

Phase clustering and transition to phase synchronization in a large number of coupled nonlinear oscillators

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The transition to phase synchronization in systems consisting of a large number (N) of coupled nonlinear oscillators via the route of phase clustering (phase synchronization among subsets of oscillators) is investigated. We elucidate the mechanism for the merger of phase clusters and find an algebraic scaling between the critical coupling parameter required for phase synchronization and N . Our result implies that, in realistic situations, phase clustering may be more prevalent than full phase synchronization.

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Recently, the phenomenon of phase synchronization [1] in weakly coupled nonidentical chaotic oscillators has received a great deal of attention [2–11]. Consider the situation where each individual oscillator exhibits a chaotic attractor in the phase space. Due to the recurrence of chaotic trajectories, the motion resembles that of a complicated rotation and, as such, a proper angle of rotation, or phase, can be defined [1,12]. When two such chaotic oscillators are coupled, their phases, denoted by $\theta_1(t)$ and $\theta_2(t)$, tend to follow each other in the sense that the phase difference remains bounded even when the coupling is weak, in contrast to the uncoupled case where the phase difference increases approximately linearly with time [3]. The amplitudes of the chaotic rotations, however, remain uncorrelated despite coherence in their phases. Chaotic phase synchronization appears to be a general phenomenon in systems of coupled nonlinear oscillators, and it has been observed in laboratory experiments [13], in biomedical systems [4,8], and in population data in ecology [9]. The phenomenon is also closely related to phase-locked loops that are highly relevant to engineering applications [14] and to neuroscience [15].

Most existing work on phase synchronization focuses on systems consisting of two or a few coupled chaotic oscillators. Phase synchronization of a large number of *globally* coupled oscillators is studied in Ref. [2]. More recently, the phenomenon of phase clustering has been discovered in a system of a large number of locally coupled phase oscillators [7]. In particular, in Ref. [7], it is found that phase synchronization of N ($N \gg 2$) coupled, nonidentical oscillators is often preceded by the presence of *partial* phase synchronization among various subsets of oscillators (clustering) as the coupling parameter, say K , is increased from zero. When the mean frequencies of the oscillators are plotted versus K , a treelike structure appears in the sense that full phase synchronization has only one value of the average frequency for all oscillators, phase clustering corresponds to several values, and the merger of two values of average frequency marks the disappearance of one phase cluster [7]. In this Rapid Communication, we address the following questions: (1) How do phase clusters merge? (2) How much coupling, say K_c^0 , is

required for a system of N coupled chaotic oscillators to synchronize in phase? The latter question is motivated by the consideration that phase synchronization of a few coupled oscillators usually occurs in a weak coupling regime. Our principal results are as follows. (1) The merger of phase clusters is typically preceded by a plus-minus bursting behavior in the instantaneous frequency (angular velocity) of each cluster. In particular, at the burst, there is a sudden increase of the angular velocity of one cluster, accompanied by a sudden decrease of the angular velocity of another cluster, leading to a 2π phase jump in the phase difference between the two clusters. As the coupling parameter is increased, the bursts become rare. The disappearance of bursts marks the merger of the two phase clusters. (2) The scaling of the critical coupling K_c^0 with N is algebraic,

$$K_c^0 \sim N^\alpha, \quad (1)$$

where $\alpha > 0$ is the scaling exponent. One implication of scaling relation (1) is that full phase synchronization of a large number of oscillators typically requires an enormous amount of coupling. Thus, for example, for a network of coupled neurons, what can typically be expected is phase clustering instead of a full phase synchronization.

