

## Noise-enhanced temporal regularity in coupled chaotic oscillators

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Existing works on coherence resonance, i.e., the phenomenon of noise-enhanced temporal regularity, focus on excitable dynamical systems such as those described by the FitzHugh-Nagumo equations. We extend the scope of coherence resonance to an important class of nonexcitable dynamical systems: coupled chaotic oscillators. In particular, we argue that, when a system of coupled chaotic oscillators in a noisy environment is viewed as a signal processing unit, the degree of temporal regularity of certain output signals may be modulated by noise and may reach a maximum value at some optimal noise level. Implications to signal processing in biological systems are pointed out.

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## I. INTRODUCTION

This paper addresses systems of coupled chaotic oscillators as possible devices for signal modulation and processing under the influence of noise. Consider, for instance, a one-dimensional array of  $N$  coupled, nearly identical nonlinear oscillators, each described by the differential equation of the type:  $d\mathbf{x}/dt = \mathbf{F}(\mathbf{x})$ , as shown schematically in Fig. 1. Assume that a signal  $x_i (i = 1, \dots, N)$ , is measured from each oscillator and the measurements are further combined by a processing unit to yield an output that is a function of all the measured signals:  $U = U(x_1, x_2, \dots, x_N)$ . Because of the chaotic nature of the underlying dynamics in each oscillator, the output signal  $U(t)$  is typically random in time. The purpose of this paper is to show that the *temporal regularity* of the output signal  $U(t)$  can be greatly modulated by noise. In particular, there exists an optimal noise amplitude for which  $U(t)$  can be made predominantly regular, despite the local chaotic dynamics in each individual oscillator.

Our motivations are twofold. First, for the purpose of processing random signals utilizing an engineering system consisting of many interconnected subsystems, it is of interest to know how the inevitable presence of noise influences the temporal regularity of the signal. Second, networks of coupled nonlinear oscillators are extremely common in biology, such as neural networks. The range in which the parameters of each individual element in the network can change reasonably can include regimes for chaotic dynamics. Yet the overall output of the network may often be regular. One example is the sino-atrial node, the natural pacemaker of a cardiac system, which consists of millions of cells with distinct intrinsic oscillating properties. Local chaotic dynamics are possible and have been observed in numerical simulations of a pair of coupled, biophysically detailed sino-atrial node cells [1]. Despite the possibility of local chaotic dynamics, the output of the pacemaker is often regular in time.

The phenomenon of noise-enhanced performance in nonlinear systems has been known since the pioneering work by Benzi *et al.* on stochastic resonance [2], where the signal-to-noise ratio of a nonlinear system is found to be sensitive to the noise amplitude and can reach a maximum value at an optimal noise level. The area of stochastic resonance has

been extremely active [3–6], with phenomena reported recently such as extensive stochastic resonance, i.e., stochastic resonance without the need to tune the noise level, if the signal of the system is defined in a particular way [7–9]. All these works concern the enhancement of the amplitude of certain dynamical variables of the system relative to the noise amplitude. The phenomenon of noise-induced enhancement of temporal regularity of a physical signal was first noticed by Sigeti *et al.* [10–12]. This phenomenon was rediscovered and renamed as *coherence resonance* in excitable nonlinear dynamical systems [13]. Since then, there have been a plethora of theoretical works [14–25] and experimental investigations as well [26–28].

Typically, in excitable dynamical systems, the time trace of dynamical variables of physical interest consists of an infinite sequence of bursts occurring at random time intervals. Coherence resonance is referred to the fact that noise can actually be utilized to improve the temporal regularity

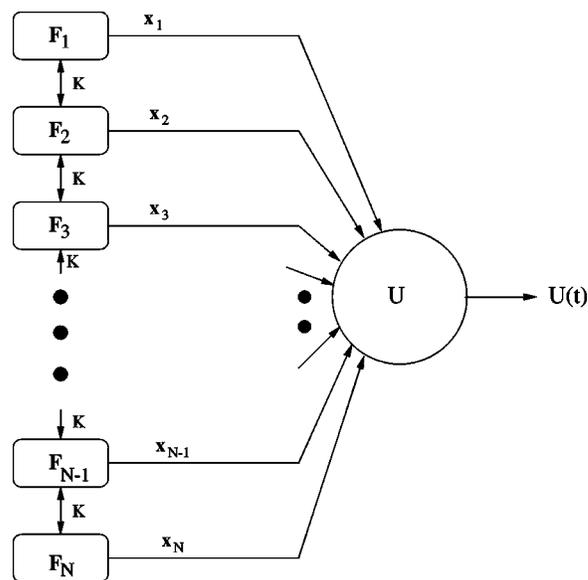


FIG. 1. A schematic illustration of a system of coupled nonlinear oscillators as a signal processing unit.

of the bursting time series [13]. In particular, at both small and large noise levels, the time series appear random in the sense that their Fourier spectra are broad band and apparently exhibit no pronounced peaks (we exclude the dc component). At some intermediate noise levels, the bursting time series appears more regular, which is characterized by the appearance of a finite set of peaks at certain frequencies. If one defines a measure, say the ratio of the height of the most pronounced peak in the Fourier spectrum to its width, to quantify the temporal regularity of the bursting time series, then one finds that the measure tends to increase as the noise level is raised and reaches a maximum value at some optimal noise level. In addition to the difference between coherence resonances and stochastic resonances that the former concern the temporal aspect of the signal while the latter deal with quantities related to amplitude such as the signal-to-noise ratio, coherence resonances do not require an external periodic driving [13,29], versus stochastic resonances that usually do [2–6].

Most existing works on coherence resonance focus exclusively on excitable systems [13] such as those described by the FitzHugh-Nagumo equations [30,31] in which the dynamics typically consist of a slow motion near some fixed point and rapid excursions away from it. In such systems, the measured time series usually consist of a silent phase and a bursting one, corresponding, respectively, to motions near the fixed point and the excursion. Temporal regularities of random signals, and consequently, coherence resonances, are apparently of great physical, biological, and engineering importance. The purpose of this paper is to point out that coherence resonance can actually be expected in another important class of dynamical systems: coupled nonlinear oscillators, which are relevant to a variety of physical and biological situations [32,33]. In particular, we argue that, when identical or slightly nonidentical chaotic oscillators are coupled together, the temporal regularity of some measured signal characterizing the degree of the synchronization among the oscillators can be modulated by external noise in the sense of coherence resonance. Such signals, for example, can simply be the difference among or its function of the corresponding dynamical variables of the oscillators. We give numerical examples and a quantitative analysis elucidating the dynamical mechanism for coherence resonances in coupled oscillator systems. Because of the ubiquity of the occurrence of such systems in nature and in engineering, a correct identification of coherence resonance will be both theoretically interesting and practically useful for applications such as signal processing. A short account of some part of this work has been published recently [34].

