Catastrophic bifurcation from riddled to fractal basins

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Most existing works on riddling assume that the underlying dynamical system possesses an invariant subspace that usually results from a symmetry. In realistic applications of chaotic systems, however, there exists no perfect symmetry. The aim of this paper is to examine the consequences of symmetry-breaking on riddling. In particular, we consider smooth *deterministic* perturbations that destroy the existence of invariant subspace, and identify, as a symmetry-breaking parameter is increased from zero, two distinct bifurcations. In the first case, the chaotic attractor in the invariant subspace is transversely stable so that the basin is riddled. We find that a bifurcation from riddled to fractal basins can occur in the sense that an arbitrarily small amount of symmetry breaking can replace the riddled basin by fractal basins. We call this a *catastrophe of riddling*. In the second case, where the chaotic attractor in the invariant subspace is transversely unstable so that there is no riddling in the unperturbed system, the presence of a symmetry breaking, no matter how small, can immediately create fractal basins in the vicinity of the original invariant subspace. This is a smooth-fractal basin boundary metamorphosis. We analyze the dynamical mechanisms for both catastrophes of riddling and basin boundary metamorphoses, derive scaling laws to characterize the fractal basins induced by symmetry breaking, and provide numerical confirmations. The main implication of our results is that while riddling is robust against perturbations that preserve the system symmetry, riddled basins of chaotic attractors in the invariant subspace, on which most existing works are focused, are *structurally unstable* against symmetry-breaking perturbations.

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I. INTRODUCTION

When a nonlinear dynamical system possesses multiple coexisting attractors [1,2], the basin boundaries between the attractors can be either smooth or quite complicated. A physically important question concerns the predictability of asymptotic attractors when initial conditions are chosen in the vicinity of the boundaries. Smooth boundaries are simple sets whose dimensions are one less than that of the phase space. For these boundaries, an improvement in the precision to specify the initial conditions results in an equal amount of improvement in the predictability of the asymptotic attractor. Fractal basins are open (e.g., contain open areas in two dimensions) but their boundaries contain fractal sets [1]. Typically, the dimension of a fractal basin boundary is a fraction less than the phase-space dimension. As a consequence, a more precise specification of the initial conditions often results in a much smaller improvement in the probability to predict the attractor correctly. Riddled basins contain no open sets (e.g., no open area in two dimensions) and have dimensions close to that of the phase space [3–10]. For riddled basins, a vast reduction in the uncertainty to specify the initial conditions results in hardly any improvement in ability to predict the final attractor. Because of this serious physical consequence, the phenomenon of riddling has received quite a lot of recent attention [3–10]. This paper concerns bifurcation from riddled basins to fractal ones (see Sec. II for more precise meanings of riddled versus fractal basins).

Riddling was first analyzed for general chaotic systems by Alexander *et al.* [3]. The dynamical conditions for riddling to occur were first described in Ref. [3] where it was shown that for systems with an invariant subspace $\mathcal{S}$, (i) if there is a chaotic attractor in $\mathcal{S}$, and (ii) if a typical trajectory in the chaotic attractor is stable with respect to perturbations transverse to $\mathcal{S}$, then the basin of the chaotic attractor in $\mathcal{S}$ can be riddled with holes that belong to the basin of another attractor off $\mathcal{S}$, provided that such an attractor exists. Mathematically, a riddled basin is the complement of a dense open set belonging to the basin of the other coexisting attractor. Thus a riddled basin contains no open sets, in contrast to fractal basins that usually do [1]. Physically, the presence of a riddled basin means that, for every initial condition that goes to the chaotic attractor in $\mathcal{S}$, there are initial conditions arbitrarily nearby that approach the attractor off $\mathcal{S}$ and, as a consequence, prediction of the asymptotic attractor for specific initial conditions and parameters becomes practically impossible. The bifurcation that leads to riddling was studied in Ref. [7]. A signature of a riddled basin was found in an experimental system consisting of coupled chaotic electronic circuits [5]. The effect of noise on riddling was also investigated [8]. More recently, conditions for riddling were reexamined, and it was found that physical signatures of riddling persist even if the chaotic set in the invariant subspace is nonattracting [10]. In most existing works on riddling, a common assumption was that the system possesses a perfect invariant subspace. Such an invariant subspace usually results from a simple symmetry in the system.

While symmetry and invariance are common in mathematical models of physical systems and such intrinsic properties in the system's equations can have intriguing and interesting dynamical consequences, the notion of symmetry and invariance is *nongeneric* because, in a physical reality, imperfections or perturbations that destroy the system symmetry are always expected to be present. Consider, for example, the following system of N coupled oscillators [11,12]:

$$\frac{d\mathbf{x}_i}{dt}=\mathbf{F}_i(\mathbf{x}_i)+K\sum_j \mathbf{H}(\mathbf{x}_i-\mathbf{x}_j), \quad i=1,\ldots,N, \tag{1}$$

where $\mathbf{F}_i(\mathbf{x}_i)$ is the velocity field of each individual oscillator when uncoupled, and the coupling is represented by the strength K and the function $\mathbf{H}(\mathbf{x}_i-\mathbf{x}_j)$ satisfying the condition $\mathbf{H}(\mathbf{0})=\mathbf{0}$. When the individual oscillators are identical, i.e., $\mathbf{F}_i=\mathbf{F}_j$ for $i,j=1,\ldots,N$, the synchronous state $\mathbf{x}_i(t)=\mathbf{x}_j(t)$ $(i,j=1,\ldots,N)$ is a solution to Eq. (1). In this case, the dynamical equation is identical for each oscillator so that oscillators starting synchronized remain so forever. The subspace defined by $\mathbf{x}_i(t)=\mathbf{x}_j(t)$ $(i,j=1,\ldots,N)$ is therefore invariant. Indeed, the existence of such an invariant subspace appears to be the starting point for analyzing the dynamics of coupled chaotic oscilltors in most existing works [11,12]. In realistic situations as in laboratory experiments, imperfections such as parameter mismatches among oscillators are inevitable. The presence of nonidentity among the velocity fields $\mathbf{F}_i$'s, no matter how small, immediately destroys the original invariant subspace defined by the synchronous state.

Motivated by the consideration that symmetry breaking is inevitable in physical situations, in this paper we address the following important question: can riddling be observed? Our principal result is that riddling is typically destroyed by symmetry-breaking perturbations, no matter how small (*catastrophe of riddling*). In a fairly general setting, a riddled basin, after its destruction, is converted into a fractal one, which is a common type of basin expected in nonlinear systems [1]. However, when the perturbations are small, the resulting fractal basin may appear, at least visually, similar to a riddled one. Thus the difference between this type of fractal basins and riddled basins is rather subtle, which is mathematically important but perhaps not so from the practical standpoint of physical observation. We establish our result by performing mathematical analysis of a class of representative analyzable models, by utilizing an approximate physical theory to derive scaling laws to characterize the effect of symmetry-breaking perturbations on riddling, and by extensive numerical computations. A short account of part of the results was recently published [13].

The rest of the paper is organized as follows. First we consider an analyzable model and present a detailed argument for the destruction of riddling under symmetry breaking (Sec. II). Because of the simplicity of the model, the subtle difference between riddled and fractal basins, and how the former is converted into the latter by symmetry-breaking perturbations, can be understood explicitly. We then derive scaling laws of physically measurable quantities for the catastrophe of riddling (Sec. III). In Sec. IV, we provide numerical support. In Sec. V, we present a discussion from the standpoint of prediction and observation.

II. AN ANALYTICAL MODEL

We consider the following two-dimensional map $\mathbf{F}(\mathbf{x})$, where $\mathbf{x}\equiv(x,y)\in\mathbb{R}^2$:

$$x_{n+1}=T(x_n)=\begin{cases}2x_n, & 0\leq x<1/2\\ 2(1-x_n), & 1/2\leq x\leq 1\end{cases}$$
$$y_{n+1}=f(x_n,y_n)=\begin{cases}ax_ny_n+\epsilon, & 0\leq y<1\\ \lambda y_n, & y\geq 1,\end{cases} \tag{2}$$

where $T(x)$ is the tent map, a, ϵ, and $\lambda>1$ are parameters, and the phase-space region of interest is $\{0\leq x\leq 1, -\infty<y<\infty\}$. When $\epsilon=0$, the system possesses the one-dimensional invariant subspace $y=0$, which is caused by the reflecting symmetry $y\rightarrow -y$. That is, if $y_0=0$, then $y_n=0$ for all $n>0$. The symmetry is broken when $\epsilon\neq 0$. Because $\lambda>1$, the map has at least two attractors: $y=\pm\infty$. When $\epsilon=0$, the chaotic attractor of the tent map in $y=0$ can be the third attractor of the full system if it is transversely stable, say, for $a<a_c(\epsilon=0)\equiv a_c^0$. When ϵ is increased from zero, no matter how small, the chaotic attractor of the tent map is no longer an attractor of the whole system. The catastrophic bifurcation of riddling occurs for $a<a_c^0$ as $|\epsilon|$ is increased from zero, in which the riddled basin of the $y=0$ chaotic attractor for $\epsilon=0$ is replaced by fractal basins of either the $y=+\infty$ or the $y=-\infty$ attractor, depending on the sign of ϵ. For $a>a_c^0$, the basins of the $y=\pm\infty$ attractors are $y>0$ and $y<0$, respectively, if $\epsilon=0$. A smooth-to-fractal basin boundary metamorphosis [14] occurs because the two simple basins ($y>0$ and $y<0$) are replaced by fractal ones as $|\epsilon|$ is increased from zero. Because of the simplicity of Eq. (2), these bifurcations can in fact be understood, to certain extent, analytically.

A. Riddling for $\epsilon=0$

When a basin is riddled, it does not contain any open sets and is full of holes (in the measure-theoretic sense), yet it must have a positive Lebesgue measure. Alexander *et al.* offered the following definition for a riddled basin [3]: *The basin of attraction of an attractor is riddled if its complement intersects every disk in a set of positive measure*. Roughly, the term ''disk'' here refers to N-dimensional phase-space volumes of all sizes. In order to argue that the basin of an attractor is riddled, the following two conditions must be established: (i) a set of positive measure is attracted to the attractor and (ii) sufficiently many points near the attractor are repelled away from it. In Ref. [3], several analytical examples were constructed for which these two conditions can be tested rigorously. In particular, to prove condition (i), one can compute the transverse Lyapunov exponent (to be defined below) and show that it is negative [3]. To prove condition (ii), it is necessary to show that there exists an open *dense* set near the attractor, points in which asymptote to another coexisting attractor. In contrast, a fractal basin is open and it is defined with respect to the basin boundary: a basin is fractal if its boundary is a fractal set. *A mathematical feature that distinguishes a riddled basin from a fractal one is then that, the former is a closed set of positive measure while the latter is open*.

In Eq. (2), for $\epsilon=0$, $y=0$ is invariant, so a transverse Lyapunov exponent can be defined, as follows:

$$h_T = \lim_{K\to\infty} \sum_{n=0}^{K-1} \ln\left|\frac{dy_{n+1}}{dy_n}\right| = \lim_{K\to\infty} \sum_{n=0}^{K-1} \ln(ax_n)$$
$$= \int_0^1 \ln(ax)dx$$
$$= \ln a - 1. \tag{3}$$

We see that a blowout bifurcation occurs at $a_c^0 = e$ where $h_T \leqslant 0$ for $a \leqslant a_c^0$ and $h_T > 0$ for $a > a_c^0$. The existence of riddling for $a \leqslant a_c^0$ can be established through the following theorem.

Theorem: Let $\mathcal{A}$ be the chaotic attractor in the invariant subspace $y=0$. The basin of $\mathcal{A}$ is riddled for $1 < a < a_c^0$.

Proof: To establish that when the transverse Lyapunov exponent h_T is negative, there is a set of positive measure that appears asymptotic to $\mathcal{A}$ [condition (i) for riddling], we follow the steps in Ref. [3]. Specifically, for fixed $C>0$, let $\mathcal{M}_C = \mathcal{A} \times \{y: |y| < C\}$ and consider the set of points: $\mathcal{P}_C = \{(x,y) \in \mathcal{M}_C : \mathbf{F}^n(x,y) \in \mathcal{M}_C$ for all n and $L^+\{(x,y)\} \subset \mathcal{A}\}$, where $L^+\{(x,y)\}$ is the forward limit set of (x,y). Let μ_x be the Lebesgue measure of the chaotic attractor in $y=0$, which is absolutely continuous, and let μ_y denote the Lebesgue measure in the y direction. Associate $\mathcal{M}_C$ with the measure: $\mu = \mu_x \times \mu_y$. For any positive $\epsilon_0 < C$, the ratio $\hat{\mu} = \mu(\mathcal{M}_C \cap \mathcal{M}_{\epsilon_0})/\mu(\mathcal{M}_{\epsilon_0})$ defines the ϵ-relative measure of a set $\mathcal{M}$.