The phenomenon of phase clustering can be illustrated by using the system of N locally coupled chaotic Rössler [16] oscillators, written in the cylindrical coordinate (r, θ, z) , as follows: $dr_i/dt = 0.15r_i \sin^2 \theta_i + \cos \theta_i [K(r_{i+1} \cos \theta_{i+1} + r_{i-1} \cos \theta_{i-1} - 2r_i \cos \theta_i) - z_i]$, $d\theta_i/dt = \omega_i + 0.15 \sin \theta_i \cos \theta_i - \sin \theta_i [K(r_{i+1} \cos \theta_{i+1} + r_{i-1} \cos \theta_{i-1} - 2r_i \cos \theta_i) - z_i]/r_i$ and $dz_i/dt = 0.2 + z_i(r_i \cos \theta_i - 10.0)$, where $i = 1, \dots, N$, and the coupling at the i th oscillator is modeled by a term $K(x_{i+1} - 2x_i + x_{i-1})$ in the Cartesian coordinate. The Rössler chaotic attractor naturally possesses a well defined rotational structure [1] and its phase variable is θ . The natural frequency of rotation of each individual oscillator, when uncoupled, is ω_i . Figure 1(a) shows the average frequencies versus the coupling parameter K for the coupled Rössler system with $N = 15$, where ω_i 's are chosen randomly from the interval: $\omega \in [0.9, 1.1]$. We see that, for $K = 0$, there are 15

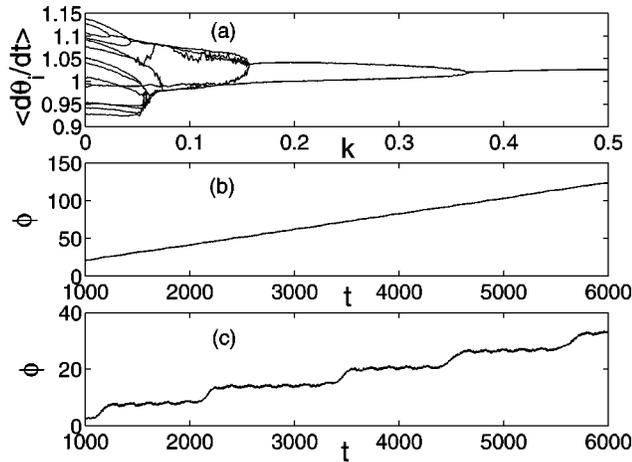


FIG. 1. For a system of 15 coupled Rössler chaotic oscillators: (a) the average frequencies vs the coupling parameter K , (b) linear growth of the phase difference between two clusters for $K=0.005 < K_m=0.024$, and (c) 2π phase jumps preceding the merger of the clusters for $K=0.022$.

distinct values of the average frequency, indicating a lack of phase coherence among oscillators in the absence of coupling. As K is increased, the number of distinct average frequencies decreases as subsets of oscillators begin to synchronize in phase, forming clusters. A treelike structure can be seen from Fig. 1(a), where a branching point represents a merger of distinct phase clusters. The disappearance of phase clusters can also be understood from the behavior of the Lyapunov exponents that are zero for $K=0$. When there is no coupling, there are N zero Lyapunov exponents corresponding to the eigendirections along the flow in each individual oscillator and N positive Lyapunov exponents. Partial phase coherence among oscillators is established when some of these exponents become negative or cross zero [1]. Thus, corresponding to each branching point in Fig. 1(a), one of the originally zero exponents becomes negative, as our numerical simulation has confirmed.

It is known that for a system of two coupled chaotic oscillators, phase synchronization is preceded by the occurrence of phase slips (jumps) in units of 2π [1,5]. Such phase slips also occur preceding the merger of two distinct phase clusters. In particular, let K_m be the value of the coupling parameter for the merger, let $\phi(t)$ be the phase difference between the two clusters. Then for $K < K_m$, $\phi(t)$ increases approximately linearly with time, as shown in Fig. 1(b). For $K \lesssim K_m$, ϕ tends to remain constant except at the moments when it suddenly jumps by 2π , as shown in Fig. 1(c). As K gets closer to K_m , the average time interval between successive phase jumps increases in a way [5,10] that can be described as chaotic transients [17].

