, then we have the following.

(i) It is possible to characterize an ''off'' state in a manner that yields an exponential probability distribution for the lengths of the laminar phases. In addition, our definition of the ''off'' state generates a probability measure for the iterates of the dynamical system whose limit (as the perturbation tends to 0) is the natural measure of the iterates of the original on-off intermittent system.

(ii) The lengths of the observed laminar phases are short on the average.

When the threshold amplitude is orders of magnitude greater than the size of the perturbation, then we have the following.

(iii) The distribution of the laminar phase lengths is exponential, in contrast to the algebraic distribution when there is an invariant manifold.

(iv) The fractal dimension of the level sets of the intermittent time series changes discontinuously from $D_0 = 1/2$ in the unperturbed case to $D_0 = 1$ in the perturbed case.

Our characterization of the probability distributions of the laminar phase lengths has an interesting practical application. Suppose that in a laboratory experiment, limited measurement accuracy leads to uncertainty as to whether a symmetry breaking has occurred. If it is possible to run the experiment long enough to collect reasonable statistics on the distribution of the laminar phase lengths, then one can check whether the distribution is exponential. An exponential distribution implies that the symmetry has been broken.

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[3] J.F. Heagy, N. Platt, and S.M. Hammel, Phys. Rev. E **49**, 1140 (1994).

[4] S.C. Venkataramani, T.M. Antonsen, Jr., E. Ott, and J.C. Sommerer, Physica D **96**, 66 (1996).

[5] The mechanism for on-off intermittency to occur in chaotic systems can be understood as follows. Since the chaotic set in $\mathcal{M}$ is only weakly unstable in the transverse subspace, all invariant sets embedded in the chaotic set, such as unstable periodic orbits, can be classified into two subsets: one transversely stable and another transversely unstable, the latter ‘‘weighs’’ slightly more than the former [Y. Nagai and Y.-C. Lai, Phys. Rev. E **56**, 4031 (1997)]. A trajectory can be attracted towards $\mathcal{M}$ and stays in the vicinity of $\mathcal{M}$ near some of the transversely stable invariant sets. This corresponds to the ‘‘off’’ state. Due to the existence of the transversely unstable invariant subset, the off state cannot be sustained indefinitely. In particular, when the trajectory comes close to some of the transversely unstable sets so that, in the time interval starting from the beginning of the off state, the trajectory becomes transversely unstable; it can then leave the off state, leading to a burst that corresponds to the ‘‘on’’ state.

[6] L.M. Pecora and T.L. Carroll, Phys. Rev. Lett. **64**, 821 (1990); For a recent review, see Chaos Focus Issue **7**, 4 (1997).

[7] J.F. Heagy, T.L. Carroll, and L.M. Pecora, Phys. Rev. Lett. **73**, 3528 (1994); J.F. Heagy, L.M. Pecora, and T.L. Carroll, *ibid.* **74**, 4185 (1995); M. Ding and W. Yang, Phys. Rev. E **54**, 2489 (1996); L.M. Pecora and T.L. Carroll, Phys. Rev. Lett. **80**, 2109 (1998).

[8] Say we consider the following model of synchronization between nonidentical chaotic oscillators:

$$d\mathbf{x}/dt = \mathbf{f}_1(\mathbf{x}) + \mathbf{C}\cdot(\mathbf{y}-\mathbf{x}),$$

$$d\mathbf{y}/dt = \mathbf{f}_2(\mathbf{x}) + \mathbf{C}\cdot(\mathbf{x}-\mathbf{y}),$$

where $\mathbf{f}_1 \approx \mathbf{f}_2$ are velocity fields that both generate chaotic flows, and $\mathbf{C}$ is the coupling matrix. We introduce two new variables, $\mathbf{u} = (\mathbf{x}+\mathbf{y})/2$ and $\mathbf{v} = (\mathbf{x}-\mathbf{y})/2$, so that a perfect synchronization state is defined by $\mathbf{v} = \mathbf{0}$. The differential equations for $\mathbf{u}$ and $\mathbf{v}$ are given by

$$d\mathbf{u}/dt \approx [\mathbf{f}_1(\mathbf{u}) + \mathbf{f}_2(\mathbf{u})]/2,$$