For every $(x,0) \in \mathcal{A}$, define

$$\tau(x) = \sup_{n \geqslant 0} \left| \exp\left(-\frac{3}{4} n h_T\right) [D_y f^{(n)}](x,0) \right| \leqslant \infty.$$

Since the limit in Eq. (3) exists for μ almost every $x \in \mathcal{A}$, $\tau(x)$ is finite for μ almost every $x \in \mathcal{A}$. Let $R(s) = \{x \in \mathcal{A}: \tau(x) \leqslant s\}$. For sufficiently small $C>0$ and for all $(x,y) \in \mathcal{M}_C$, we have

$$|f(x,y)| \leqslant \exp(-\tfrac{1}{2} h_T) |[D_y f](x,0)| |y|.$$

Now fix C. Then the existence of h_T implies that if $x \in R(s)$ and $|f^{(k)}(x,y)| \leqslant C$ for all k, we have, by induction, the following:

$$|f^{(n)}(x,y)| \leqslant \exp(-\tfrac{1}{2} n h_T) |[D_y f^{(n)}](x,0)| |y|$$
$$\leqslant s \exp(\tfrac{1}{4} n h_T) |y|.$$

Thus, if $x \in R(s)$ and $|y| < C/s$, then $\mathbf{F}^{(n)}(x,y) \in \mathcal{M}_C$ for all $n \geqslant 0$ and $\lim_{n\to\infty} f^{(n)}(x,y) = 0$, which means that, if $\epsilon_0 \leqslant C/s$, then the ϵ relative measure $\hat{\mu}(\mathcal{P}_C) \geqslant \mu(R(s))/\mu(\mathcal{A})$. Therefore, for a given ϵ_0, if we set $s = C/\epsilon_0$, then $\lim_{\epsilon_0 \to 0} \hat{\mu}(\mathcal{P}_C) = 1$. That is, most points (with respect to the product measure of μ_y and μ_x) sufficiently close to $\mathcal{A}$ stay C-near $\mathcal{A}$ and eventually approach $\mathcal{A}$. This establishes condition (i) for riddling.

To establish condition (ii) for riddling, we consider a small interval $[x_F - \delta_-, x_F + \delta_+]$, located at distance y_0 above the fixed point $x_F = 2/3$ embedded in $\mathcal{A}$, where $0 < y_0 \ll 1$ and δ_- and δ_+ are infinitesimally small. Since $a \lesssim a_c^0$, we have $ax_F > 1$. Let k be the number of iterations that the initial condition (x_F, y_0) is mapped beyond $y=1$, i.e., $k = \text{int}[|\ln y_0|/\ln(ax_F)] + 1$. After k iterations, the initial condition $(x_F + \delta_+, y_0)$ maps to (x_k^+, y_k^+), where $x_k^+ = x_F + (-1)^k 2^k \delta_+$ and

$$y_k^+ = a^k y_0 \prod_{i=0}^{k-1} [x_F + (-1)^i 2^i \delta_+].$$

Since δ_+ is infinitesimal, we have

$$\ln y_k^+ = k \ln(ax_F) + \ln y_0 + \sum_{i=0}^{k-1} \ln\left[1 + \frac{(-2)^i \delta_+}{x_F}\right]$$
$$\approx \frac{(-1)^k 2^{k-1} \delta_+}{3x_F}. \tag{4}$$

Similarly, for the initial condition $(x_F - \delta_-, y_0)$, we have

$$\ln y_k^- \approx \frac{(-1)^{k-1} 2^{k-1} \delta_-}{3x_F}. \tag{5}$$

Equations (4) and (5) are valid when δ_+ and δ_- are chosen such that $2^k \delta_\pm \ll 3x_F$, i.e.,

$$\delta_\pm = 3\alpha x_F 2^{int[|\ln y_0|/\ln(ax_F)]+1} \equiv \alpha g(y_0), \tag{6}$$

where $\alpha \ll 1$ and $g(0) = 0$. If k is even (odd), we have $\ln y_k^+ > 0$ ($\ln y_k^- > 0$) and thus $y_k^+ > 1$ ($y_k^- > 1$). In any case, after k iterations, either $[x_F - \delta_-)$ or $(x_F + \delta_+]$ maps to $y > 1$ and asymptotes to the $y = \infty$ attractor. Let

$$W = \{(x,y)|_{y>0} : x_F - \alpha g(y) \leqslant x < x_F$$
$$\text{or } x_F < x < x_F + \alpha g(y)\}. \tag{7}$$

Thus any point in W maps to $y > 1$ under Eq. (2), and we have $W^+ \equiv W \cup \{y > 1\} \subset B(+\infty)$, where $B(+\infty)$ denotes the basin of the $+\infty$ attractor. Let B_W be the union of all preimages of W^+ under map (2):

$$B_W = \cup_{n=0}^{\infty} \mathbf{F}^{-n}(W^+).$$

Note that $\{T^{-n}\}_{n=0}^{\infty}(x_F)$ consists of points with binary representations all ending in an infinite string 010101 . . . , where a trajectory point x of the tent map is assigned the symbol 0 (1) if $x < 1/2$ ($x > 1/2$). Thus $\{T^{-n}\}_{n=0}^{\infty}(x_F)$ is dense in the unit interval $x \in [0,1]$. Since $\{T^{-n}\}_{n=0}^{\infty}(x_F)$ are the roots of the region B_W in the invariant subspace, B_W is an open dense set. This completes the proof.

After the blowout bifurcation, the system has two attractors at $\pm\infty$ with basin $y > 0$ and $y < 0$, respectively. There is no riddling in this case.

B. Catastrophic bifurcation of riddled basin

The replacement of the riddled basin by fractal ones in the presence of a symmetry-breaking perturbation can be seen qualitatively, as follows. As discussed above, for $\epsilon = 0$, the basin of the chaotic attractor $\mathcal{A}$ in $y=0$ is a closed set with positive Legesgue measure, which is the complement set of two symmetric open dense sets belonging to the attractors at $y=\pm\infty$, respectively. While initial conditions with $y_0>0$ or $y_0<0$ can go to $\mathcal{A}$, they cannot cross the invariant line $y=0$. For $\epsilon\neq 0$, the dense set of unstable periodic orbits originally embedded in $\mathcal{A}$ in $y=0$ spreads out in the vicinity of $y=0$, converting $\mathcal{A}$ into a chaotic transient. Because of this spread of unstable periodic orbits, a trajectory initiated in $y>0$ can penetrate the originally invariant line $y=0$ and approach the $y=-\infty$ attractor, and vice versa. The basin of the $y=-\infty$ attractor in $y>0$ must be open and therefore must be fractal, the same for the basin of the $y=+\infty$ attractor in $y<0$. This can be understood as follows. Consider an open neighborhood $\mathcal{B}$ of one of the attractors at infinity. Choose a point p in its basin and evolve it forward in time. Eventually the resulting trajectory will approach the attractor, which means that, at some finite time, the trajectory will enter $\mathcal{B}$, say at point p'. The point p' in $\mathcal{B}$ must then have an open neighborhood. Since p' is iterated from p in finite time, p must also have an open neighborhood in the basin [15]. Thus, as soon as ϵ becomes nonzero, the riddled basin of $\mathcal{A}$ is destroyed and simultaneously, two fractal basins arise. In what follows we analyze how unstable periodic orbits embedded in the original chaotic attractor in $y=0$ are perturbed by the symmetry breaking, based on which we can establish the existence of open, but not dense, sets that belong to the basins of the attractors at infinities.

1. Unstable periodic orbits and their stabilities under the symmetry-breaking perturbation

For concreteness, we consider map (2) with $\epsilon<0$ and $a\approx a_c^0$. Since unstable periodic orbits are structurally stable, we expect that they shift to a small neighborhood about the original invariant subspace $y=0$ for $\epsilon\neq 0$. In particular, the original fixed point $(x_F,0)$ (a repeller with two unstable directions in x and y), is shifted to: (x_F,y_F), where y_F is given by

$$y_F=\frac{-|\epsilon|}{1-ax_F}. \tag{8}$$

For $a\approx a_c^0$, we have $ax_F>1$ and, hence, $y_F>0$. The Jacobian matrix of Eq. (2), evaluated at (x_F,y_F), is given by

$$\begin{bmatrix} -2 & 0 \\ ay_F & ax_F \end{bmatrix}.$$

We see that due to the skew-product structure of Eq. (2), the eigenvalues of the perturbed fixed point (x_F,y_F) are: $L_x=-2$ and $L_y=ax_F>1$. Thus under the symmetry-breaking perturbation, the shifted fixed point is still a repeller. Now consider period-2 orbits $(x_1^{(2)},0)$ and $(x_2^{(2)},0)$, where $T(x_1^{(2)})=x_2^{(2)}$ and $T(x_2^{(2)})=x_1^{(2)}$. For $\epsilon\neq 0$, the y coordinates of the orbits become

$$y_1^{(2)}=\frac{-|\epsilon|(ax_2^{(2)}+1)}{1-a^2x_1^{(2)}x_2^{(2)}} \quad \text{and} \quad y_2^{(2)}=\frac{-|\epsilon|(ax_1^{(2)}+1)}{1-a^2x_1^{(2)}x_2^{(2)}}.$$

We observe that (1) the orbit is shifted upward (downward) from $y=0$ if it is a repeller (saddle), and (2) the eigenvalues of the orbit remain unchanged.

In general, for a periodic orbit of period p (say, the jth one among all 2^p orbits in the tent map),

$$(x_1^{(p)},0),(x_2^{(p)},0),\ldots,(x_p^{(p)},0),$$

where $T(x_i^{(p)})=x_{i+1}^{(p)}$ $(i=1,\ldots,p-1)$ and $T(x_p^{(p)})=x_1^{(p)}$. For $|\epsilon|\neq 0$, the y locations of the orbit points are given by

$$y_i^{(p)}=\left[-|\epsilon|(1+ax_{i-1}^{(p)}+a^2x_{i-1}^{(p)}x_{i-2}^{(p)}+\cdots+a^{p-1})\times\prod_{l=1,l\neq i}^{p}x_l^{(p)}\right]\Bigg/\left(1-a^p\prod_{m=1}^{p}x_m^{(p)}\right). \tag{9}$$

If the original periodic orbit is a repeller (saddle), i.e., $a^p\prod_{m=1}^{p}x_m^{(p)}>1$ $(a^p\prod_{m=1}^{p}x_m^{(p)}<1)$, it remains a repeller (saddle) but its location is shifted upward (downward), i.e. $y_i^{(p)}>0$ $(y_i^{(p)}<0)$. Since all repellers are located in $y>0$, a trajectory starting in $y<0$ cannot cross $y=0$, but since all saddles are located in $y<0$, a trajectory starting in $y>0$ can move across the x axis and go to the $y=-\infty$ attractor. Thus, due to the symmetry breaking, the $y=-\infty$ attractor acquires basins in $y>0$. In so far as the original symmetric system possesses two distinct classes of unstable periodic orbits (repellers and saddles), the basin of the $y=-\infty$ attractor has a component in $y>0$, regardless of whether $a<a_c^0$ or $a>a_c^0$. In what follows we will argue that the symmetry-breaking induced basin component is not riddled but fractal.

The picture dipicted above, i.e., saddles shifted downward and repellers upward, is specific for our model system [Eq. (2)] for the case of $\epsilon<0$. For $\epsilon>0$, saddles will shift upward and repellers will shift downward. In general, in two dimensions we expect to observe saddles and repellers on both sides of the original invariant subspace when there is a symmetry breaking. Thus there will be fractal basins both above and below the original invariant subspace. In higher dimensions, unstable periodic orbits with different unstable dimensions (a type of nonhyperbolicity known as *unstable dimension variability* [16]), which are originally all located in the invariant subspace, will be shifted to its neighborhood under a symmetry-breaking perturbation.