To make a physical analysis of the phenomenon of phase clustering feasible, we wish to focus on the phase variables. While in general, the differential equations for the phase and amplitude variables are coupled in a nonlinear fashion, the time scales of these variables are different. In particular, for a typical chaotic rotation, such as one produced by the Rössler chaotic oscillator, the amplitude variables are

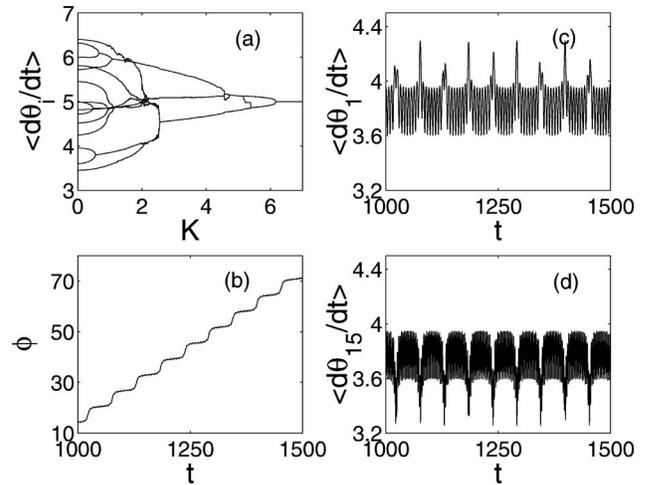


FIG. 2. For the Kuramoto paradigm Eq. (2) with $N=15$: (a) average frequencies of the oscillators vs the coupling parameter, (b) 2π phase jumps preceding the merger of the clusters for $K=0.52$, and (c,d) pairing of the bursts in the instantaneous frequencies of the two clusters preceding their merger.

“slow” and the phase variables are “fast” [1]. Thus, in a time scale where the phase variable changes appreciably, say from 0 to 2π , the amplitude variables can be roughly treated as constants [18]. Under this approximation, the phase variables of a system of coupled chaotic oscillators obey a canonical equation of the following form: $d\theta_i/dt \approx \omega_i + G_1(\theta_i) + KG_2(\theta_{i-1}, \theta_i, \theta_{i+1})$, where G_1 and G_2 are nonlinear functions, and the dependences on the phase variables in G_1 and G_2 occur through the sine or cosine functions. A well known form of the canonical equation is the Kuramoto model [19].

We are thus motivated to study the following version of the Kuramoto model [7]:

$$\dot{\theta}_i = \omega_i + \frac{K}{3} [\sin(\theta_{i+1} - \theta_i) + \sin(\theta_{i-1} - \theta_i)], \quad i = 1, \dots, N, \quad (2)$$

where the coupling is the nearest-neighboring type and ω_i 's ($i = 1, \dots, N$) are the natural frequencies of the individual oscillators. The Kuramoto model [19] describes the dynamics of a population of periodic oscillators globally coupled via a mean field and it arises in models in neuroscience [15]. In our numerical experiments, ω_i 's are chosen randomly from a probability distribution (Gaussian or uniform) centered at ω_0 with variance σ . Figures 2(a) and 2(b) show, for $N=15$, $\omega_0=5.0$, and $\sigma=1.0$, the distinct average frequencies of oscillators versus K and the 2π jumps in the time evolution of the phase difference from a pair of clusters about to be synchronized, respectively. Physically, a sudden change of 2π in ϕ means that one cluster of oscillators rotates faster, which is accompanied by a simultaneous slowing down in the rotation of the other cluster. As a result, we expect the instantaneous frequency of one cluster to increase suddenly, and that of the other cluster to decrease suddenly at the same time, which can also be seen by noting, from Eq.

(2), that $\sum_{i=1}^N \dot{\theta}_i = \sum_{i=1}^N \omega_i = \text{const}$. For the pairing cluster that experiences a sudden decrease in the instantaneous frequency, an increase in the frequency must follow immediately so that its constant average frequency can be maintained. As a result, a pair of bursts is created: at each 2π jump, one cluster of oscillators experiences a positive burst in its instantaneous frequency, accompanied by a simultaneous negative burst in that of its pairing cluster, as shown in Figs. 2(c) and 2(d). We mention that the behavior of the bursting pairs preceding phase synchronization has not been noted previously, and as we will argue later, the behavior can be better understood from a physical picture. From Fig. 2(a), we observe a treelike structure similar to that in Fig. 1(a). In what follows we will concentrate on the Kuramoto model Eq. (2) for the derivation and numerical support of the scaling law Eq. (1). In addition, we will discuss a potential model that is capable of yielding a physical picture for the phenomena of phase clustering and synchronization.