The rest of the paper is organized as follows. In Sec. II, we present a physical theory to explain why coherence resonances can occur in wide parameter regimes in coupled chaotic oscillators, and how to identify and characterize them. In Sec. III, we give numerical examples and, particularly, describe how the phenomenon may be tested in laboratory experiments using coupled chaotic electronic circuits. A discussion is offered in Sec. IV.

## II. THEORY

Here, we establish the theoretical foundation of coherence resonance in coupled chaotic oscillators. In particular, given such a system, we address the following questions: (1) what type of signals, say, a combination of the dynamical variables from the oscillators, can exhibit coherence resonance? and (2) how to quantify the resonance? In what follows, we will describe a hierarchy of systems consisting of two coupled identical oscillators, two coupled nonidentical oscillators, and  $N(N \geq 3)$  coupled oscillators, and discuss the general features of these systems. We will then construct a one-dimensional analyzable model, which captures the common ingredients of coupled chaotic oscillators required for coherence resonance, and derive explicit solutions demonstrating quantitative features of the resonance.

### A. Two coupled identical chaotic oscillators

We consider the following system:

$$\begin{aligned}\dot{\mathbf{x}}_1 &= \mathbf{f}(\mathbf{x}_1) + \mathbf{K} \cdot (\mathbf{x}_2 - \mathbf{x}_1), \\ \dot{\mathbf{x}}_2 &= \mathbf{f}(\mathbf{x}_2) + \mathbf{K} \cdot (\mathbf{x}_1 - \mathbf{x}_2),\end{aligned}\quad (1)$$

where  $\mathbf{x}_{1,2} \in \mathcal{R}^d$ ,  $\mathbf{f}$  is the vector velocity field of both chaotic oscillators when uncoupled, and  $\mathbf{K}$  is the coupling matrix. Since the oscillators are identical, the synchronization state (or the synchronization manifold, denoted by  $\mathcal{M}$ ):  $\mathbf{x}_1 = \mathbf{x}_2$ , is a solution of Eq. (1), which is invariant under its time evolution. That is, in a noiseless situation, if the two oscillators start out being synchronized, they remain so forever. Introducing two variables:  $\mathbf{u} = (\mathbf{x}_1 + \mathbf{x}_2)/2$  and  $\mathbf{v} = (\mathbf{x}_1 - \mathbf{x}_2)/2$ , we obtain, near the synchronization manifold  $\mathcal{M}$  ( $\mathbf{v} = \mathbf{0}$ ), the following equations:

$$\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u}), \quad (2)$$

$$\dot{\mathbf{v}} = \left[ \frac{\partial \mathbf{f}}{\partial \mathbf{u}} - 2\mathbf{K} \right] \cdot \mathbf{v}, \quad (3)$$

where  $\partial \mathbf{f} / \partial \mathbf{u}$  is the Jacobian matrix of the velocity field  $\mathbf{f}$ . From Eqs. (2) and (3), we see that, in the synchronization manifold, the dynamics is described by the velocity field  $\mathbf{f}$ , which generates a chaotic flow. In the absence of coupling, i.e.,  $\mathbf{K} = \mathbf{0}$ , the evolution of the variable  $\mathbf{v}$  is governed by the Jacobian matrix  $\partial \mathbf{f} / \partial \mathbf{u}$  that possesses at least one time-averaged eigenvalue that is positive (i.e., a positive transverse Lyapunov exponent), as the flow generated by  $\mathbf{f}$  is chaotic. Thus, the synchronization solution  $\mathbf{v} = \mathbf{0}$  is unstable. In a mathematical term, we say that the synchronization manifold is transversely unstable. As the coupling parameter  $K$  is increased, the eigenvalues of the matrix  $(\partial \mathbf{f} / \partial \mathbf{u} - 2\mathbf{K})$  decrease. Eventually, for  $K > K_c$ , where  $K_c$  is a critical coupling value, all time-averaged eigenvalues of the matrix become negative, signifying an asymptotic stability of the synchronization state. Since the dynamics in the synchronization manifold is chaotic for  $K > K_c$ , the synchronism so achieved is chaotic [35–41].

The above discussion is for noiseless situations. Our point is, in the presence of noise (of amplitude  $D$ ), for  $K$  near  $K_c$ , the coupled system exhibits dynamical characteristics required for coherence resonance. In particular for  $K \leq K_c$ , the synchronization state is unstable so that the vector signal  $\mathbf{v}(t)$ , which is simply proportional to the difference between the dynamical variables  $\mathbf{x}_1(t)$  and  $\mathbf{x}_2(t)$ , exhibits an on-off intermittent behavior [42–46], regardless of whether noise is absent or present. For  $K \geq K_c$ , the synchronization state is stable so that  $\mathbf{v}(t) \rightarrow \mathbf{0}$  asymptotically for  $D=0$  (noiseless situation), but for  $D \neq 0$ ,  $\mathbf{v}(t)$  exhibits, again, on-off intermittency. To understand the origin of the intermittent behavior, we focus on the noiseless situation with  $K \leq K_c$ . Let  $\lambda_T \geq 0$  be the largest time-averaged eigenvalue of the matrix  $(\partial \mathbf{f} / \partial \mathbf{u} - 2\mathbf{K})$ . Since  $\lambda_T$  is an asymptotic quantity, i.e., it is defined in the infinite time limit, in any finite times it is a random variable with a probability distribution. Suppose we distribute a large number of initial conditions in the synchronization manifold  $\mathcal{M}$ , compute  $\lambda_T(t)$  for each trajectory at time  $t$ , and then construct the histogram of these finite time exponents. Typically, the histogram is centered at  $\lambda_T$  with a width proportional to  $1/\sqrt{t}$ . Thus, at any finite time, the distribution of  $\lambda_T(t)$  has a tail on the negative side, indicating that some trajectories actually experience attraction towards  $\mathcal{M}$ . By the ergodicity of chaotic trajectories in  $\mathcal{M}$ , we see that a single trajectory, while in general repelled from  $\mathcal{M}$ , will experience episodes of time during which it actually is attracted toward  $\mathcal{M}$ . Thus, the trajectory tends to stay near  $\mathcal{M}$  with bursts away from it at random times, signifying on-off intermittency. At the onset of on-off intermittency, i.e., when  $\lambda_T=0$ , the time interval between two successive bursts, or the *laminar phase*, obeys a power-law probability distribution with the exponent  $-3/2$  [43]. The mechanism for on-off intermittency may also be understood by analyzing the transverse stabilities of the infinite set of unstable periodic orbits embedded in the chaotic attractor in  $\mathcal{M}$  [36,47]. For  $K \geq K_c$  with noise, the above dynamical picture remains qualitatively similar. For  $K \geq K_c$  without noise, on-off intermittency occurs only for a finite amount of time because, asymptotically,  $\mathcal{M}$  is stable. However, noise may cause the trajectories to be desynchronized, sustaining the on-off intermittent behavior.