$$d\mathbf{v}/dt \approx [\mathbf{G}(\mathbf{u}) - 2\mathbf{C}]\cdot\mathbf{v} + [\mathbf{f}_1(\mathbf{u}) - \mathbf{f}_2(\mathbf{u})]/2,$$

where $\mathbf{G}(\mathbf{u})$ is the average Jacobian matrix of the velocity fields $\mathbf{f}_1$ and $\mathbf{f}_2$. If $\mathbf{f}_1 = \mathbf{f}_2$, $\mathbf{v} = \mathbf{0}$ is the invariant subspace and on-off intermittency can occur in $\mathbf{v}$ if the largest Lyapunov exponent evaluated from the product of matrices $[\mathbf{G}(\mathbf{u}) - 2\mathbf{C}]$ is slightly positive. However, the invariant subspace is destroyed when there is a systematic difference between $\mathbf{f}_1$ and $\mathbf{f}_2$.

[9] When the symmetry-breaking perturbation term is additive to the dynamical equations, as we demonstrate in this paper, two regimes of dynamical interest can be defined, depending on the size of the perturbation relative to that of a typical dynamical variable exhibiting on-off intermittency. If the perturbation is not additive, then generally it may be difficult to distinguish the corresponding dynamical regimes. However, in a system of coupled chaotic oscillators, which is perhaps the most commonly utilized class of physical systems in the study of on-off intermittency, symmetry-breaking perturbations are simply proportional to the mismatch between the parameters of the coupled oscillators. In this case, it is then straightforward to define the two distinct dynamical regimes introduced in this paper, as one can compare the size of the parameter mismatch to that of an on-off intermittent dynamical variable.

[10] When $\eta = 0$, as a is increased from a_c, the laminar-phase distribution begins to develop an exponential tail, but the onset of the exponential behavior typically occurs at large time so that in short-time scales, the dominant behavior is still algebraic. For instance, it was estimated that for $a = 2.75$, the onset of the exponential distribution occurs for time $n \geqslant 2200$, which was not observed in numerical experiments [3].

[11] N. Platt, S.M. Hammel, and J.F. Heagy, Phys. Rev. Lett. **72**, 3498 (1994).

[12] It is known that a numerical trajectory from a chaotic map will eventually become periodic due to the round-off error. Typically, the number of iterations T_N for the onset of artificial periodicity scales with the round-off error δ as $T_N \sim \delta^{-D_2/2}$, where D_2 is the correlation dimension of the chaotic set [C. Grebogi, E. Ott, and J.A. Yorke, Phys. Rev. A **38**, 3688 (1988)]. In our numerical model, $D_2=1$ and we use double precision arithmetic ($\delta \sim 10^{-16}$), so we have $T_N \sim 10^8$. To accumulate 10^7 laminar phases, we utilize 10^4 numerical trajectories from randomly chosen initial conditions, each of length 10^7 and, hence, the computer round-off will have no effect on the chaoticity of the trajectories.

[13] If v comes from a chaotic process, the corresponding probability density function cannot be written down because typically it contains an infinite number of singularities [18]. However, we still expect to be able to define statistical averages such as drift.

[14] If an average drift $\overline{\nu}$ can be defined, the Fokker-Planck equation is a reasonable model that describes the evolution of the probability distribution insofar as the random walk occurs in many time steps so that the time can be treated approximately as a continuous variable, regardless of the details of the probability distribution of ν_n [19].

[15] In the case treated by Venkataramani *et al.* [4], the random walk is approximately unbiased so that $\tau \gg 1$. In our case, this approximation is no longer valid because the walk is significantly biased.

[16] J. Mathews and R. L. Walker, *Mathematical Methods of Physics* (Addison-Wesley, Reading, MA, 1970).

[17] Our analysis is based on a random-walk model, solved by using the diffusion equation under a series of crude approximations. It is thus not surprising that the model does not yield the correct value of the fractal dimension *in a direct way*. Instead, it yields a dimension value that is larger than 1, which, *indirectly*, implies $D_0=1$, as the fractal set is embedded in a one-dimensional line segment.

[18] E. Ott, *Chaos in Dynamical Systems* (Cambridge University Press, Cambridge, 1993).

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[20] In addition to the fit suggested by the scaling relation (24), we have tried both power-law and exponential scaling to fit the numerical data in Fig. 9. The scaling (24) apparently gives the best fit.