Let K_c^0 be the critical value of the coupling parameter for phase synchronization, i.e., for $K > K_c^0$, the average rotational frequencies of all oscillators are the same. Let $\xi_i \equiv \theta_{i+1} - \theta_i$, and we have $\langle \dot{\xi}_i \rangle = 0$ for $K \geq K_c^0$, where $\langle \cdot \rangle$ denotes the time average. We thus obtain the following upper bound for K_c^0 from Eq. (2):

$$K_c^0 = \frac{-3 \max(\delta\omega_i)}{\min(\sin \xi_{i+1} - 2 \sin \xi_i + \sin \xi_{i-1})}. \quad (3)$$

In the regime of phase synchronization, a strong coherence among the phase variables exists and it is reasonable to assume that $\xi_1, \xi_2, \dots, \xi_N$ are identically distributed random variables. Assuming that for large N , they are distributed uniformly in $[a, b]$ (e.g., $[0, 2\pi]$), we obtain, for N large, the following: $\sin \xi_{i+1} - 2 \sin \xi_i + \sin \xi_{i-1} = -2[(a-b)/N]^2 \sin \xi_i$, which yields: $K_c^0 = [3N^2 \max(\delta\omega_i)]/[2(a-b)^2 \min(\sin \xi_i)]$. If N is fixed, we have: $K_c^0 \sim \delta\omega$. That is, the critical coupling required for phase synchronization is proportional to the width of the probability distribution from which the intrinsic frequencies of the oscillators are drawn. If, on the other hand, $\delta\omega$ is fixed, we obtain the algebraic scaling law (1) with exponent $\alpha = 2$. We note that the quadratic scaling law is specific for the Kuramoto model Eq. (2). In general, the scaling exponent α will be different from 2.

We have performed a series of systematic numerical experiments to verify scaling relation (1) and that between K_c^0 and $\delta\omega$. In particular, we choose ω_i 's ($i = 1, \dots, N$) from a uniform distribution of mean $\omega_0 = 5.0$ and standard deviation $\delta\omega$, and compute the expected value $E[K_c^0]$ of the critical coupling parameter from 20 realizations of the set of intrinsic frequencies. Figure 3(a) shows, for fixed $N = 50$, $E[K_c^0]$ versus $\delta\omega$, which is apparently linear. Figure 3(b) shows, for fixed $\delta\omega = 1.0$, $E[K_c^0]$ versus N for $30 < N < 600$ on a logarithmic scale. The slope of the fitted line is $\alpha = 1.8 \pm 0.2$ for $N \geq 100$, suggesting the validity of scaling relation (1) for the Kuramoto model. Similar results are obtained when the distribution of the intrinsic frequencies is Gaussian. A remarkable feature of the algebraic scaling relation (1) is that it is

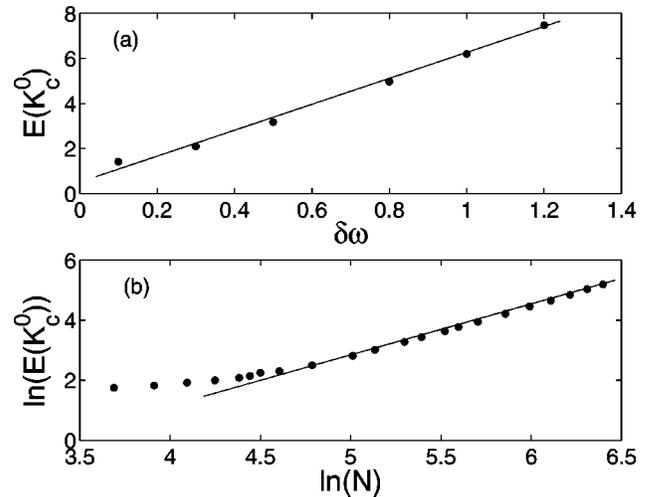


FIG. 3. For the Kuramoto paradigm Eq. (2): (a) $E(K_c^0)$ vs $\delta\omega$ for $N = 50$, and (b) $\ln E(K_c^0)$ vs $\ln N$, where the individual ω_i frequencies are chosen randomly from a uniform distribution.

apparently valid for the system of coupled Rössler oscillators, as shown in Fig. 4 for $10 \leq N \leq 600$, where $\delta\omega = 0.01$. We observe that $E(K_c^0)$ indeed scales with N algebraically, but the scaling exponent (about 0.4) apparently depends on the details of the system.