A characteristic feature of on-off intermittency is the existence of two distinct states: the “off” state in which  $\mathbf{v}(t) \approx \mathbf{0}$  and the “on” state where  $\mathbf{v}(t)$  deviates significantly away from the “off” state. Typically, the system tends to reside in the “off” state for a certain amount of random time with intermittent bursts away from the “off” state (the “on” state). Roughly, the “off” and “on” states here correspond to the motion near the fixed point and the excursion away from it, respectively, in a typical excitable system that exhibits coherence resonance. Thus, qualitatively, under the influence of noise, we expect coherence resonance to occur in coupled chaotic systems.

### B. Two coupled nonidentical chaotic oscillators

We now consider the following system:

$$\dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1) + \mathbf{K} \cdot (\mathbf{x}_2 - \mathbf{x}_1),$$

$$\dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x}_2) + \mathbf{K} \cdot (\mathbf{x}_1 - \mathbf{x}_2), \quad (4)$$

where  $\mathbf{x}_{1,2} \in \mathcal{R}^d$ ,  $\mathbf{f}_1$ , and  $\mathbf{f}_2$  are the velocity fields of the chaotic oscillators when uncoupled. We consider the situation where the two oscillators are slightly nonidentical:  $\mathbf{f}_1 \approx \mathbf{f}_2$ , and explore the dynamics near the approximate synchronization state:  $\mathbf{x}_1 \approx \mathbf{x}_2$ . In variables  $\mathbf{u}$  and  $\mathbf{v}$ , we obtain, near the approximate synchronization state  $\mathbf{v} \approx \mathbf{0}$ , the following equations:

$$\dot{\mathbf{u}} \approx \frac{1}{2} [\mathbf{f}_1(\mathbf{u}) + \mathbf{f}_2(\mathbf{u})] + \frac{1}{2} \left( \frac{\partial \mathbf{f}_1}{\partial \mathbf{u}} - \frac{\partial \mathbf{f}_2}{\partial \mathbf{u}} \right) \cdot \mathbf{v}, \quad (5)$$

$$\begin{aligned} \dot{\mathbf{v}} &\approx \left[ \frac{1}{2} \left( \frac{\partial \mathbf{f}_1}{\partial \mathbf{u}} + \frac{\partial \mathbf{f}_2}{\partial \mathbf{u}} \right) - 2\mathbf{K} \right] \cdot \mathbf{v} + \frac{1}{2} [\mathbf{f}_1(\mathbf{u}) - \mathbf{f}_2(\mathbf{u})], \\ &\equiv [-\mathbf{A} + \mathbf{N}(t)] \cdot \mathbf{v} + \boldsymbol{\xi}_i(t), \end{aligned} \quad (6)$$

where  $\partial \mathbf{f}_{1,2} / \partial \mathbf{u}$  are the Jacobian matrices of the velocity fields  $\mathbf{f}_{1,2}$ , respectively,  $-\mathbf{A}$  is a matrix whose elements are the average values of the corresponding elements of the matrix in front of  $\mathbf{v}$  in Eq. (6),  $\mathbf{N}(t)$  is a zero-mean random matrix, and  $\boldsymbol{\xi}_i(t)$  stands for the small chaotic modulation term in Eq. (6), which vanishes if the oscillators are identical. Since both oscillators are chaotic, we see from Eq. (5) that the variables  $\mathbf{u}$  is typically chaotic because approximately, it is the average of  $\mathbf{x}_1$  and  $\mathbf{x}_2$  under a small chaotic modulation term proportional to  $\mathbf{v}$ . The variable  $\mathbf{v}$ , on the other hand, obeys the equation that describes on-off intermittency [48] under “noise” because  $\mathbf{u}$  is chaotic. Thus, we expect to observe coherence resonance in the signal  $\mathbf{v}(t)$  under the influence of external noise.

### C. $N(N \geq 3)$ coupled chaotic oscillators

We consider the following system of  $N$  coupled, identical chaotic oscillators:

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{f}(\mathbf{x}_i) + K \sum_{j=1}^N G_{ij} \mathbf{H}(\mathbf{x}_j), \quad i = 1, \dots, N, \quad (7)$$

where  $G_{ij}$ 's are elements of the normalized coupling matrix  $\mathbf{G}$  and  $\mathbf{H}(\mathbf{x})$  is a smooth function. The synchronization manifold  $\mathcal{M}$  is defined by:  $\mathbf{x}_1 = \mathbf{x}_2 = \dots = \mathbf{x}_N$ . If  $G_{ij}$ 's satisfy the condition  $\sum_j G_{ij} = 0$  for all  $i$ , then  $\mathcal{M}$  is an invariant subspace of Eq. (7). The stability of  $\mathcal{M}$  may be assessed by examining the variational equation derived from Eq. (7), as follows [49]:

$$\frac{d\delta \mathbf{x}_i}{dt} = \mathbf{Df}(\mathbf{x}_i) \cdot \delta \mathbf{x}_i + K \sum_{j=1}^N G_{ij} \mathbf{DH}(\mathbf{x}_i) \cdot \delta \mathbf{x}_j, \quad (8)$$

where  $\mathbf{Df}$  and  $\mathbf{DH}$  denote the partial derivatives. On  $\mathcal{M}$ , where  $\mathbf{x}_1 = \dots = \mathbf{x}_N = \mathbf{x}$ , this may be written concisely as

$$\frac{d\delta \mathbf{X}}{dt} = [\mathbf{I}_N \otimes \mathbf{Df}(\mathbf{x}) + K \mathbf{G} \otimes \mathbf{DH}(\mathbf{x})] \cdot \delta \mathbf{X}, \quad (9)$$

where  $\delta\mathbf{X}=(\delta\mathbf{x}_1, \dots, \delta\mathbf{x}_N)^T$ , and  $\mathbf{I}_N$  denotes the  $N \times N$  identity matrix. If  $\mathbf{G}=\mathbf{T}^{-1}\mathbf{\Gamma}\mathbf{T}$  with  $\mathbf{\Gamma}=\text{diag}(\gamma_0, \dots, \gamma_{N-1})$ , then the system (9) may be decoupled into the following block diagonal form:

$$\frac{d\mathbf{V}}{dt}=[\mathbf{I}_N \otimes \mathbf{Df}(\mathbf{x}) + K\mathbf{\Gamma} \otimes \mathbf{DH}(\mathbf{x})] \cdot \mathbf{V}, \quad (10)$$

where  $\mathbf{V}=(\mathbf{v}_1, \dots, \mathbf{v}_N)^T$  and  $\mathbf{v}_i=\sum_j T_{ij}\delta\mathbf{x}_j$ . In terms of the individual components, we have  $N$  variational equations:

$$\frac{d\mathbf{v}_k}{dt}=[\mathbf{Df}(\mathbf{x}) + K\gamma_k\mathbf{DH}(\mathbf{x})] \cdot \mathbf{v}_k, \quad k=0,1, \dots, N-1. \quad (11)$$

