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Effect of smoothing on robust chaos

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In piecewise-smooth dynamical systems, situations can arise where the asymptotic attractors of the system in an open parameter interval are all chaotic (e.g., no periodic windows). This is the phenomenon of robust chaos. Previous works have established that robust chaos can occur through the mechanism of border-collision bifurcation, where border is the phase-space region where discontinuities in the derivatives of the dynamical equations occur. We investigate the effect of smoothing on robust chaos and find that periodic windows can arise when a small amount of smoothness is present. We introduce a parameter of smoothing and find that the measure of the periodic windows in the parameter space scales linearly with the parameter, regardless of the details of the smoothing function. Numerical support and a heuristic theory are provided to establish the scaling relation. Experimental evidence of periodic windows in a supposedly piecewise linear dynamical system, which has been implemented as an electronic circuit, is also provided.

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I. INTRODUCTION

Chaotic systems have been exploited for applications such as control [1], communication [2,3], enhancing mixing in chemical processes [4], spectrum spreading [5,6], and random number generation [7], etc. In these applications, it is desirable that the underlying chaotic attractor is *robust* so that random fluctuations or small disturbances would not cause the system to transit to a periodic state. Take as an example control of a chaotic system. The major key ingredient for the control of chaos is the observation that any chaotic set has embedded within it a large number of unstable periodic orbits of low periods. Because of ergodicity, the trajectory visits or accesses the neighborhood of each periodic orbit. Some of these periodic orbits may correspond to desired system's performance according to some criterion. The second ingredient is the realization that chaos, while signifying sensitive dependence on small changes to the current state and henceforth rendering unpredictable the system state in the long time, implies that the system's behavior can be altered by using small perturbations. The accessibility of the chaotic system to many different periodic orbits combined with its sensitivity to small perturbations allows for the control and manipulation of the chaotic process [1,8,9]. The main idea behind control is then to take the advantage of chaos, i.e., it offers an infinite number of possibilities for periodic motions, to select the one that yields the best performance. A prerequisite for control is then the availability of a robust chaotic attractor that will not disappear under small perturbations to the system.

Mathematically, a chaotic attractor is said to be robust if, about its parameter values, there exists an open neighborhood in the parameter space with no periodic attractor and the chaotic attractor is unique in the neighborhood [10]. In smooth dynamical systems, periodic windows are dense in parameter regions where there are chaotic attractors [11], so robust chaos is not expected to occur in such systems. The seminal work by Banerjee, Yorke, and Grebogi [10,12] showed that robust chaos can arise in nonsmooth dynamical systems such as piecewise-smooth systems. In particular, one can imagine dividing the phase space into two (or more) non-overlapping regions. In each region, the dynamical system is described by some smooth functions that have continuous derivatives. These functions are different in different regions and, hence, their derivatives are typically not continuous at the border between adjacent phase-space regions. When there is a fixed point on the border and when it goes through a bifurcation as a parameter changes, there is a discontinuous change in the elements of the Jacobian matrix evaluated at the fixed point. Such a bifurcation is called a *border-collision bifurcation* [13]. It has been established mathematically that, under fairly general conditions, robust chaos can arise in the neighborhood of border-collision bifurcations [10,12].

In this paper, we investigate the *robustness* of robust chaos. This issue was motivated by our experience with certain experimental electronic circuits whereas, even when the circuit equations are strictly piecewise-smooth (such as the Chua's circuit [14]), periodic windows are always observed in measurements. Intuitively, we expect that a small amount of smoothing can destroy robust chaos by inducing periodic windows. However, previous works by Tse and co-workers have established that *nonlinearity in pieces* can also induce periodic windows [15,16]. Thus two factors can contribute to the rising of periodic windows: smoothing and nonlinearity in pieces. In realistic physical systems such as electronic circuits, these two factors can be entangled and it is of fundamental interest to disentangle their effects on robust chaos. In general this is an extremely challenging problem as the two factors are not easily separated for any realistic dynamical systems. In this paper we thus take the following approach in an attempt to partially address the disentanglement problem: we consider piecewise-smooth dynamical systems and investigate the effect of smoothing on robust chaos. In particular, for a strictly piecewise-smooth system the derivatives of the mathematical functions that describe the dynamical system are discontinuous on the border. We then introduce a small amount of smoothing to the border characterized by a parameter d, where $d=0$ corresponds to the idealized case of strict nonsmooth dynamical systems. Given a large parameter interval of interest, we let μ be the measure of periodic windows in the interval. For $d=0$, we have $\mu=0$ so that there is robust chaos. Our main finding is that μ scales linearly with d, indicating that an arbitrarily small amount of smoothness around the border region in the system equations can destroy robust chaos. We shall provide numerical evidence and a heuristic argument to establish the scaling law. We also provide evidence of periodic windows from an experimental electronic circuit that is supposedly piecewise linear. The linear scaling law could thus provide a clue in the aforementioned disentanglement problem. For example, in an experimental situation where the underlying systems is piecewise smooth, if periodic windows are observed and their measure appears to increase linearly with some parameter, then it is likely that smoothing can be responsible for the windows. To pin down the exact origin of the periodic windows, however, is challenging and apparently an unsolved problem at the present.