2. Occurrence of fractal basins

We show that there is still an open set in $0\leqslant(x,y)\leqslant 1$ that goes to the $y=+\infty$ attractor. Consider a horizontal infinitesimal interval slightly above the perturbed fixed point (x_F,y_F): $[x_F-\delta_-,x_F+\delta_+]$ at $y_0\gtrsim y_F$. The trajectory from the initial condition (x_0,y_0), where $x_0=x_F+\delta_+$, is

$$x_n^+ = x_F + (-2)^n \delta_+,$$
$$y_n^+ = \left(a^n \prod_{i=0}^{n-1} x_i \right) y_0 - |\epsilon| \left(1 + \sum_{l=1}^{n-1} a^l \prod_{m=n-l+1}^{n} x_m \right). \tag{10}$$

For $|\epsilon| \gtrsim 0$ and $\delta_+ \gtrsim 0$, we have

$$\ln y_n^+ \approx n \ln(a x_F) + \ln y_0 + \frac{(-1)^n 2^{n-1} \delta_+}{3 x_F} - |\epsilon| \gamma(n),$$

where $\gamma(n) = (1 + \Sigma_{l=1}^{n-1} a^l \Pi_{m=n-l+1}^{n} x_m)/(a x_F)^n$. Let k be the time that the trajectory from the initial condition (x_F, y_0) reaches $y=1$, i.e., $k \ln(ax_F) + \ln y_0 - |\epsilon|\gamma(k) = 0$. We obtain $\ln y_k^+ \approx (-1)^k 2^{k-1} \delta_+ /(3x_F)$. Similarly, after k iterations, the y coordinate of the trajectory point from the initial condition: $(x_F - \delta_-, y_0)$ is: $\ln y_k^- \approx (-1)^{k-1} 2^{k-1} \delta_- /(3x_F)$. Regardless of whether k is even or odd, we see that either $y_k^- > 1$ or $y_k^+ > 1$. Thus an open area exists immediately above the shifted fixed point (x_F, y_F) which belongs to the basin of the $y = +\infty$ attractor. The set of infinite number of preimages of this area is thus an open set that goes to the $y = +\infty$ attractor.

When $\epsilon = 0$, the ''roots'' of the open set, i.e., the fixed point $(x_F, 0)$ and all its preimages, are located in the invariant subspace $y=0$ and are dense, because the tent map in Eq. (2) is noninvertible [17]. For $\epsilon \neq 0$, these ''roots'' are shifted and are distributed in the two-dimensional phase-space region about $y=0$. Thus, the open set is no longer dense. The set of initial conditions in the unit square $0 \leqslant (x,y) \leqslant 1$ that go to the $y = -\infty$ attractor is now open. In fact, it is straightforward to see that the region bounded by the curve $xy < |\epsilon|/a$ in the unit square maps to $y<0$ after one iteration. The basin of the $y = -\infty$ attractor thus consists of this bounded region and all its preimages. The boundaries between the basins of the $y = \pm\infty$ attractors are fractal. We remark, however, that in this case, the basin in $y>0$ of the $y = -\infty$ attractor may appear indistinguishable from that of a riddled basin because most unstable periodic orbits in the original invariant subspace are perturbed only slightly. The above arguments can be casted into the following conjecture.

Conjecture: For $a \lesssim a_c^0$ and $\epsilon \neq 0$ in Eq. (2), the chaotic attractor in the invariant subspace, together with its riddled basin for $\epsilon = 0$, is replaced by a chaotic transient and fractal basins of the attractors at infinities, respectively.

III. CRITICAL BEHAVIORS AND SCALING LAWS

From Sec. II B 1 we see that the presence of a small amount of symmetry-breaking causes a spread of unstable periodic orbits in the neighborhood, of size about ϵ, about the original invariant subspace. The dynamics outside the neighborhood can be approximately described by that of a random walk. To see this, we rewrite the y equation in the analytical model [Eq. (2)] as follows:

$$-\ln y_{n+1} = -\ln y_n - \ln(a x_n + \epsilon / y_n).$$

Letting $Y_n \equiv -\ln y_n$, we obtain

$$Y_{n+1} = Y_n + \nu_n, \tag{11}$$

where $\nu_n = -\ln(ax_n + \epsilon/y_n)$ is a random variable because x_n comes from a chaotic process. For $\epsilon \sim 0$, ν_n is approximately independent of y_n most of time (except when y_n gets close to the original invariant subspace). Equation (11) thus describes, approximately, a random walk. If the average drift $\nu \equiv \langle (Y_{n+1} - Y_n) \rangle = \langle \nu_n \rangle$ is small, the random walk model can be solved by using the diffusion approximation, from which various scaling relations can be derived. Specifically, since ν is small, the evolution of the probability as a function of discrete time n in space Y can be approximated as an evolution in continuous time t. Let $P(Y,t)dY$ be the probability of finding the walking in the interval $[Y, Y+dY]$ at time t, then $P(Y,t)$ obeys, *approximately,* the diffusion equation [18]

$$\frac{\partial P}{\partial t} + \nu \frac{\partial P}{\partial Y} = D \frac{\partial^2 P}{\partial Y^2}, \tag{12}$$

where D is the diffusion coefficient, defined as follows:

$$D = \langle (Y_n - \nu)^2 \rangle. \tag{13}$$

Adopting the above diffusive picture, we see that ν and D are the two key parameters that determine the dynamics. In fact, the average drift ν is analogous to the transverse Lyapunov exponent which can be defined only when $\epsilon = 0$, and the diffusion coefficient D characterizes the degree of the fluctuations of the finite time transverse Lyapunov exponent [4]. Our viewpoint is that, when there is a symmetry breaking so that the notions of invariant subspace and transverse Lyapunov exponent no longer hold, we can still use ν and D to characterize the dynamics in the vicinity of the original invariant subspace. In particular, regarding the ϵ neighborhood of the original invariant suspace as a *pseudoinvariant manifold* under a symmetry breaking, the stability of this manifold can be quantified by ν and D. Defining the *pesudotransverse Lyapunov exponent*

$$\Lambda_q \equiv -\nu, \tag{14}$$

we see that if $\Lambda_q > 0$ $(\nu < 0)$, the pseudoinvariant manifold is transversely unstable because a trajectory leaves the pseudoinvariant manifold exponentially fast. If, however, $\Lambda_q < 0$ $(\nu > 0)$, a trajectory can spend a long time (to be quantified below) near the pseudoinvariant manifold, although the trajectory will eventually leave it. In this sense, the manifold is *quasistable* with respect to transverse perturbations. Thus, we see that, introducing the pseudotransverse Lyapunov exponent, with the parameter D's characterizing its finite-time fluctuations, enables us to quantify the dynamical property of the pseudoinvariant manifold. This may provide a general approach to addressing problems such as the stability of the synchronization manifold in coupled *nonidentical* chaotic oscillators.

A detailed discussion about the validity of the diffusion approximation near the transition point to a chaotic attractor with a riddled basin, at which the average drift (or the transverse Lyapunov exponent) is nearly zero, can be found in

Ref. [4]. In our case, because of the symmetry breaking, the range for the validity of the diffusion approximation is limited. In particular, we note that, approximately, a trajectory cannot enter the ϵ neighborhood of the original invariant subspace $y=0$. However, for $y>\epsilon$, the trajectory experiences both repulsion from and attraction toward the ϵ neighborhood of $y=0$ due to the existence of periodic orbits with different unstable dimensions, namely, repellers and saddles. If $\nu\approx 0$, the amount of repulsion is approximately equal to that of attraction and, hence, we expect the diffusion picture to be valid for $\epsilon\lesssim y<1$. In the walk's space, the range is $Y\in(0,\overline{\epsilon})$, where $\overline{\epsilon}=-\ln\epsilon\gg 1$. As we will describe, by imposing different boundary conditions at $\overline{\epsilon}$, distinct scaling relations can be derived. In the following we consider three quantities that are measureable in numerical or laboratory experiments. For the clarity of presentation, we consider the case where $\epsilon\lesssim 0$ so that the symmetry-breaking induced basin of the $y=-\infty$ attractor lies in $y>0$.

A. Fraction of symmetry-breaking induced basin

We concentrate on the phase-space region $[0\leqslant x\leqslant 1, |\epsilon|\leqslant y\leqslant 1]$, and fix a line segment $0\leqslant x\leqslant 1$ at $y=y_0>0$, and uniformly choose a large number of initial conditions from it. Thus we have the initial condition

$$P(Y,0)=\delta(Y-Y_0), \tag{15}$$

where $Y_0=-\ln y_0$. Since a trajectory reaching $y=1$ quickly approaches the $y=+\infty$ attractor, we have the following absorbing boundary condition at $Y=-\ln 1=0$:

$$P(0,t)=0. \tag{16}$$

Roughly, a trajectory entering the $|\epsilon|$ neighborhood of $y=0$ is lost to the basin of the $-\infty$ attractor. A realistic picture is that the Y location of the absorbing boundary depends on x. For instance, from the model of Eq. (2), we see that a trajectory goes to the $y=-\infty$ attractor whenever $ax_ny_n<|\epsilon|$. In so far as x_n is not too small, this happens when $y_n<|\epsilon|/ax_n\sim|\epsilon|$. Thus, as a crude approximation, we impose an absorbing boundary at $\overline{\epsilon}$:

$$P(\overline{\epsilon},t)=0. \tag{17}$$

Let $F(|\epsilon|)$ be the fraction of initial conditions from the line segment at y_0 that are asymptotic to the $y=-\infty$ attractor. As $|\epsilon|$ is increased, we expect $F(|\epsilon|)$ to increase [note that $F(0)=0$]. For $|\epsilon|\sim 0$, we obtain, by solving the diffusion equation Eq. (12), together with the initial and boundary conditions [Eqs. (15)–(17)], the following scaling law (see Appendix A):

$$F_{|\epsilon|}=\frac{y_0^{\nu/D}-1}{|\epsilon|^{\nu/D}-1}. \tag{18}$$

If $\nu\gtrsim 0$, we have $|\epsilon|^{\nu/D}-1\approx -1$ for $|\epsilon|\sim 0$ and, hence $F_{|\epsilon|}\approx 1-y_0^{\nu/D}=\text{const}$. If $\nu\lesssim 0$, we have: $|\epsilon|^{\nu/D}-1\approx|\epsilon|^{\nu/D}$ and, hence

$$F_{|\epsilon|}\sim|\epsilon|^{-\nu/D}=|\epsilon|^{|\nu|/D}\quad\text{for } \nu\lesssim 0. \tag{19}$$

Thus we see that in the parameter regime where $\nu\approx 0$, the fraction remains roughly constant, regardless of the amount of the symmetry breaking. This also implies the catastrophic nature of the symmetry breaking: riddling is destroyed and a fractal basin component is immediately induced as the system deviates away from the symmetric one (say, in the functional space of system equations), no matter how small the deviation is.

B. Lifetime of chaotic transient

Consider a trajectory originated from the symmetry-breaking induced fractal basin of the $y=-\infty$ attractor in $y>0$. After it falls in the negative vicinity of $y=0$, it typically experiences a chaotic transient there. In particular, if $\nu<0$ ($\Lambda_q>0$), the transient time is short. If, however, $\nu>0$ ($\Lambda_q<0$), the time can be extraordinarily long. Specifically, say we sprinkle a large number N_0 of initial conditions in the $|\epsilon|$ neighborhood of $y=0$, which is equivalent to the following initial condition:

$$P(Y,0)=\delta(Y-\overline{\epsilon}). \tag{20}$$

Then the number of trajectories that still remain in the neighborhood decays exponentially with time,

$$N(t)=N_0e^{-t/\tau}, \tag{21}$$

where τ is the lifetime of the chaotic transient near $y=0$. To obtain the scaling of τ, we note that, due to the symmetry breaking, a trajectory can never reach $y=0$. Thus, roughly, the boundary at $|y|\sim|\epsilon|$ is impenetrable. In the walker's space, there is then no probability flux into the boundary. Thus we have the following reflecting boundary condition at $Y=\overline{|\epsilon|}$:

$$\nu P(\overline{|\epsilon|},t)-D\frac{dP}{dY}_{Y=\overline{|\epsilon|}}=0. \tag{22}$$

Solving the diffusion equation under the initial and boundary conditions [Eqs. (16), (20), and (22)], we obtain the following expression for τ (see Appendix B):

$$\tau\approx\frac{D}{\nu^2}|\epsilon|^{-\nu/D}. \tag{23}$$

We stress that, due to symmetry breaking, no typical trajectory can remain near $y=0$ forever, so the average transient lifetime is finite. Thus Eq. (23) is not valid for $\nu=0$ because of the failure of the diffusion approximation to include the dependence of the step of the random walk ν_n on its position Y_n in Eq. (11). Nonetheless, for $\nu\neq 0$, we expect Eq. (23) to capture, qualitatively, the behavior of the chaotic transients caused by symmetry breaking. In particular, for $|\epsilon|\gtrsim 0$, τ is short if $\nu<0$ and it can be long for $\nu>0$. In the simple model [Eq. (2)], $\nu<0$ for $a>a_c^0$ and $\nu>0$ for $a<a_c^0$. Thus we have $\nu\sim(a_c^0-a)$. For $a>a_c^0$, $|\epsilon|^{-\nu/D}$ is small, and Eq. (23) gives

$$\tau \sim (a - a_c^0)^{-2} \quad \text{for} \quad a > a_c^0. \tag{24}$$

For $a < a_c^0$, the factor $|\epsilon|^{-\nu/D}$ dominates τ because $|\epsilon|$ is small. We have

$$\tau \sim (a_c^0 - a)^{-2} e^{(a_c^0 - a)|\ln|\epsilon||/D}. \tag{25}$$

We see that τ increases *exponentially* as a is decreased from a_c^0. To obtain a rough idea about how long the transient lifetime can be, say we have $D=1$ and $|\epsilon|=10^{-10}$. Thus if $a_c^0 - a = 0.1$, we have $\tau \sim 10^3$. But if $a_c^0 - a = 1.5$, then $\tau \sim 10^{15}$. Because of the long transient, the pseudoinvariant manifold for $a < a_c^0$ can be regarded as quasistable.