Note that the condition  $\sum_j G_{ij}=0$  implies that  $G$  has at least one zero eigenvalue, which we take to be  $\gamma_0$ ; the corresponding equation determines the Lyapunov exponents of the chaotic attractor in  $\mathcal{M}$ . The remaining  $N-1$  equations determine the stability of the orbit in the  $d(N-1)$  directions transverse to  $\mathcal{M}$ . There exists a critical value  $K_c$  of the coupling parameter, where for  $K > K_c$ , the chaotic synchronization state becomes transversely stable. We expect coherence resonance to occur in a parameter regime about  $K_c$ .

From the above analysis, we see that all signals  $\mathbf{v}_i(t)$  ( $i=1, \dots, N$ ), except the one in  $\mathcal{M}$ , exhibits on-off intermittency for  $K$  about  $K_c$  under the influence of noise. As a practical matter, we write

$$\mathbf{v}_i(t)=\sum_{j=1}^N T_{ij} \cdot (\mathbf{x}_j - \mathbf{x}_s), \quad (12)$$

where  $\mathbf{x}_s$  is an arbitrary point in  $\mathcal{M}$ . If the phase-space region in which the chaotic attractor of each individual oscillator lies contains the origin  $\mathbf{x}=\mathbf{0}$ , we may conveniently choose  $\mathbf{x}_s=\mathbf{0}$ . In the presence of noise, coherence resonance may then be observed in any component of  $\mathbf{V}$  that is not in  $\mathcal{M}$ .

If the oscillators coupled are not identical, the synchronization state  $\mathbf{x}_1=\dots=\mathbf{x}_N$  is no longer invariant. Analogous to the case of two coupled nonidentical oscillators, we expect the signals defined in Eq. (12) to exhibit coherence resonance under the influence of noise.

#### D. An analyzable model

To make the analysis feasible, we consider one scalar variable that exhibits on-off intermittency. That is, we consider the one-dimensional version of Eq. (6). Under the influence of external noise  $\xi_e(t)$ , we have

$$\dot{v}=[-\lambda + N(t)]v + \xi_i(t) + \xi_e(t) \equiv [-\lambda + N(t)]v + \xi(t), \quad (13)$$

where  $\xi(t)$  now stands for the combination of internal chaotic modulation and external noise. Note that Eq. (13) is similar to the paradigmatic model for analyzing on-off intermittency under the influence of noise [48]. To quantify how

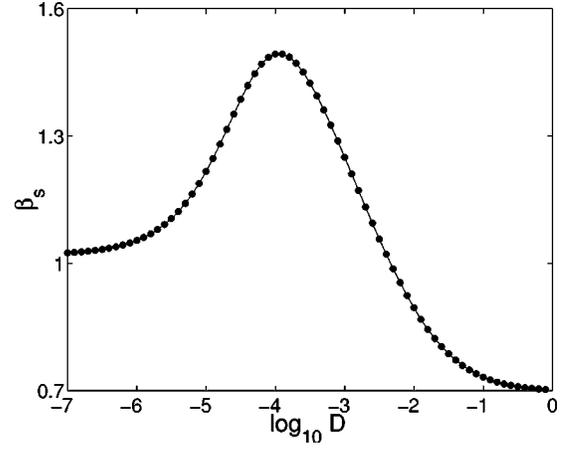


FIG. 2. Theoretical prediction of the measure of coherence resonance  $\beta_T$  versus noise amplitude for a general system of coupled chaotic oscillators.

the temporal regularity of  $v(t)$  is modulated by noise, we use the following measure introduced in Ref. [13], for convenience:

$$\beta_T = \frac{\langle T \rangle}{\sqrt{\text{Var}(T)}}, \quad (14)$$

where  $T$  is the interval between the bursts, and  $\langle T \rangle$  and  $\text{Var}(T)$  are the average value and variance of  $T(t)$ , respectively. To obtain  $\langle T \rangle$  and  $\text{Var}(T)$ , we consider the following Fokker-Planck equation associated with Eq. (13):

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial v} \left[ \left( -\lambda v + \frac{1}{2} \epsilon v \right) P \right] + \frac{1}{2} \frac{\partial^2}{\partial v^2} [(\epsilon v^2 + D)P], \quad (15)$$

where  $P(v, t)$  is the time-dependent probability distribution function of the random variable  $v(t)$ ,  $D$  is the noise amplitude, and  $\epsilon$  is the amplitude of  $N(t)$ . Noting that the intermittent interval  $T$  is in fact the first-passage time, we solve Eq. (15) for quantities that are required for characterizing the time regularity of  $v(t)$  under the conditions that there is an absorbing boundary at  $v=a$  and a reflecting one at  $v=b$ . We obtain [50], for the first and second moments of  $T$ , the following:

$$\begin{aligned} \langle T(v_0) \rangle &= 2 \int_{v_0}^a dy (\epsilon y^2 + D)^{(\lambda/\epsilon) - (1/2)} \int_b^y (\epsilon z^2 \\ &\quad + D)^{-(1/2) - (\lambda/\epsilon)} dz, \\ \langle T^2 \rangle &= 4 \int_{v_0}^a dy (\epsilon y^2 + D)^{(\lambda/\epsilon) - (1/2)} \\ &\quad \times \int_b^y (\epsilon z^2 + D)^{-(1/2) - (\lambda/\epsilon)} \langle T(z) \rangle dz, \end{aligned}$$

where  $v_0$  is the initial value of  $v(t)$ . The quantity  $\beta_T$  may then be obtained from these moments. Figure 2 shows a typical behavior of  $\beta_T$  as a function of  $D$  that we obtain by

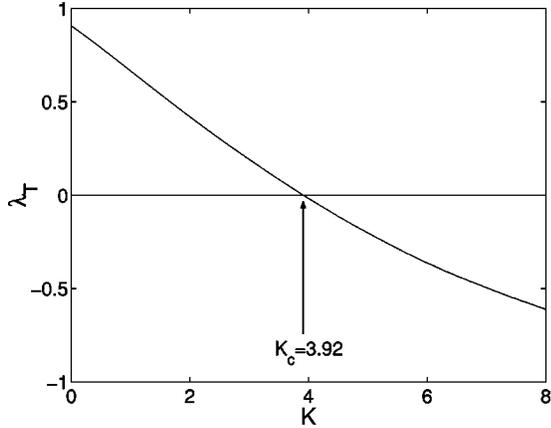


FIG. 3. For a pair of two-coupled identical Lorenz chaotic oscillators, the largest transverse Lyapunov exponent versus the coupling parameter  $K$ .

numerically evaluating the integrals contained in the moments with the following parameters (arbitrary):  $v_0 = -5$ ,  $a = 1$ ,  $b = -20$ , and  $\lambda = \epsilon = 10^{-4}$ . Signature of coherence resonance is seen clearly from Fig. 2 where  $\beta_T$  attains a maximum value at some optimal noise amplitude. The theoretical prediction (Fig. 2) thus suggests that a dynamical system with on-off intermittency can exhibit coherence resonance.