We now offer a physical explanation for the phenomenon of phase clustering. Intuitively, when the intrinsic frequencies of two oscillators are close, it is easier for them to be synchronized in phase under coupling. In contrast, if the difference in the frequencies is large, a stronger coupling is required for phase synchronization. Thus, in a given range of the coupling parameter, the oscillators with close frequencies are synchronized in phase, forming a cluster. The average frequency of the cluster, however, changes slowly as the coupling is increased. Synchronization (merger) among clusters occurs when their frequencies get close. These consider-

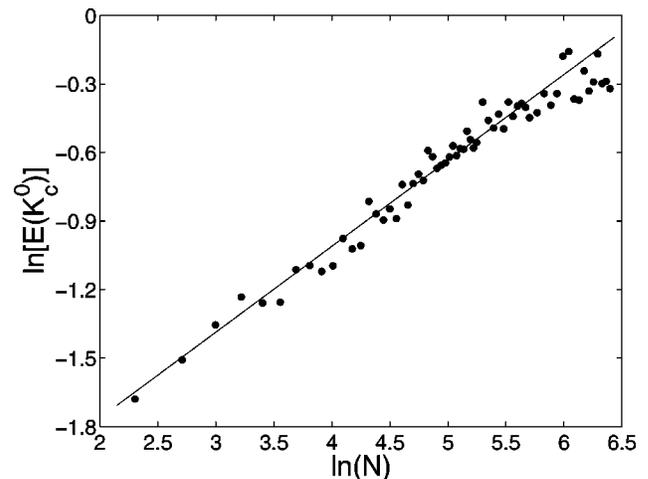


FIG. 4. For the system of coupled Rössler oscillators: $E(K_c^0)$ vs N on a logarithmic scale for $10 \leq N \leq 600$ (over one order of magnitude in N). We see that the algebraic scaling relation between $E(K_c^0)$ and N persists.

ations can also be seen from a mechanical picture implied by the general canonical equation, which can be written as: $\dot{\xi} = F(\xi_{i-1}, \xi_i, \xi_{i+1})$, where F is the force acting on an *overdamped* particle [5]. The corresponding potential function is: $V(\xi_i) = -\int F d\xi_i$ (for the Kuramoto model, the potential function can be obtained explicitly). The dynamics of the set of coupled oscillators can thus be regarded as the motion of a set of equal number of mechanical particles in the potential field $V(\xi)$. The potential function possesses a series of local minima (wells). A number of particles trapped in one of the local wells corresponds to a phase cluster. Increasing the coupling parameter is equivalent to lowering the height of the potential hill between two neighboring wells. A merger of two phase clusters corresponds to the merger of two potential wells. Complete phase synchronization corresponds to the collective motion of particles in a single dominant potential well.

We summarize by stressing that, while phase synchronization in systems of two (or a few) coupled chaotic oscillators is well understood, far less has been achieved for sys-

tems of a large number of coupled oscillators. In particular, the *tendency* for an array of nonlinear oscillators to synchronize in phase in domains or clusters is not well understood. Our paper makes a contribution to this understanding by presenting a clear dynamical picture for the formation and parametric evolution of phase clusters. In addition, we work out an explicit scaling relation between the amount of coupling required for the synchronization and the number of oscillators coupled. An implication of the scaling analysis is that enormous coupling is typically required for complete phase synchronization and, hence, we expect phase clustering to be common in networks of coupled oscillators. Our argument justifying the utilization of the Kuramoto model as an analysis paradigm to understand coupled chaotic oscillators suggests that the results of this paper are relevant to more realistic physical systems.

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