It should be noted that both Eqs. (24) and (25) are valid in the parameter region where $a \neq a_c^0$. In general, if we compute τ in a parameter region about a_c^0, we expect to see a *crossover* behavior from short to long lifetime near a_c^0. Also note from Eq. (25) that, a long transient is expected only when the amount of the symmetry breaking is small. The transient, while long, is characteristically different from the superpersistent chaotic transient [19].

C. Fractal dimension

To assess the dimensionality of the boundary between the basin of the $y = +\infty$ attractor and the symmetry-breaking induced basin, we fix a line segment at $y = y_0$, where $\epsilon \ll y_0 < 1$, and examine the set of intersecting points with it of the basin boundary. Let d_0 be the box-counting dimension of the set. We expect $0 < d_0 \leqslant 1$ and, the dimension of the boundary is $D_0 = 1 + d_0$ in the two-dimensional phase space. For a riddled basin, D_0 is typically close to the phase-space dimension [4,20]. Our point is that, despite the presence of a small amount of symmetry breaking, D_0 is still close to 2. *Thus, in a practical sense, the symmetry-breaking induced fractal basin resembles a riddled one.*

The box-counting dimension d_0 can be computed by using the uncertainty algorithm [1,21]. Specifically, let $\Phi(\delta)$ be the probability that two initial conditions of distance δ chosen from the line segment at y_0 go to different attractors, namely, one to $y = +\infty$ and another to $y = -\infty$. Then typically, $\Phi(\delta)$ scales with δ as [1]

$$\Phi(\delta) \sim \delta^{\alpha}, \tag{26}$$

where $0 \leqslant \alpha \leqslant 1$ is the uncertainty exponent [1]. The uncertainty dimension d_u of the basin boundary is defined to be

$$d_u = N - \alpha, \tag{27}$$

where N is the phase-space dimension. For hyperbolic systems, it can be shown rigorously that $d_u = d_0$ [21]. The same relation is conjectured to hold for nonhyperbolic systems as well [21]. Following the analysis detailed in Ref. [4], it can be shown, by utilizing the solution to the diffusion equation Eq. (12) described in Appendix A, that the uncertainty exponent is independent of the symmetry-breaking parameter ϵ and is given by

$$\alpha = \frac{\nu^2}{4D}. \tag{28}$$

Thus in the regime where $\nu \approx 0$ (but $\nu \neq 0$) so that the diffusion approximation is valid, we expect $\alpha \approx 0$ and hence $d_0 \approx 1$. Regarding δ as the accuracy in the specification of the initial condition, a fractal basin boundary with dimension close to that of the phase space (or a near zero uncertainty exponent) means that the uncertainly probability $\Phi(\delta)$ remains approximately constant, regardless of how accurately we can specify the initial condition. Thus, realistically, it is impossible to predict, from a given initial condition, the asymptotic attractor. This fundamental obstacle to prediction is common for riddled basins [4,20] and persists even when the riddled basin is replaced by a fractal one due to the symmetry-breaking.

IV. NUMERICAL EXAMPLES

A. A two-dimensional map

We consider the following two-dimensional map:

$$\begin{aligned} x_{n+1} &= f(x_n) + b y_n^2, \\ y_{n+1} &= a x_n y_n + y_n^3 - \epsilon, \end{aligned} \tag{29}$$

where $f(x_n) = r x_n(1 - x_n)$ is the logistic map, and a, b, r, and $\epsilon \geqslant 0$ are parameters. We fix $r = 3.8$ so that the logistic map apparently possesses a chaotic attractor. Parameter b is fixed at $b = 0.1$. The symmetry-breaking parameter is ϵ.

1. Riddled basin for $\epsilon = 0$

For $\epsilon = 0$, the system possesses a simple reflecting symmetry about $y = 0$ and, hence, it is the one-dimensional invariant subspace in which there is a chaotic attractor given by the logistic map. The y^3 term stipulates that trajectories with large values of y are asymptotic to $|y| = \infty$ rapidly. Thus Eq. (29) possesses two additional attractors: one at $y = +\infty$ and another at $y = -\infty$. There is a blowout bifurcation at $a_c^0 \approx 1.726$, where $h_T \lesssim 0$ for $a \lesssim a_c^0$ and $h_T \gtrsim 0$ for $a \gtrsim a_c^0$, as shown in Fig. 1. Thus for $a \lesssim a_c^0$, the basin of the chaotic attractor in $y = 0$ is riddled, as shown in Fig. 2(a), where $a = 1.7$, a grid of 1000×1000 initial conditions is chosen in $(0 < x < 1, -1 < y < 1)$, and black dots denote initial conditions whose trajectories stay within 10^{-10} of $y = 0$ for successive 1000 iterations (which are numerically considered as having approached the $y = 0$ chaotic attractor). To demonstrate that there are initial conditions arbitrarily near $y = 0$ that are asymptotic to either the $y = \pm\infty$ attractors, we choose a grid of 1000×1000 initial conditions but in the region $(0.5 < x < 0.6, -0.01 < y < 0.01)$ and plot the initial conditions that are asymptotic to the $y = \pm\infty$ attractors, as shown in Fig. 2(b), where black dots in $y > 0$ $(y < 0)$ denote initial conditions to the $y = +\infty$ $(y = -\infty)$ attractor. Figures 2(a) and 2(b) exhibit features typical of riddling [3–10].

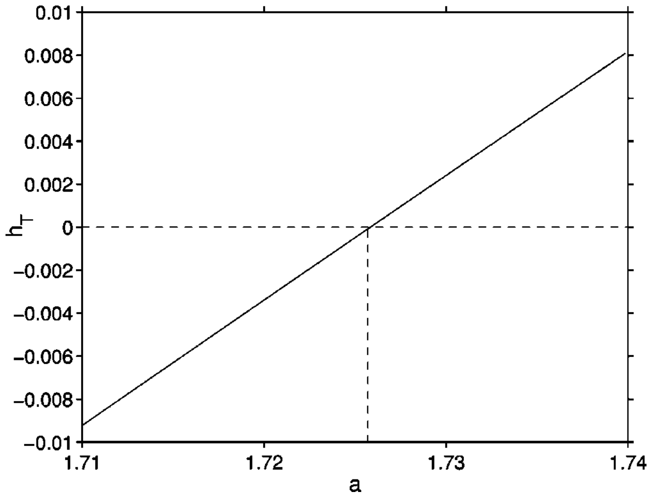

FIG. 1. For $\epsilon=0$ in the numerical model [Eq. (29)], the transverse Lyapunov exponent h_T vs the parameter a.

2. Riddledlike fractal basins in the presence of small symmetry breaking

When $\epsilon \neq 0$, $y=0$ is no longer invariant and, the riddling observed in Figs. 2(a) and 2(b) is destroyed and is replaced by fractal basins, no matter how small ϵ is. To be concrete, we consider $\epsilon \gtrsim 0$. In this case, the $y<0$ region still belongs to the basin of the $y=-\infty$ attractor, but now it has a basin component in $y>0$ due to the symmetry breaking. The basin boundary between the $y=+\infty$ and the $y=-\infty$ attractors in $y>0$ is a fractal, as shown in Fig. 3(a) for $a=1.72 \lesssim a_c^0$, where the black dots denote initial conditions that are asymptotic to the $y=-\infty$ attractor. Although, as we have argued, the basins are fractal, they mimic riddled basins, which is demonstrated by Fig. 3(b), where a small region near $y=0$ in Fig. 3(a) is magnified but now the black dots denote initial conditions that go to the $y=+\infty$ attractor. We

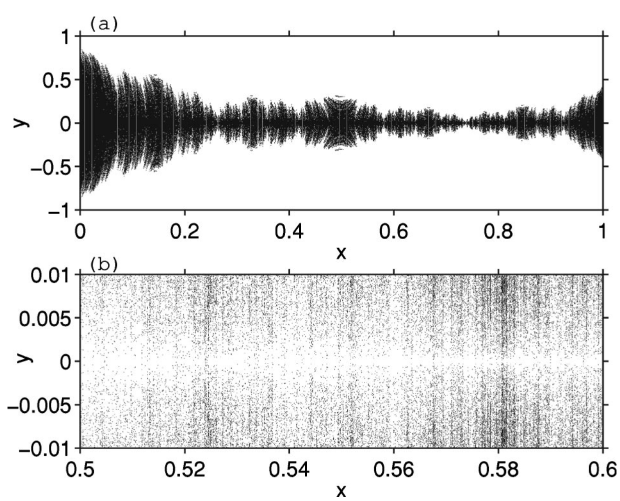

FIG. 2. For $\epsilon=0$ in the numerical model [Eq. (29)], (a) the basin of the $y=0$ chaotic attractor (black dots), and (b) the basin of the $y=\pm\infty$ attractors (black dots) in a small region near $y=0$. Apparently, there are initial conditions arbitrarily near the $y=0$ chaotic attractor that are asymptotic to the $y=\pm\infty$ attractors.

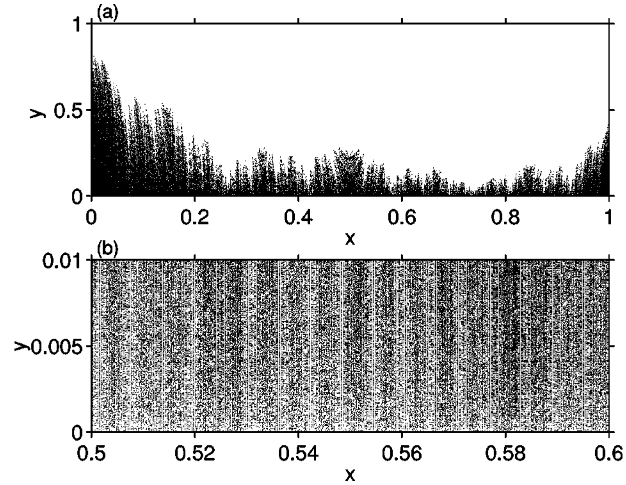

FIG. 3. For $\epsilon=10^{-10}$ and $a=1.72<a_c^0$ in the numerical model [Eq. (29)], (a) the basin of the $y=-\infty$ attractor (black dots), and (b) the basin of the $y=+\infty$ attractor (black dots) in a small region near $y=0$.

see that there are initial conditions arbitrarily near $y=0$ that are asymptotic to the $y=+\infty$ attractor, similar to the behavior depicted in Fig. 2(b). While Figs. 3(a) and 3(b) are for a parameter value below the blowout bifurcation point for $\epsilon=0$, similar basin structures exist for $a \gtrsim a_c^0$, as shown in Figs. 4(a) and 4(b), where Fig. 4(b) is now a blowup of part of Fig. 4(a) because in this case, the majority of the initial conditions go to the $y=+\infty$ attractor. Comparing Figs. 3 and 4 with Figs. 2(a) and 2(b), we observe two features: (1) the symmetry-breaking induced fractal basins are visually similar to riddled basins and (2) the fractal basins exist on both sides of the original blowout bifurcation point (defined when there is no symmetry breaking).

3. Drift and diffusion coefficient

When $\epsilon \neq 0$ so that a symmetry breaking is present, the invariant subspace is destroyed and the notion of the trans-

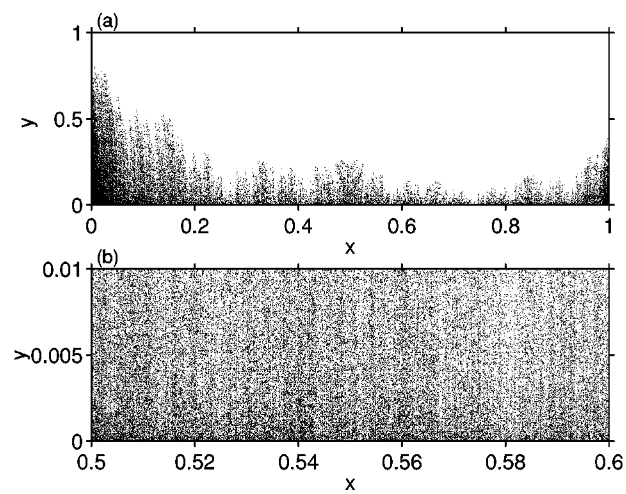

FIG. 4. For $\epsilon=10^{-10}$ and $a=1.73>a_c^0$ in the numerical model [Eq. (29)], (a) the basin of the $y=-\infty$ attractor (black dots), and (b) the basin of the $y=-\infty$ attractor (black dots) in a small region near $y=0$.