### III. NUMERICAL EXAMPLES

#### A. Two coupled identical Lorenz oscillators

We consider the following system of two coupled Lorenz oscillators:

$$\begin{aligned}\dot{x}_{1,2} &= \sigma_{1,2}(y_{1,2} - x_{1,2}) + K(x_{2,1} - x_{1,2}) + D\xi_x(t), \\ \dot{y}_{1,2} &= \gamma_{1,2}x_{1,2} - y_{1,2} - x_{1,2}z_{1,2} + D\xi_y(t), \\ \dot{z}_{1,2} &= -b_{1,2}z_{1,2} + x_{1,2}y_{1,2} + D\xi_z(t),\end{aligned}\quad (16)$$

where  $\sigma_{1,2}$ ,  $\gamma_{1,2}$ , and  $b_{1,2}$  are the parameters of the Lorenz oscillator [51],  $K$  is the coupling parameter,  $\xi_{x,y,z}(t)$  are independent Gaussian random processes that simulate the external noise, and  $D$  is the noise amplitude. Here, we consider the case where the two Lorenz oscillators are identically chaotic: we set  $\sigma_{1,2} = 10.0$ ,  $\gamma_{1,2} = 28.0$ , and  $b_{1,2} = 8/3$  so that each Lorenz oscillator, when uncoupled, exhibits a chaotic attractor. This identity stipulates that the asymptotic synchronization state:  $\mathbf{x}_1(t) = \mathbf{x}_2(t)$ , where  $\mathbf{x} = \{x, y, z\}$ , is a solution of Eq. (16) in the absence of noise. Figure 3 shows the largest transverse Lyapunov exponent versus the coupling parameter (For a rigorous definition of the transverse Lyapunov exponents, see Refs. [52–55,49]). We see that, in the noiseless situation, the synchronization state is unstable for  $K < K_c$  and stable for  $K > K_c$ , where  $K_c \approx 3.92$ . Thus, for  $K$  near  $K_c$ , under the influence of noise, the coupled system exhibits dynamical characteristics required for coherence resonance.

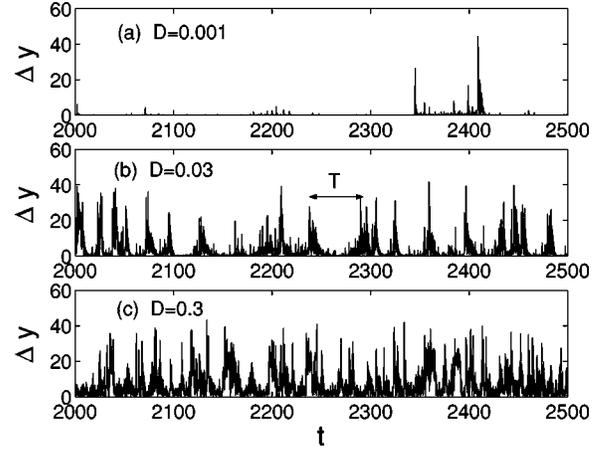


FIG. 4. For a pair of two-coupled identical Lorenz chaotic oscillators at coupling  $K = 4.0 > K_c$ : time series  $\Delta y(t)$  at three different noise levels:  $D = 0.001$  (a),  $D = 0.03$  (b), and  $D = 0.3$  (c).

Figures 4(a)–4(c) show, for  $K = 4.0 \geq K_c$ , the time series  $\Delta y(t) \equiv |y_1(t) - y_2(t)|$  at three different noise levels:  $D = 0.001$ ,  $D = 0.03$ , and  $D = 0.3$ , respectively. Visually, there appears to be a strong temporal regularity in  $\Delta y(t)$  at the intermediate noise level [Fig. 4(b)], compared with those in Figs. 4(a) and 4(c). To characterize the degree of this regularity, we compute the corresponding Fourier power spectra. Figures 5(a)–5(c) show the power spectra of the signals in Figs. 4(a)–4(c), respectively. For small noise [Fig. 5(a)], the spectrum exhibits no peak except for the one at  $\omega = 0$  (the dc component), indicating a lack of the temporal regularity in the bursting time series. The situation is similar for large noise [Fig. 5(c)]. A pronounced peak at  $\omega \neq 0$  does exist at the intermediate noise level [Fig. 5(b)], indicating the existence of a strong time periodic component in the time series. The apparent temporal regularity seen in Fig. 5(b) can be quantified by the characteristics of the peak at the nonzero frequency  $\omega_p$  in the spectrum. In particular, we utilize the following quantity:

$$\beta_S \equiv HQ_S \equiv H\omega_p / \Delta\omega, \quad (17)$$

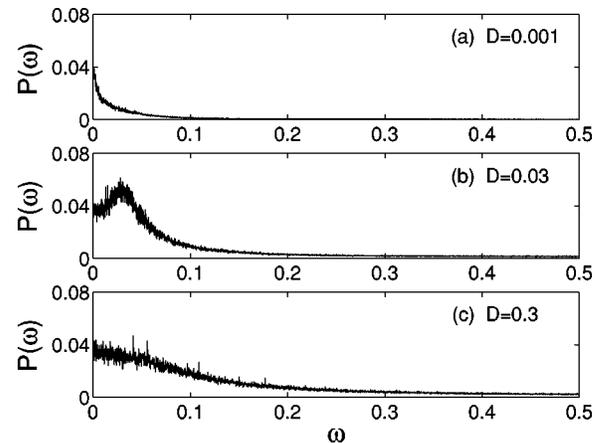


FIG. 5. Fourier power spectra of  $\Delta y(t)$  for the noise-driven coupled identical Lorenz chaotic oscillators for  $K = 4.0$  and the following noise levels: (a)  $D = 0.001$ , (b)  $D = 0.03$ , and (c)  $D = 0.3$ .