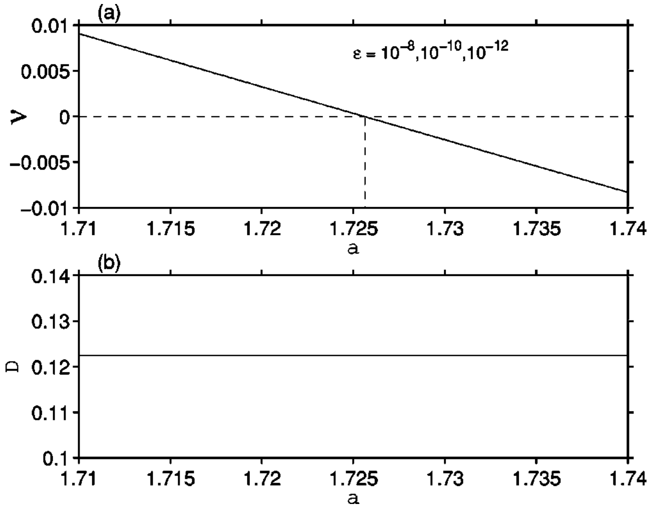

FIG. 5. The average drift ν and the diffusion coefficient D vs a in the numerical model [Eq. (29)] for three values of ϵ.

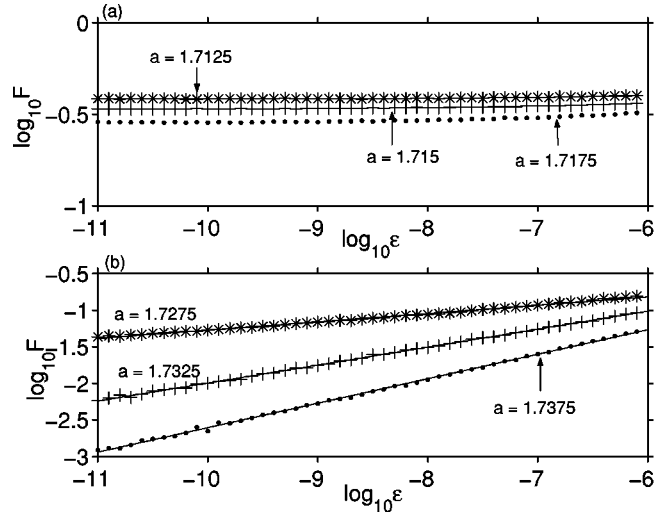

FIG. 6. Scaling of the fraction of the symmetry-breaking induced fractal basin: (b) $\log_{10}F_{|\epsilon|}$ vs $\log_{10}\epsilon$ for three values of $a<a_c^0$ for which $\nu>0$, and (b) algebraic scaling between $F_{|\epsilon|}$ and ϵ for three values of $a>a_c^0$ ($\nu<0$).

verse Lyapunov exponent no longer holds. Our analysis of the simple model [Eq. (2)] indicates, however, that one can use the drift coefficient ν and the diffusion coefficient D to characterize the dynamics in the vicinity of the original invariant subspace. These two parameters are universal in the sense that they determine scaling properties for the symmetry-breaking induced fractal basins, regardless of the details of the system. To compute ν and D for Eq. (29), we fix a horizontal line segment at $1 \gg y_0 \gg \epsilon \gtrsim 0$ and distribute a large number of initial conditions (x_0,y_0) on it. The parameters are then given by

$$\nu = \langle \Delta Y \rangle \equiv \langle Y_1 - Y_0 \rangle \equiv \langle -\ln|y_1(x_0)/y_0| \rangle, \qquad (30)$$

$$D = \frac{1}{2}\langle (\Delta Y - \nu)^2 \rangle,$$

where $\langle \cdot \rangle$ is the ensemble average. Note that when $\epsilon=0$, due to the ergodicity of the chaotic attractor in the invariant subspace, we have $\nu=-h_T$. Figures 5(a) and 5(b) show ν and D versus the parameter a in Eq. (29), respectively, for three values of ϵ: 10^{-8}, 10^{-10}, and 10^{-12}, where 10^5 initial conditions are chosen from the line segment $x_0 \in [0,1]$ at $y_0=10^{-3}$. We see that ν changes sign at $a \approx a_c^0$, while D remains essentially constant about a_c^0. The curves at different values of ϵ are indistinguishable, indicating that ν and D are ''invariant'' parameters to characterize the dynamics under small symmetry-breaking.

4. Fraction of symmetry-breaking induced basin [Eqs. (18) and (19)]

To numerically compute the fraction of the basin induced by the symmetry breaking, we choose 10^7 initial conditions at $y_0=10^{-1}$ and compute the number of trajectories that appear asymptotic to the $y=-\infty$ attractor. Our theory predicts that, for $\nu \gtrsim 0$, the fraction is almost independent of the magnitude of the symmetry breaking ϵ. This is shown in Fig. 6(a) for three values of $a \lesssim a_c^0$ for which $\nu \gtrsim 0$. However, for $\nu \lesssim 0$ [$a \gtrsim a_c^0$], the fraction scales algebraically with ϵ, as verified for three values of a in Fig. 6(b). The numerically obtained algebraic exponents are, however, several times larger than the predicted one (ν/D). The source of deviation comes from the usage of the diffusion approximation to solve the random-walker problem [Eq. (11)], which is crude in the sense that the amount of the random walk ν_n is not completely independent of position variable Y_n. Nonetheless, the predicted algebraic behavior appears valid.

5. Chaotic transient lifetime

Solutions based on the diffusion approximation suggest that the lifetime of the chaotic transients induced by symmetry breaking, as a function of its magnitude, is short and remains roughly constant for $\nu \lesssim 0$ ($a \gtrsim a_c^0$). However, the lifetime can be long and increases algebraically with ϵ for $\nu \gtrsim 0$ ($a \lesssim a_c^0$). These behaviors are shown in Fig. 7, where for each pair of fixed a and ϵ, τ is the numerical average lifetime computed from 10^7 initial conditions chosen from the line $y_0=10^{-1}$. Again, although the algebraic scaling behavior is predicted correctly using the diffusion theory, it fails to predict the scaling exponent, for the reason quoted in Sec. IV A 4.

For fixed ϵ, Eq. (23) predicts that the scaling behaviors of the transient lifetime with the parameter variation are characteristically different for $\nu<0$ and $\nu>0$. In particular, for $\nu<0$ ($a>a_c^0$), the lifetime scales algebraically with the parameter variation, while for $\nu>0$ ($a<a_c^0$), the scaling behavior is exponential and, therefore, the lifetime can be long. Near $a=a_c^0$, as we explain in Sec. III B, Eq. (23) fails but we expect to see a crossover between the distinct scaling behaviors. Figure 8 shows the distinct scaling behaviors, together with the crossovers, for three values of ϵ, where we see that for small ϵ, the lifetime of the chaotic transient can indeed be extraordinarily long for $a<a_c^0$.

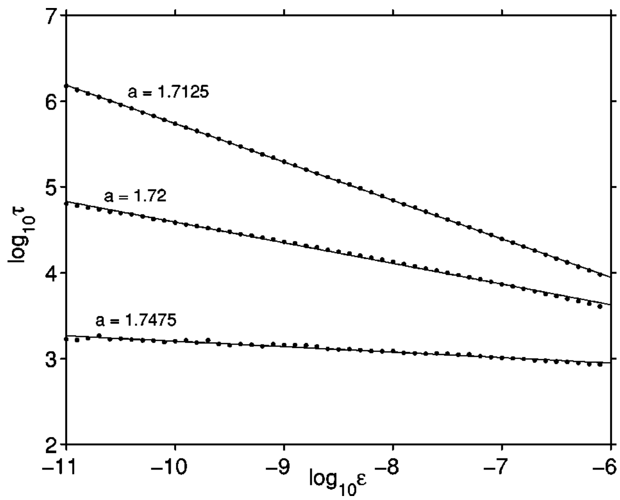

FIG. 7. For three fixed values of parameter a, the scaling of the chaotic transient lifetime with ϵ.

6. Fractal dimension

The main feature of the fractal dimension of the basins induced by symmetry breaking is that the dimension is close to the phase-space dimension and, therefore, these basins are indistinguishable from riddled basins based on a dimension measurement. Numerically, for fixed values of ϵ and a, we randomly distribute a large number N_0 of initial-condition pairs at $y_0=10^{-3}$. A pair is said to be uncertain if the two initial conditions are asymptotic to different attractors. For a fixed distance δ between the two initial conditions, we increase N_0 until the the number of uncertain initial-condition pairs reaches a fixed number, say 2000. The uncertain probability is given, approximately, by $\Phi(\delta)\approx 2000/N_0$. Figures 9(a) and 9(b) show, for $a=1.73>a_c^0$ and $a=1.72<a_c^0$, respectively, $\Phi(\delta)$ versus δ on a logarithmic scale, where $\epsilon=10^{-8}$ is fixed for both figures. A linear fit gives the approximate value of the uncertainty exponent. We obtain, for both Figs. 9(a) and 9(b), $\alpha\approx 0.002\pm 0.01$. The fractal dimensions of the basin boundaries are thus $d_u=2-\alpha\approx 1.998\approx 2$

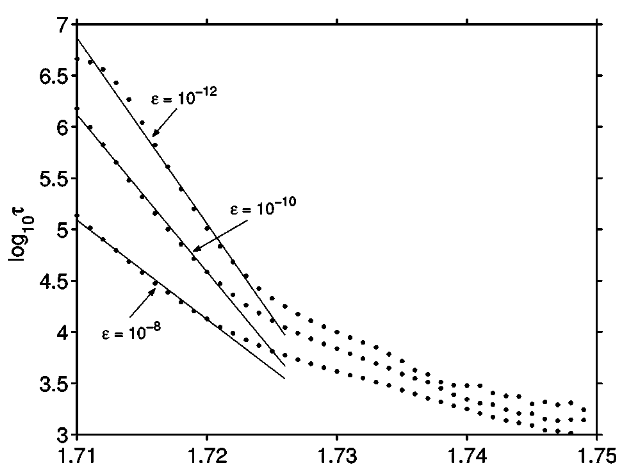

FIG. 8. For three fixed values of ϵ, the scaling of the chaotic transient lifetime with a parameter a.

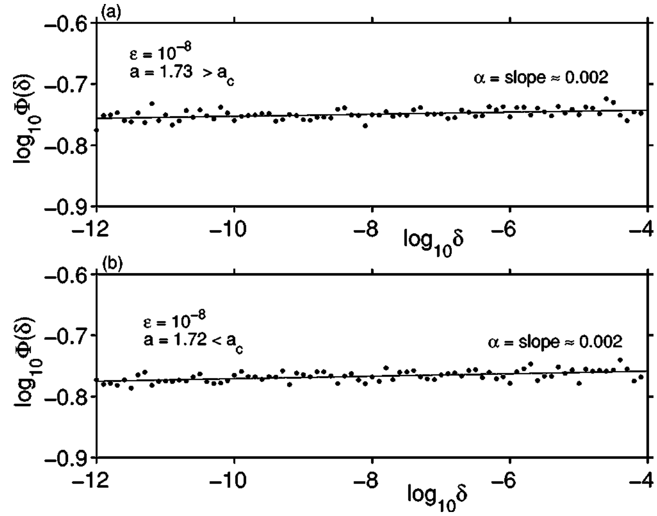

FIG. 9. Scaling of the uncertain probability $\Phi(\delta)$ (plotted on a logarithmic scale) for $\epsilon=10^{-8}$: (a) $a=1.73>a_c^0$ and (b) $a=1.72<a_c^0$.

for both $a\gtrsim a_c^0$ and $a\lesssim a_c^0$. The closeness of the dimensions of the fractal basin boundaries to that of the phase space signifies an extreme difficulty in predicting the asymptotic attractor for given initial conditions, which is typical for riddled basins [4].

B. A system of coupled chaotic oscillators

We now examine the effect of symmetry breaking on riddling in continuous-time dynamical systems. The example we study is the system of four coupled Rössler-like electronic oscillators utilized by Heagy *et al.* [5] in the first experimental investigation of riddled basins. The differential equations of the system are

$$\frac{dx_i}{dt}=-\omega_i y_i - z_i - r x_i + K(x_{i+1}+x_{i-1}-2x_i),$$

$$\frac{dy_i}{dt}=x_i+a y_i, \tag{31}$$

$$\frac{dz_i}{dt}=-b z_i + c g(x_i), \quad i=1,2,3,4,$$

where $(x_i, y_i, z_i)=\mathbf{x}_i\in R^3$ are the dynamical variables of individual oscillators with frequencies ω_i $(i=1,\dots,4)$, K is the linear coupling parameter, and the source of nonlinearity comes from the piecewise linear function $g(x)$: $g(x)=x-d$ for $x>d$ and $g(x)=0$ otherwise; r, a, b, c, and d are intrinsic parameters of the oscillators. Nonidentities among oscillators are stipulated by setting ω_i's $(i=1,\dots,4)$ slightly different. Following Heagy *et al.* [5], we use periodic boundary conditions and, for concreteness, we fix $a=0.13$, $b=1.0$, $c=15.0$, $r=0.05$, and $\omega_i\approx 0.5$ $(i=1,\dots,4)$, and choose K as the bifurcation parameter. At this parameter setting, each oscillator exhibits a chaotic attractor when uncoupled [5].