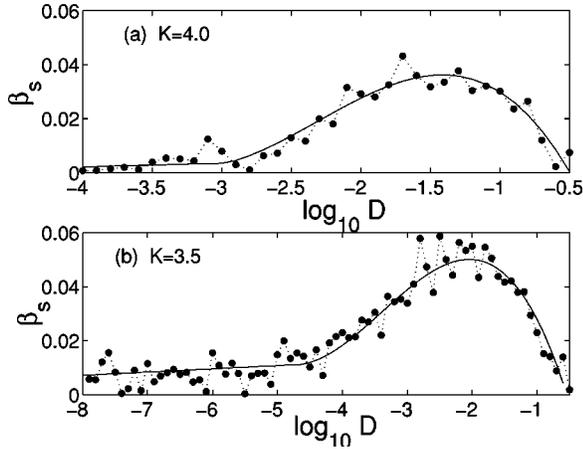


FIG. 6. Coherence-resonance measure  $\beta_S$  versus the noise amplitude for the pair of coupled identical Lorenz chaotic oscillators with coupling parameter (a)  $K=4.0 > K_c$  and (b)  $K=3.5 < K_c$ , where the solid lines are Polyfit curves.

where  $H$  is the height of the spectral peak and  $\Delta\omega$  is its half width [13–25]. The equivalence between  $\beta_S$  and  $\beta_T$  in Eq. (14) can be seen, as follows. A physical process may be described either in the time domain:  $f(t)$ , or in the frequency domain by its Fourier transform  $F(\omega)$ . When  $f(t)$  is approximately periodic, its Fourier spectrum exhibits a peak at  $\omega_p = 1/\langle T \rangle$  with width  $\Delta\omega$ . Since  $T \sim 1/\omega$ , we have:  $\langle T \rangle + \Delta T \sim 1/(\omega_p + \Delta\omega) \approx 1/\omega_p - \Delta\omega/\omega_p^2$ . Thus,  $\Delta T \sim \Delta\omega/\omega_p^2$  and, hence,  $\beta_T = \langle T \rangle / \Delta T \sim \omega_p / \Delta\omega \sim \beta_S$ .

By its definition, a high value of  $\beta_S$  indicates a strong temporal regularity in the bursting time series. Figures 6(a) and 6(b) shows, for Eq. (16) at  $K=4.0 > K_c$  and  $K=3.5 < K_c$ , respectively,  $\beta_S$  versus the noise amplitude  $D$ . We see that  $\beta_S$  is small at small noise levels, increases as the noise is increased, reaches a maximum at an optimal noise level, and decreases as the noise is increased further. These are features that are typically associated with stochastic resonance where typically, a signal-to-noise ratio is plotted against the noise level, but here the relevant quantity concerns the time regularity.

### B. Two coupled nonidentical Lorenz oscillators

When the oscillators are not identical, the synchronization state is no longer invariant. Our analysis in Sec. II indicates that, if the mismatch between the oscillators is small, on-off intermittency persists in a wide parameter regime even in a noiseless situation [43]. With the intermittency, we expect that noise can regulate effectively the temporal characteristics of the bursting time series and, consequently, coherence resonance can occur. To stipulate the nonidentity between the two oscillators in Eq. (16), we set:  $\sigma_1 = 10.0$  and  $\sigma_2 = 11.0$ , which corresponds to a 10% parameter mismatch. The remaining parameters are the same as those in Sec. III A. Figures 7(a)–7(c) show, for  $K=4.0$  and noise amplitudes of  $D=0.001$ ,  $D=0.03$ , and  $D=0.3$ , respectively, the output signal  $\Delta y(t)$ . Even visually, we notice a strong temporal regularity at the intermediate noise amplitude [Fig. 7(b)]. Figures 8(a)–8(c) show the Fourier power spectra of the signals in

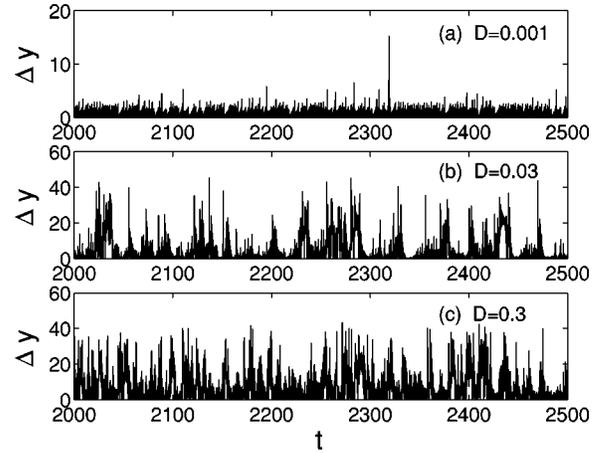


FIG. 7. For a pair of two coupled nonidentical Lorenz chaotic oscillators at coupling  $K=4.0$  with a 10% parameter mismatch, time traces of the signal  $\Delta y(t)$  at three different noise levels: (a)  $D=0.001$ , (b)  $D=0.03$ , and (c)  $D=0.3$ .

Figs. 7(a)–7(c), respectively. Figure 9 shows the coherence-resonance measure  $\beta_S$  versus the noise amplitude, where we observe typical features of coherence resonance, as those in Figs. 6(a) and 6(b).

### C. $N \geq 3$ coupled Lorenz oscillators

We consider the following system of  $N$ , nearest-neighbor coupled identical Lorenz chaotic oscillators:

$$\begin{aligned} \dot{x}_i &= \sigma(y_i - x_i) + K(x_{i+1} - 2x_i + x_{i-1}) + D\xi_x(t), \\ \dot{y}_i &= \gamma x_i - y_i - x_i z_i + D\xi_y(t), \\ \dot{z}_i &= -bz_i + x_i y_i + D\xi_z(t), \quad i = 1, \dots, N, \end{aligned} \quad (18)$$

where periodic boundary condition is assumed. The matrix  $\mathbf{G}$  and the coupling function  $\mathbf{H}(\mathbf{x})$  in Eq. (7) are given by:

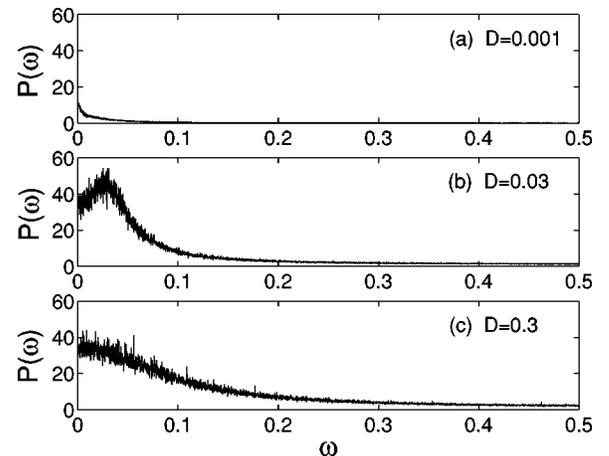


FIG. 8. (a)–(c) Fourier power spectra of the signals in Figs. 7(a)–7(c), respectively.