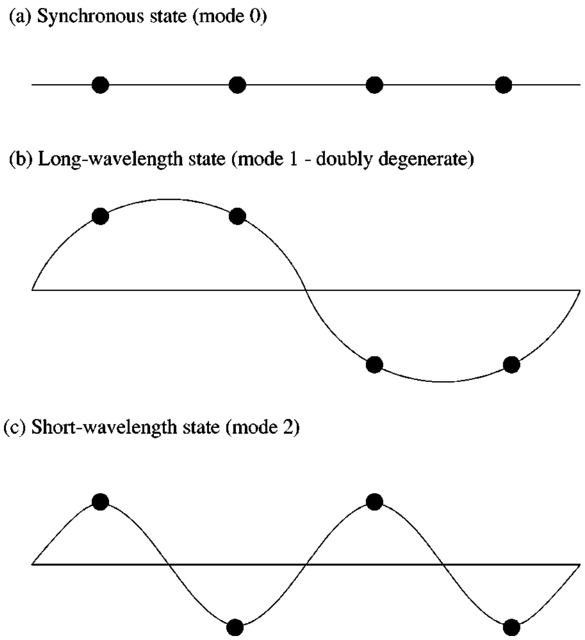

FIG. 10. Schematic illustration of the spatially distinct configurations of attractors in a system of four coupled Rössler-like chaotic oscillators [Eq. (31)].

We first consider the case of identical oscillators: $\omega_i = \omega_0 = 0.5$ $(i=1,\ldots,4)$. It can be seen easily that the coupled system possesses the following shift symmetry [11]: the system is invariant when the oscillator indices are uniformly shifted. Because of this, the synchronous state: $\mathbf{x}_1 = \cdots = \mathbf{x}_4$ lies in a three-dimensional invariant manifold in which there is a chaotic attractor (the one from individual, uncoupled oscillators). The dynamical behavior of oscillators near the synchronous state can be characterized by the spatial Fourier modes. In particular, for N (even) oscillators, there are N Fourier modes with mode indices ranging from 0 to $(N-1)$: the zeroth mode represents the motion on the synchronization manifold and the rest are transverse to it, and the first through the $(N/2-1)$th modes are doubly degenerate. Thus, for $N=4$, there are three spatially distinct states: a synchronous state (mode 0), two long-wavelength states (doubly degenerate), and a short-wavelength state (mode 2), as shown schematically in Fig. 10, where the vertical direction corresponds to some dynamical variables of the oscillators, say $y_i(t)$, and the horizontal axis represents the relative location of the oscillators (or i, the oscillator index). Since both modes 1 and 2 correspond to motion in the transverse direction, their stabilities can be quantified by the transverse Lyapunov exponents [5]. Figure 11 shows the first two transverse Lyapunov exponents of both modes as a function of the coupling parameter K, where for reference, the first two exponents of the synchronous chaos (mode 0) are also shown, which do not depend on K due to the invariance of the synchronization manifold. In the parameter interval $[K_s, K_d]$ (denoted by the two vertical dashed lines in the figure), all transverse exponents are negative, indicating that the chaotic attractor in the synchronization manifold is also an attractor in the full phase space of the coupled system.

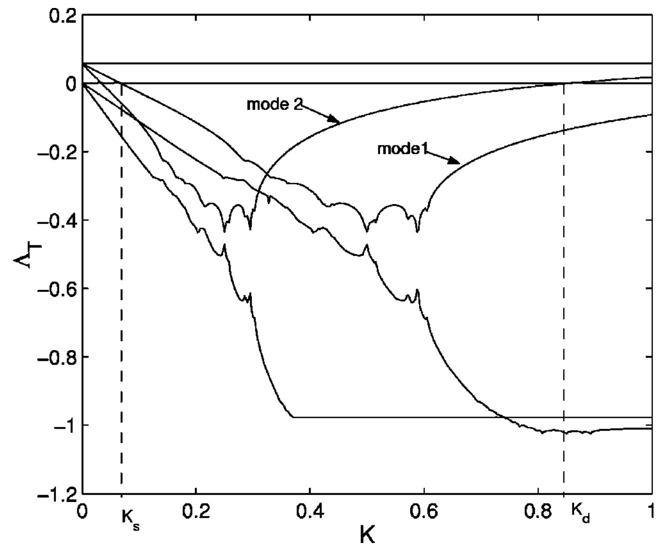

FIG. 11. For a system of four coupled identical Rössler-like chaotic oscillators [Eq. (31)], the first two transverse Lyapunov exponents for modes 1 and 2. The horizontal lines denote the first two Lyapunov exponents of the chaotic attractor in the synchronization manifold, which do not depend on the coupling parameter.

Riddled basin can therefore occur if there are other coexisting attractors not in the synchronization manifold.

In Ref. [5], it is shown that for $K \approx K_d$, indeed there exist two period-1 coexisting attractors that belong spatially to mode 2, as shown schematically in Fig. 12. Because of the fixed range of their dynamical variables in the lattice, these two attractors can be easily distinguished by using the spatial Fourier coefficient of mode 2: $S_2 = y_1(t) - y_2(t) + y_3(t) - y_4(t)$. Thus the periodic attractors in Figs. 12(a) and 12(b) satisfy $\langle S_2 \rangle > 0$ and $\langle S_2 \rangle < 0$, respectively. (The synchronous chaotic attractor has $\langle S_2 \rangle = 0$.) To visualize riddled basins,

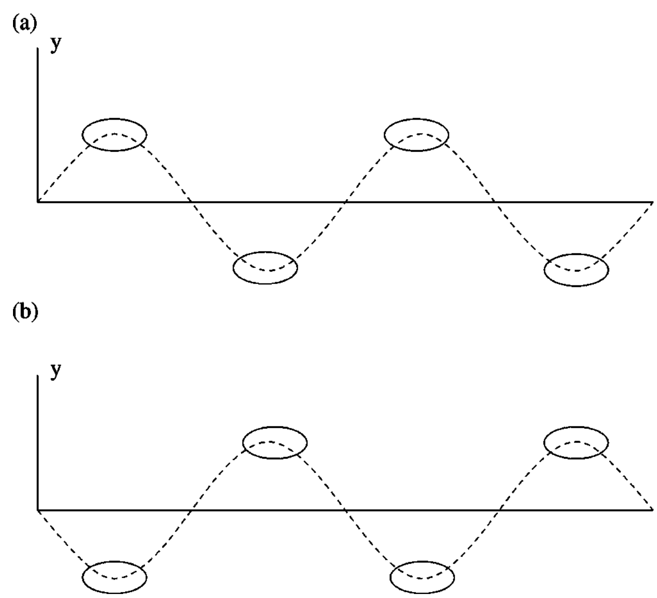

FIG. 12. Schematic illustration of the two coexisting period-1 attractors (not in the synchronization manifold) that have a mode-2 spatial structure.

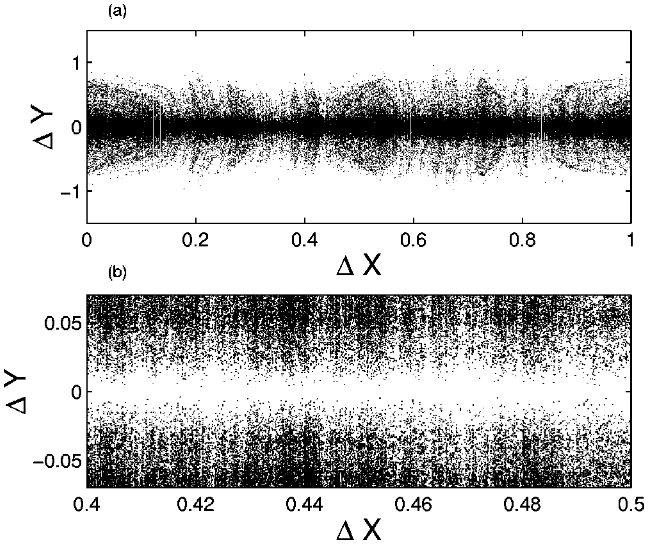

FIG. 13. For a system of four coupled identical Rössler-like chaotic oscillators [Eq. (31)], for $K=0.85<K_d$. (a) Riddled basins of the chaotic attractor (black dots) obtained in the $(\Delta x,\Delta y)$-plane; (b) blowup of part of (a), where now the black dots above and below $\Delta y=0$ denote basins of the period-1 attractors with $\langle S_2\rangle>0$ and $\langle S_2\rangle<0$, respectively.

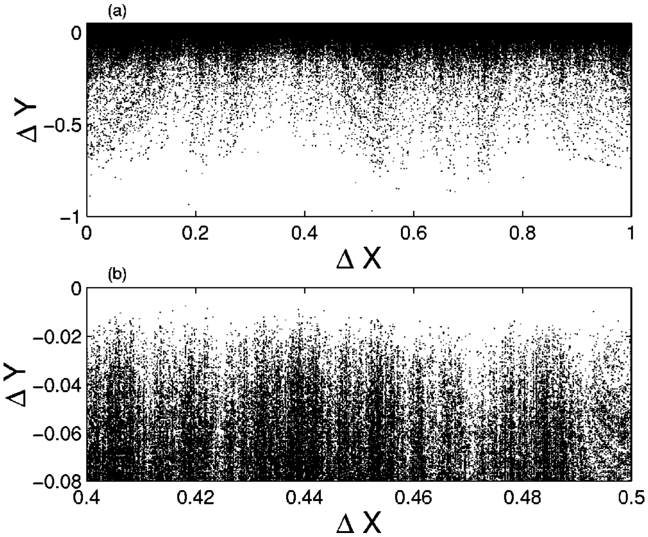

FIG. 14. Fractal basins in the $(\Delta X,\Delta Y)$ plane of the two period-1 attractors in a system of four coupled nonidentical Rössler-like chaotic oscillators [Eq. (31)] when there is a symmetry breaking ($\epsilon=10^{-2}$). (a) Basin (black dots) of the period-1 attractor with $\langle S_2\rangle>0$. (b) A blowup of part of (a) near $\Delta Y=0$, where now the black dots denote the basin of the period-1 attractor with $\langle S_2\rangle<0$. The coupling parameter is $K=0.85<K_d$.

we construct a two-dimensional grid $(\Delta X,\Delta Y)$, following Ref. [5], where the synchronization manifold corresponds to $(\Delta X,\Delta Y)=(0,0)$, and for a given point (x_s,y_s,z_s) on the chaotic attractor in the three-dimensional synchronization manifold, initial conditions for each oscillators are chosen, as

$$x_i=x_s+\Delta X,$$

$$y_i=y_s+(-1)^i\Delta Y, \tag{32}$$

$$z_i=z_s,$$

for $i=1,\ldots,4$. In our numerical experiments, we distribute a grid of 700×700 of initial points in the $(\Delta X,\Delta Y)$ plane in the range: $0\leqslant\Delta X\leqslant1$ and $-1\leqslant\Delta Y\leqslant1$. For each point, the initial conditions for the oscillators are chosen according to Eq. (32), and Eq. (31) is integrated. The time average of the mode-2 Fourier coefficient $\langle S_2\rangle$ is then computed, after disregarding a finite amount of transient, to determine to which attractor the initial point is asymptotic to. Figure 13(a) shows the basin of the chaotic attractor in the synchronization manifold (black dots), which appears similar to Fig. 2(a) and is apparently riddled. A blowup of part of Fig. 13(a) is shown in Fig. 13(b), where, for clarity of presentation, the black dots above and below $\Delta Y=0$ belong to the basins of the period-1 attractors with $\langle S_2\rangle>0$ and $\langle S_2\rangle<0$, respectively. We again note the similarity between Fig. 13(b) and Fig. 2(b).

We now describe the effect of symmetry breaking on riddling. For a system of coupled chaotic oscillators, a convenient way to introduce a small amount of symmetry breaking is to make the oscillators slightly nonidentical, such as making the frequency parameter ω_i slightly mismatched. We choose ω_i from the interval $[\omega_0-\epsilon/2,\omega_0+\epsilon/2]$, where $\epsilon\ll\omega_0$ is equivalent to a symmetry-breaking parameter. Similar to the map example in Sec. IV A, the presence of a symmetry breaking immediately destroys the riddled basin of the chaotic attractor in the synchronization manifold and replaces it by fractal basins of the coexisting period-1 attractors. Figure 14(a) shows, for $K=0.85<k_d$ and $\epsilon=10^{-2}$, the basin (black dots) of period-1 attractor with $\langle S_2\rangle>0$, and a blowup of part of (a) near $\Delta Y=0$ is shown in Fig. 14(b), where now the black dots denote the basin of the period-1 attractor with $\langle S_2\rangle<0$. We see that Figs. 14(a) and 14(b) are similar to Figs. 3(a) and 3(b), respectively. The fractal basins persist even for $K\gtrsim K_d$, where one of the transverse Lyapunov exponents (the one associated with mode 2) is slightly positive, as shown in Figs. 15(a) and 15(b) for $K=0.875\gtrsim K_d$ and $\epsilon=10^{-2}$, where the basin of the period-1 attractor with $\langle S_2\rangle>0$ and its blowup near $\Delta Y=0$ are shown, respectively. We note that Figs. 15(a) and 15(b) are similar to Figs. 4(a) and 4(b), respectively, which are the corresponding basin plots for the map example.

We now present numerically obtained scaling laws for various critical behaviors in the system of coupled Rössler-like chaotic oscillators. Our main point is that these scaling laws are qualitatively similar to those obtained from the map example in Sec. IV A, thereby furnishing more support for the theoretical predictions in Sec. III.

(1) Fraction of symmetry-breaking induced basins. To compute $F_{|\epsilon|}$, we make use of the $(\Delta X,\Delta Y)$ plane. In particular, for a given value of ϵ, to compute the fraction of basin in the negative ΔY region of the period-1 attractor with $\langle S_2\rangle>0$, we fix a horizontal line at $\Delta Y_0\lesssim0$, randomly distribute 10^4 initial values of ΔX_0 in the interval $[-5,5]$ on the

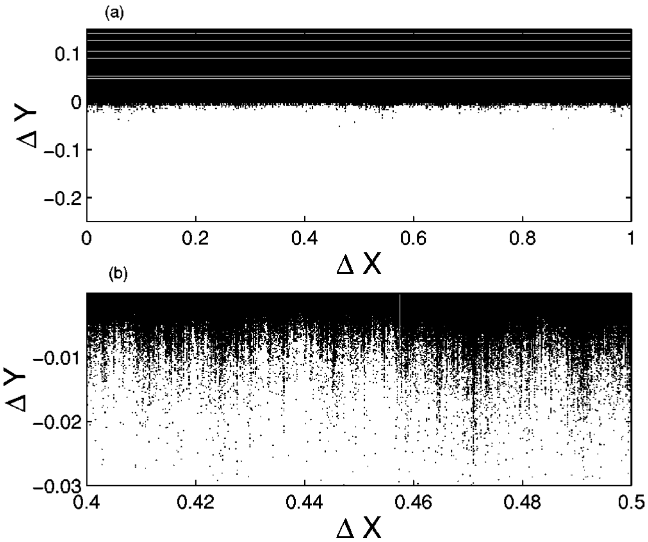

FIG. 15. Symmetry-breaking induced fractal basins for $K=0.87>K_d$ in a system of four coupled nonidentical Rössler-like chaotic oscillators ($\epsilon=10^{-2}$). (a) Basin (black dots) of the period-1 attractor with $\langle S_2\rangle>0$. (b) A blowup of part of (a) near $\Delta Y=0$.

line, and compute the number of ΔX_0 values whose corresponding initial conditions yield trajectories that are asymptotic to the period-1 attractor. The scaling behaviors of $F_{|\epsilon|}$ with ϵ for $K\lesssim K_d$ and $K\gtrsim K_d$ are shown on a logarithmic scale in Figs. 16(a) and 16(b), respectively. They are similar to those with the map example [Figs. 6(a) and 6(b)], which agree reasonably with the theoretical predictions [Eqs. (18) and (19)].

(2) Chaotic transient lifetime. We again distribute a large number of initial values of ΔX_0 (in the interval $[-5,5]$) on a line at $\Delta Y_0=0$ and for each corresponding initial condition, we compute how long it takes for the trajectory to leave the synchronization manifold. Numerically, this time is approximately the time for the value of $|\langle S_2\rangle|$ of the trajectory to exceed a small, empirically set threshold, say, 10^{-3}. The average value of all these times is taken to be the average chaotic transient lifetime τ. Figures 17(a) and 17(b) show, for $K=0.858\lesssim K_d$ and $K=0.875\gtrsim K_d$, respectively, the scaling behaviors of τ with the symmetry-breaking parameter ϵ. We observe algebraic scaling behaviors in both cases, and also a similarity between Figs. 17 and 7. The drastic increase of the transient lifetime as K is reduced through K_d is shown in Fig. 18, where $\epsilon=0.1$.

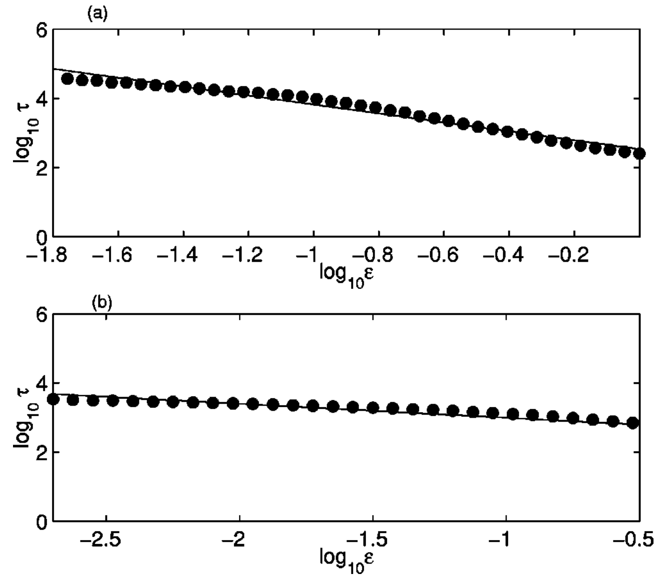

FIG. 17. For a system of four coupled nonidentical Rössler-like chaotic oscillators, the scaling of the average chaotic transient lifetime τ with the symmetry-breaking parameter ϵ: (a) $K=0.858<K_d$ and (b) $K=0.875>K_d$.

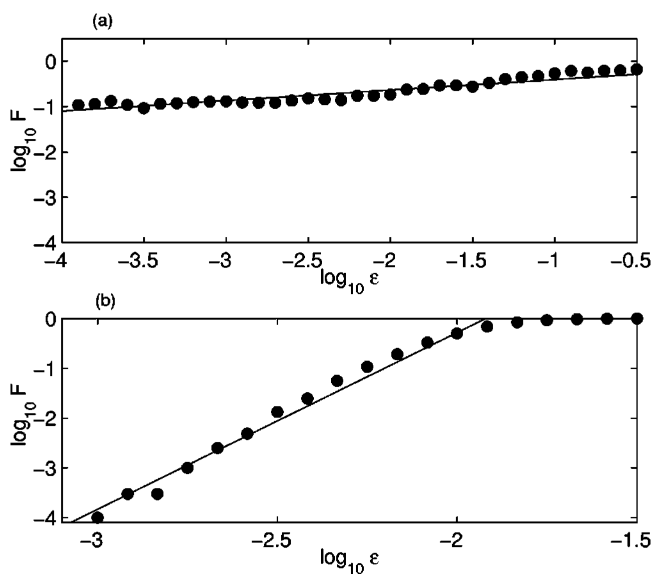

FIG. 16. For a system of four coupled nonidentical Rössler-like chaotic oscillators, the scaling of the fraction of the symmetry-breaking induced fractal basins with the symmetry-breaking parameter ϵ for (a) $K=0.858<K_d$ and (b) $K=0.875>K_d$.

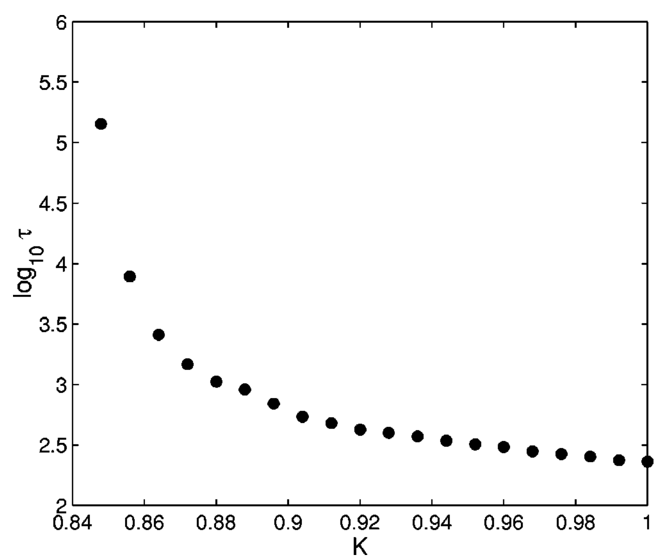

FIG. 18. For a system of four coupled nonidentical Rössler-like chaotic oscillators, the scaling of the average chaotic transient lifetime τ with the coupling parameter K for $\epsilon=0.1$. We see that τ increases dramatically as K is decreased through K_d.

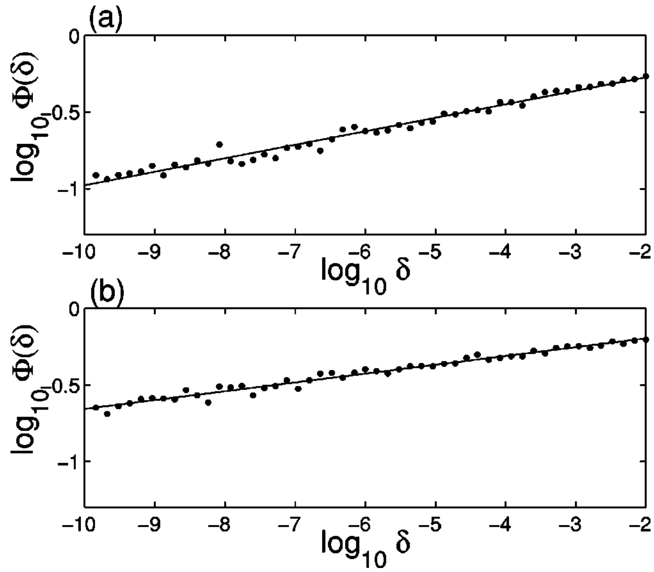

FIG. 19. For a system of four coupled nonidentical Rössler-like chaotic oscillators, we show the scaling of the uncertain probability for (a) $K=0.835$ and (b) $K=0.868$. The uncertainty exponents are approximately 0.09 and 0.06 for (a) and (b), indicating that the dimensions of the fractal basin boundaries in the full 12-dimensional phase space are approximately 11.91 and 11.94, respectively.

(3) Fractal dimension. The fractal dimension of boundaries between the symmetry-breaking induced basins is computed from the scaling of the uncertainty probability $\Phi(\delta)$. For the system of coupled Rössler oscillators, initial-condition pairs are chosen from a fixed line at $\Delta Y_0 \lesssim 0$ in the $(\Delta X, \Delta Y)$ plane. Results from two typical cases are shown in Figs. 19(a) and 19(b) for $\epsilon=0.1$, $K=0.835$, and 0.868, respectively. The uncertainty exponents are $\alpha \approx 0.09$ and $\alpha \approx 0.06$. Since the phase-space dimension of Eq. (31) is 12, these values of α imply that the dimensions of the fractal basin boundaries are 11.91 and 11.94 for $K=0.835$ and $K=0.868$, respectively.

V. DISCUSSION

The main contribution of this paper is a systematic analysis of the effect of symmetry breaking on riddling. Specifically, we consider symmetry breaking that destroys the the essential dynamical element required for riddling: the existence of an invariant subspace. Our principal result is that, while riddling is robust against perturbations that preserve the symmetry and invariance of the system, it is structurally unstable under perturbations that destroy the symmetry [22]. Such perturbations destroy riddling, create fractal basins with physical properties similar to those of a riddled one, and induce long chaotic transients that scale exponentially with parameter variations. An implication of this work is that riddled basins may not actually be observable in physical experiments, say, in systems of coupled, slightly nonidentical chaotic oscillators. What can be observed is fractal basins that appear like riddled ones [23–25].