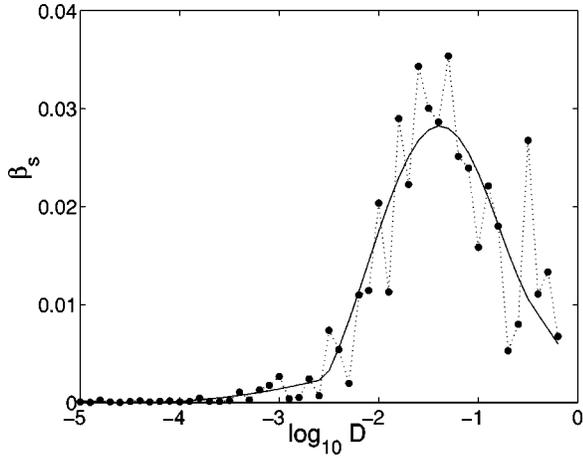


FIG. 9. Coherence-resonance measure  $\beta_S$  versus the noise amplitude for the pair of coupled nonidentical Lorenz chaotic oscillators for  $K=4.0$ , together with a Polyfit curve (solid line).

$$\mathbf{G} = \begin{pmatrix} -z & 1 & 0 & 0 & \cdots & 1 \\ 1 & -z & 1 & 0 & \cdots & 0 \\ 0 & 1 & -z & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & \cdots & -z \end{pmatrix}, \quad \mathbf{H}(\mathbf{x}) = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}. \quad (19)$$

To be illustrative, we choose  $N=5$  in our numerical experiments. The five eigenvalues of  $\mathbf{G}$  are:  $\gamma_0=0$ ,  $\gamma_1=-1.382$ ,  $\gamma_3=-2.0$ ,  $\gamma_4=-3.0$ , and  $\gamma_4=-3.618$ . The corresponding eigenvector matrix  $\mathbf{T}$  then determines the signals, as a function of linear combinations of the dynamical variables  $\mathbf{x}_i (i=1, \dots, 5)$ , which may potentially exhibit coherence resonance. For instance, we find that the following signal:

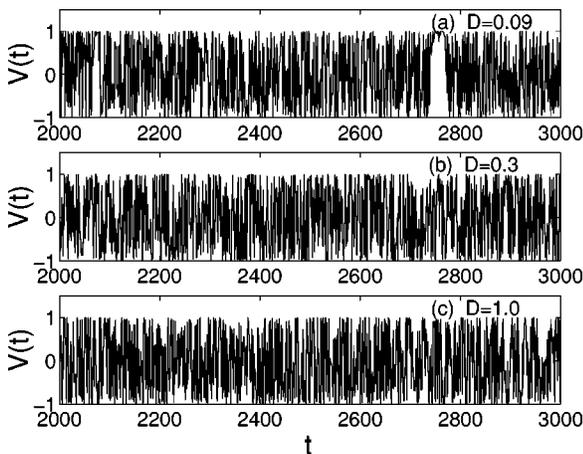


FIG. 10. For  $N=5$  coupled identical Lorenz chaotic oscillators at coupling  $K=1.1$ , time traces of the signal  $V(t)$  [Eq. (20)] at three noise levels: (a)  $D=0.09$ , (b)  $D=0.3$ , and (c)  $D=1.0$ , where  $f(x) = \cos x$ .

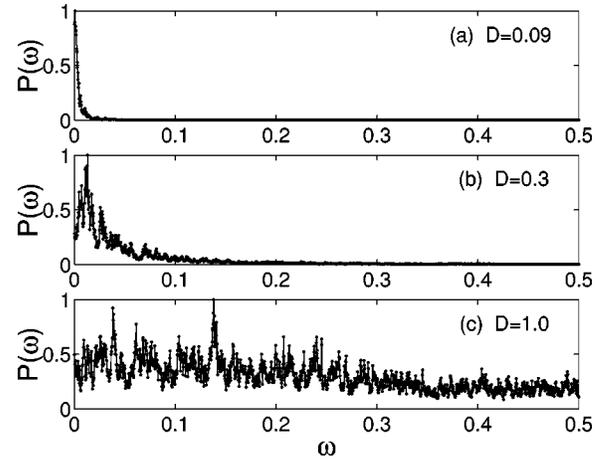


FIG. 11. (a)–(c) Fourier power spectra of the signals in Figs. 10(a)–10(c), respectively.

$$V(t) \equiv f\{a[-y_1(t) + y_2(t) + y_3(t) - y_5(t)]\}, \quad (20)$$

where  $f$  is a smooth function and  $a \neq 0$  is a constant, exhibits coherence resonance. Figures 10(a)–10(c) show, for  $f(x) = \cos x$  and  $a=1.118$ , the signal  $V(t)$  at the noise amplitudes  $D=0.09$ ,  $D=0.3$ , and  $D=1.0$ , respectively. Visually, changes in the temporal regularity of the signals in Figs. 10(a)–10(c) are not apparent. However, there are signatures of such changes and, hence, coherence resonance, in the corresponding power spectra, as shown in Figs. 11(a)–11(c). Figure 12 shows the coherence-resonance measure  $\beta_S$  versus the noise amplitude, which clearly indicates the achievement of the maximum temporal regularity of the signal at an optimal noise level.

#### D. Two coupled Chua's circuits

We consider the following system of two coupled Chua's circuits in dimensionless form [56], which can be readily implemented in laboratory experiments:

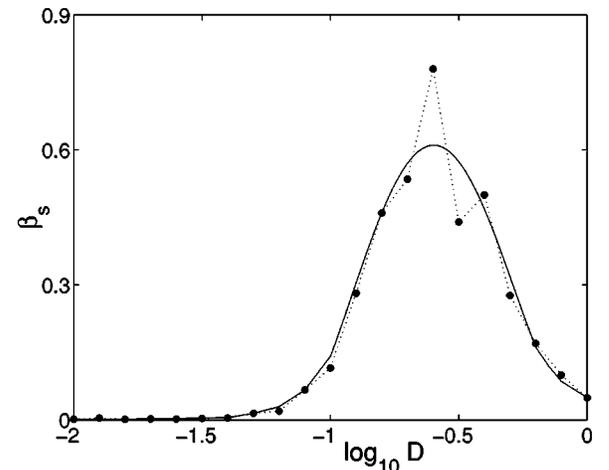


FIG. 12. Coherence-resonance measure  $\beta_S$  versus the noise amplitude for five coupled identical Lorenz chaotic oscillators with coupling parameter  $K=1.1$ , where the solid line is a Polyfit curve.