ACKNOWLEDGMENTS

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APPENDIX A: SCALING OF FRACTION OF SYMMETRY-BREAKING INDUCED BASIN

To solve the diffusion equation [Eq. (12)], we use the standard Laplace transform technique [18]. Let

$$\overline{P}(Y,s) \equiv \int_0^\infty P(Y,t) e^{-st} dt \tag{A1}$$

be the Laplace transform of $P(Y,t)$, where $\mathrm{Re}(s)>0$. Substituting Eq. (A1) into Eq. (12) and using the initial condition [Eq. (15)], we obtain

$$D\frac{d^2\overline{P}}{dY^2} - \nu\frac{d\overline{P}}{dY} - s\overline{P} = -\delta(Y-Y_0). \tag{A2}$$

Utilizing the boundary conditions [Eqs. (16) and (17)], we obtain

$$\overline{P}(Y,s) = \begin{cases} A[e^{\lambda_1 Y} - e^{(\lambda_1-\lambda_2)\overline{\epsilon}+\lambda_2 Y}], & Y>Y_0 \\ AK(e^{\lambda_1 Y} - e^{\lambda_2 Y}), & 0<Y\leqslant Y_0, \end{cases} \tag{A3}$$

where A, K, λ_1, and λ_2 are given by

$$A = \frac{e^{-\lambda_2 Y_0} - e^{-\lambda_1 Y_0}}{D(\lambda_2-\lambda_1)[e^{(\lambda_1-\lambda_2)\overline{\epsilon}} - 1]}, \tag{A4}$$

$$K = \frac{e^{\lambda_1 Y_0} - e^{(\lambda_1-\lambda_2)\overline{\epsilon}+\lambda_2 Y_0}}{e^{\lambda_1 Y_0} - e^{\lambda_2 Y_0}},$$

$$\lambda_1 = \frac{\nu + \sqrt{\nu^2+4Ds}}{2D},$$

$$\lambda_2 = \frac{\nu - \sqrt{\nu^2+4Ds}}{2D}.$$

The instantaneous probability flux per unit time through the absorbing boundary at $\overline{\epsilon}$ is

$$f_\epsilon(t) = \left[\nu P(Y,t) - D\frac{dP}{dY}\right]\bigg|_{Y=\overline{\epsilon}} = -D\frac{dP}{dt}\bigg|_{Y=\overline{\epsilon}}. \tag{A5}$$

The total probability flux through $\overline{\epsilon}$ in time t is

$$F_\epsilon(t) = \int_0^t f_\epsilon(t')dt'. \tag{A6}$$

The Laplace transform of $F_\epsilon(t)$ is given by

$$F_\epsilon(s) = -\frac{D}{s}\left.\frac{d\overline{P}(Y,s)}{dY}\right|_{Y=\overline{\epsilon}} = \frac{e^{\lambda_1\overline{\epsilon}}(e^{-\lambda_2 Y_0}-e^{-\lambda_1 Y_0})}{s[e^{(\lambda_1-\lambda_2)\overline{\epsilon}}-1]}. \tag{A7}$$

Note that $F_\epsilon(s)$ has a pole at $s=0$ and a branch singularity at $s^* = -\nu^2/(4D) < 0$ (the value of s that satisfies $\lambda_1 = \lambda_2$ is not a pole). Performing the inverse Laplace transform of $F_\epsilon(s)$,

$$F_\epsilon(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F_\epsilon(s) e^{st} ds, \tag{A8}$$

where $\sigma > 0$ so that all singularities are to the left of the integration path, we see that the contribution from the branch singularity is proportional to $e^{s^* t}$, which vanishes in the limit $t\to\infty$. Since the fraction of the symmetry-breaking induced basin of the $y=-\infty$ attractor is $F_\epsilon = \lim_{t\to\infty} F_\epsilon(t)$, we see that the only contribution to F_ϵ comes from the pole at $s=0$ [which determines $F_\epsilon(t)$ in the infinite t limit by the nature of the Laplace transform]. Evaluating the residual at $s=0$ leads to the scaling relation [Eq. (18)].

APPENDIX B: SCALING OF CHAOTIC TRANSIENT LIFETIME

We solve the diffusion equation [Eq. (12)] under the initial and boundary conditions [Eqs. (16), (20), and (22)]. The solution is

$$\overline{P}(Y,s) = \begin{cases} A_1(e^{\lambda_1 Y} + C_1 e^{\lambda_2 Y}), & Y > Y_0 \\ A_1 C_2 (e^{\lambda_1 Y} - e^{\lambda_2 Y}), & 0 < Y \leq Y_0, \end{cases} \tag{B1}$$

where the coefficients C_1, C_2, and A_1 are given by

$$C_1 = -\frac{\lambda_2}{\lambda_1} e^{(\lambda_1-\lambda_2)\overline{\epsilon}},$$

$$C_2 = \frac{e^{\lambda_1 Y_0} + C_1 e^{\lambda_2 Y_0}}{e^{\lambda_1 Y_0} - e^{\lambda_2 Y_0}}, \tag{B2}$$

$$A_1 = \frac{e^{-\lambda_1 Y_0} - e^{-\lambda_2 Y_0}}{D(C_1+1)(\lambda_2-\lambda_1)}.$$

The number of trajectories from the N_0 initial conditions distributed at ϵ which remain between $y=\epsilon$ and $y=1$ at time t, in which the corresponding random walkers are still moving diffusively, is

$$N(t) = N_0 \int_0^{\overline{\epsilon}} P(Y,t) dY. \tag{B3}$$

Its Laplace transform is given by

$$\overline{N}(s) = N_0 \int_0^{\overline{\epsilon}} \overline{P}(Y,s) dY = \frac{\lambda_2(e^{\lambda_1\overline{\epsilon}}-1) - \lambda_1(e^{\lambda_2\overline{\epsilon}}-1)}{D\lambda_1\lambda_2(\lambda_1 e^{\lambda_2\overline{\epsilon}} - \lambda_2 e^{\lambda_1\overline{\epsilon}})}. \tag{B4}$$

To determine the singularities of $\overline{N}(s)$, we note that under the transform $\sqrt{\nu^2+4Ds} \to -\sqrt{\nu^2+4Ds}$, λ_1 and λ_2 are interchanged and, hence, $\overline{N}(s)$ is invariant under the transform. As a result, $\overline{N}(s)$ is an even function of $\sqrt{\nu^2+4Ds}$ and it has no branch singularity. Possible singularities of $\overline{N}(s)$ are therefore poles determined by

$$\lambda_1 e^{\lambda_2\overline{\epsilon}} - \lambda_2 e^{\lambda_1\overline{\epsilon}} = 0. \tag{B5}$$

Since we are interested in the behavior of $N(t)$ at large times, we look for solutions of Eq. (B5) at $s\approx 0$. If $\nu \lesssim 0$ ($a \gtrsim a_c^0$), we have $\lambda_1 \approx -s/\nu$ and $\lambda_2 \approx \nu/D$. Thus the pole is $s_p \approx -\nu^2 e^{-(\nu/D)\overline{\epsilon}}/D$. If $\nu \gtrsim 0$ ($a \lesssim a_c^0$), we have $\lambda_1 \approx \nu/D$, $\lambda_2 = -s/D$ and, hence again, $s_p \approx -\nu^2 e^{-(\nu/D)\overline{\epsilon}}/D$. Thus, asymptotically, $N(t) \sim e^{s_p t}$, giving the following lifetime of the chaotic transient: $\tau = 1/|s_p|$, which is Eq. (23).

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[7] Y.-C. Lai, C. Grebogi, J. A. Yorke, and S. C. Venkataramani, Phys. Rev. Lett. **77**, 55 (1996). This paper describes how riddling can possibly arise as a system parameter changes. The key point is that the chaotic attractor $\mathcal{A}$ in the invariant subspace $\mathcal{S}$ has embedded within itself an infinite number of unstable periodic orbits, and they constitute the skeleton of the attractor [23]. Depending on the parameter, these periodic orbits can be stable or unstable with respect to perturbations *transverse* to $\mathcal{S}$. Riddling occurs when an unstable periodic orbit, typically of low period [24], first becomes transversely unstable. When this occurs, a set consisting of an infinite number of tonguelike structures is open at the location of the periodic orbit and the locations of all its preimages. The ''roots'' of these structures are thus *dense* in $\mathcal{S}$, and have a Lebesgue measure zero. The complement of the set of these roots thus assumes the full measure in $\mathcal{S}$. By continuity, in the vicinity of $\mathcal{S}$, the complement of the set of tongues, which is the basin of A in $\mathcal{S}$, must have a positive measure, thereby establishing condition (i) for riddling. Away from $\mathcal{S}$, the set of infinite number of tongues intersect with a hyperplane of the same

dimension as that of $\mathcal{S}$ in a set of positive measure. The tongues thus constitute an open set in the transverse subspace. The open set of tongues constitutes the basin of another attractor (if it exists). This establishes condition (ii) for riddling.

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[15] This argument cannot be applied to, say, the chaotic attractor in the invariant subspace when $\epsilon = 0$, because the attractor contains no open neighborhood. More precisely, the chaotic attractor is a *Milnor attractor* [J. Milnor, Commun. Math. Phys. **99**, 177 (1985); **102**, 517 (1985)], which is defined as follows. Consider a nonlinear map $\mathbf{F}(\mathbf{x})$ on an N-dimensional manifold $\mathcal{S}$. The *ω-limit* set of a point $\mathbf{x}$ is the forward limit set of $\mathbf{x}$ under iterations by $\mathbf{F}(\mathbf{x})$: $\mathbf{L}^+(\mathbf{x}) = \cap_k \overline{\cup_{l \geq k} \mathbf{F}^l(\mathbf{x})}$. That is, $\mathbf{L}^+(\mathbf{x})$ is the set of all points of the form $\lim_{n\to\infty} \mathbf{F}^{i_n}(\mathbf{x})$, where $i_n \to \infty$ with n. A close subset $A \subset \mathcal{S}$ is called an *attractor* of $\mathbf{F}(\mathbf{x})$ if it satisfies the following two conditions: (1) the basin of attraction $\beta(A)$, consisting of all points $\mathbf{x} \in \mathcal{S}$ for which $\mathbf{L}^+(\mathbf{x}) \subset A$, must have a strictly positive N-dimensional Lebesgue measure; and (2) there is no strictly smaller closed set $A' \subset A$ so that $\beta(A') = \beta(A)$ up to a set of measure zero. In two dimensions, for instance, the Lebesgue measure is essentially area. The above definition of the attractor requires that its basin in total have a positive two-dimensional area, although it is not necessary that the basin contains any open sets (open areas). When a basin is riddled, it does not contain any open sets and it is full of holes (in the measure-theoretic sense), yet it must have a positive Lebesgue measure.

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[17] The tent map in our analytical model [Eq. (2)] is noninvertible. Because of the noninvertibility, the preimages of an unstable periodic orbit are dense on the chaotic attractor. This offers a great deal of convenience in analysis, as one can simply focus on a simple periodic orbit, such as the fixed point, to establish analytically that there is a dense set embedded in the chaotic attractor that is transversely unstable, even when the attractor is transversely stable. Generally, it is not necessary to have a noninvertible map in the invariant subspace to have riddling. For invertible maps and flows, in the case of riddling, the transversely unstable dense set consists of an infinite number of unstable periodic orbits [4,25], which are, however, amenable only to numerical analysis in general.

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[22] In the absence of symmetry-breaking perturbations, riddling occurs when the chaotic attractor in the invariant subspace is transversely stable. However, this does not mean that the chaotic attractor and its riddled basin are structurally stable in the full phase space, because the attractor lies only in a subspace and its transverse stability is thus only a *local* property. As shown in this paper, the chaotic attractor and its riddled basin are typically destroyed by arbitrarily small, symmetry-breaking perturbations.

[23] It is a widely accepted notion that in deterministic chaotic systems, unstable periodic orbits constitute the *skeleton* of the chaotic sets such as chaotic attractors or nonattracting chaotic saddles. See, for example, A. Katok, Pub. Math. IHES **51**, 377 (1980); T. Morita, H. Hata, H. Mori, T. Horita, and K. Tomita, Prog. Theor. Phys. **78**, 511 (1987); G. H. Gunaratne and I. Procaccia, Phys. Rev. Lett. **59**, 1377 (1987); D. Auerbach, P. Cvitanović, J.-P. Eckmann, G. H. Gunaratne, and I. Procaccia, Phys. Rev. Lett. **58**, 2387 (1988); C. Grebogi, E. Ott, and J. A. Yorke, Phys. Rev. A **37**, 1711 (1988); D. Auerbach, B. O'Shaughnessy, and I. Procaccia, Phys. Rev. A **37**, 2234 (1988); D. Auerbach, Phys. Rev. A **41**, 6692 (1990); F. Christiansen, P. Cvitanović, and V. Putkaradze, Nonlinearity **10**, 55 (1997); Y.-C. Lai, Y. Nagai, and C. Grebogi, Phys. Rev. Lett. **79**, 649 (1997); Y.-C. Lai, Phys. Rev. E **56**, 6531 (1997); P. Schmelcher and F. K. Diakonos, Phys. Rev. Lett. **78**, 4733 (1997); Phys. Rev. E **57**, 2739 (1998); R. Davidchack and Y.-C. Lai, *ibid.* **60**, 6172 (1999); M. Dhamala and Y.-C. Lai, *ibid.* **60**, 6176 (1999).

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