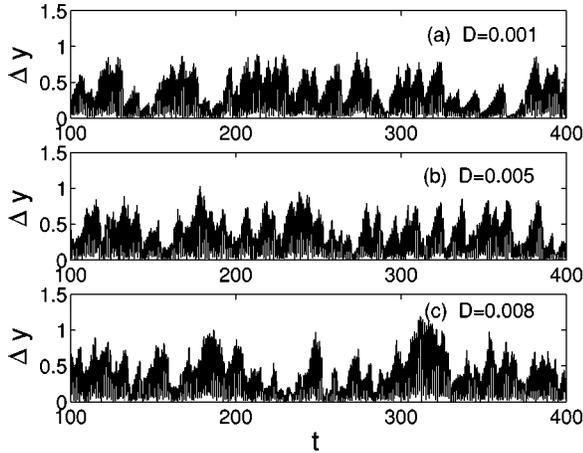


FIG. 13. For a pair of coupled identical Chua chaotic oscillators at coupling  $K=0.3$ , time traces of the signal  $\Delta y(t)$  at three different noise levels: (a)  $D=0.001$ , (b)  $D=0.005$ , and (c)  $D=0.008$ .

$$\begin{aligned} \dot{x}_{1,2} &= \alpha[y_{1,2} - h(x)] + K(x_{2,1} - x_{1,2}) + D\xi_x(t), \\ \dot{y}_{1,2} &= x_{1,2} - y_{1,2} + z_{1,2} + D\xi_y, \\ \dot{z}_{1,2} &= -\beta y_{1,2} + \epsilon_0 \sin \omega_0 t + D\xi_z(t), \end{aligned} \quad (21)$$

where  $h(x) = m_1 x + 1/2(m_0 - m_1)(|x+1| - |x-1|)$  is the piecewise linear function that describes the current-voltage relation of the nonlinear diode in the circuit, the parameters of each individual circuit are chosen to be:  $m_0 = -1/7$ ,  $m_1 = 2/7$ ,  $\alpha = 8.875$ ,  $\beta = 14.286$ ,  $\epsilon_0 = 0.15$ , and  $\omega_0 = 0.35$  so that each circuit exhibits a chaotic attractor when uncoupled. The coupling parameter is  $K$  and the noise amplitude is  $D$ .

The purpose of studying Eq. (21) is that the system represents a convenient testbed for experimentally demonstrating coherence resonance. (We are currently pursuing this.) Figures 13(a)–13(c) show, for  $K=0.3$ , the time trace of

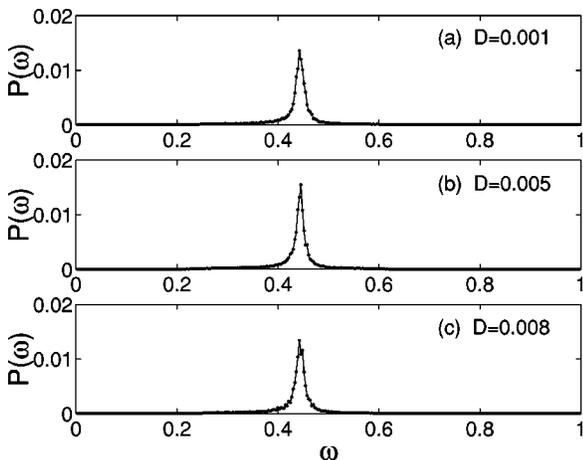


FIG. 14. (a)–(c) Fourier power spectra of the signals in Figs. 13(a)–13(c), respectively.

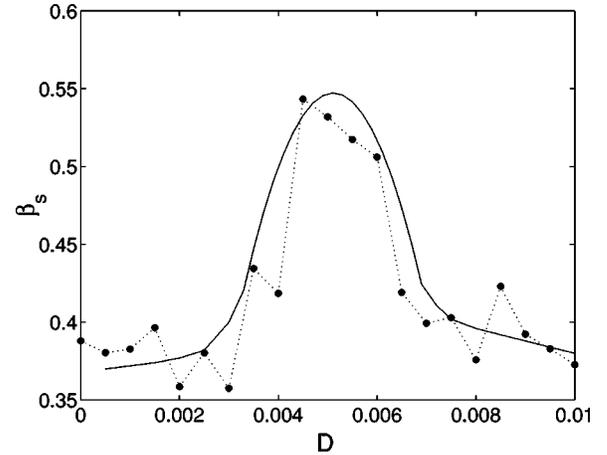


FIG. 15. Coherence-resonance measure  $\beta_S$  versus the noise amplitude for the pair of coupled identical Chua chaotic oscillators with coupling parameter  $K=0.3$ , together with a Polyfit curve (solid line).

the signal  $\Delta y(t) = y_2(t) - y_1(t)$ , for  $D=0.001$ ,  $D=0.005$ , and  $D=0.008$ , respectively. The corresponding power spectra are shown in Figs. 14(a)–14(c), respectively. There exists a spectral peak at  $\omega > 0$ , the shape of which appears to be modulated by noise. A strong signature of coherence resonance can be seen from Fig. 15, which shows the measure  $\beta_S$  versus the noise amplitude. Similar results are obtained when the circuits are slightly nonidentical, or when the number of coupled circuits is larger than two (data not shown).

#### IV. DISCUSSION

We summarize by listing the set of necessary dynamical conditions for coherence resonance: (1) there exists a reference state near which a trajectory can spend long time spans; (2) the system has the potential to temporally burst out of the reference state; and (3) the system is nonlinear. Under these conditions, it is possible for a signal characterizing the bursting behavior to become temporally more regular under the influence of noise. Excitable systems apparently satisfy these conditions. The analysis and numerical computations presented in this paper indicate that coupled chaotic oscillators, a class of dynamical systems of recent interests, also satisfy these conditions and, hence, they generically exhibit coherence resonance. Such systems may be readily constructed in laboratory, say, by using electronic circuits, for experimentally verifying the theoretical prediction of this paper.

Since coupled chaotic oscillators occur in many different contexts of natural sciences, we expect our finding to be important. For example, imagine a biological system consisting of two coupled, chaotically behaving neurons. Knowing that noise may enhance temporal regularity in some outputs of the system is clearly of importance if regular behavior is

desirable, as a tuning of an internal parameter of the system to move it into a regular regime is practically impossible. Similar applications may be anticipated in a system of two coupled chaotic lasers where a regular output signal is desirable.

## ACKNOWLEDGEMENTS

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