In Sec. II, we present numerical results with prototypical piecewise-smooth dynamical systems and show that a small amount of smoothing can immediately cause the emergence of periodic windows. Numerical calculations also point to a scaling law between the measure of periodic windows and the parameter of smoothing. In Sec. III, a physical theory is provided to explain the scaling law. In Sec. IV, we present experimental observation of the emergence of periodic windows in a chaotic electronic circuit that generates the dynamical behavior of a piecewise linear map. Conclusions and discussions are offered in Sec. V.

II. NUMERICAL RESULTS

A. One-dimensional map

We first consider the simplest piecewise-smooth dynamical system, the tent map given by

$$T(x) = \begin{cases} F(x) \equiv ax, & \text{for } 0 < x < 0.5 \\ G(x) \equiv a(1-x), & \text{for } 0.5 < x < 1, \end{cases} \tag{1}$$

where a is a parameter. The tent map exhibits robust chaos for $1 \leq a \leq 2$, as can be seen from the bifurcation diagram in Fig. 1(a) where, apparently, no periodic windows can be found. To study the effect of smoothing, we introduce the following smoothing function:

$$H(x) = \frac{e^{\beta(x-0.5)}}{1 + e^{\beta(x-0.5)}}, \tag{2}$$

where β is a parameter. A higher value of β indicates that the map function is less smooth. The behaviors of $H(x)$ for two different values of β are shown in Fig. 2, where one can see the geometry of the smoothing effect about the border region, i.e., the point 1/2, of the map. After incorporating the smoothing function $H(x)$, the map equation becomes

$$T(x) = F(x) + [G(x) - F(x)]H(x). \tag{3}$$

A typical bifurcation diagram of this smoothed system is shown in Fig. 1(b) for $\beta=100$. We observe the occurrence of periodic windows, signifying the disappearance of robust chaos.

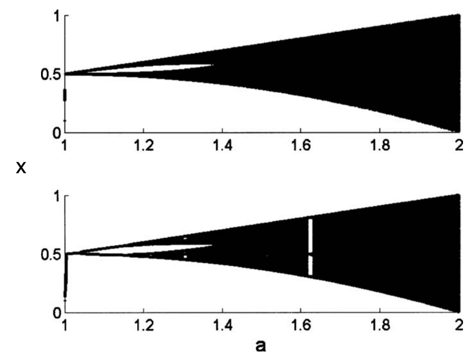

FIG. 1. For the tent map, bifurcation diagram for cases where (a) the system is strictly piecewise smooth [Eq. (1)] and (b) there is a small amount of smoothing around the border [Eq. (3)]. The ordinate is x.

To characterize the emergence of periodic windows due to smoothing, we calculate, as a function of the inverse of the parameter β, the measure of the periodic windows in a parameter interval that exhibits robust chaos when the map is strictly piecewise smooth. In particular, we vary β in the interval [100,5000] at the increment of $\Delta\beta=100$. For each fixed value of β, we generate a high-resolution bifurcation diagram by using 10^5 parameter values, whereas for each parameter value, a random initial condition is chosen and a transient of 1000 iterations is disregarded so that the system can reach the attractor for the parameter. We then calculate the fraction μ of parameter values that lead to periodic windows of all detectable sizes. For example, say there is a periodic window of period-3 that starts at $a=a_{\min}$ via a tangent bifurcation and ends at $a=a_{\max}$ via a crisis [11]. The size of the window is defined to be $\Delta a \equiv a_{\max}-a_{\min}$. For convenience, we define $d=1/\beta$ to be the *parameter of smoothing*, which characterizes the scale on which the function with break is replaced by the smoothed function (see Fig. 2). Figure 3 shows μ versus d, normalized by the maximum value of d used in the simulation. We observe an approximately proportional relation between μ and d (the up-

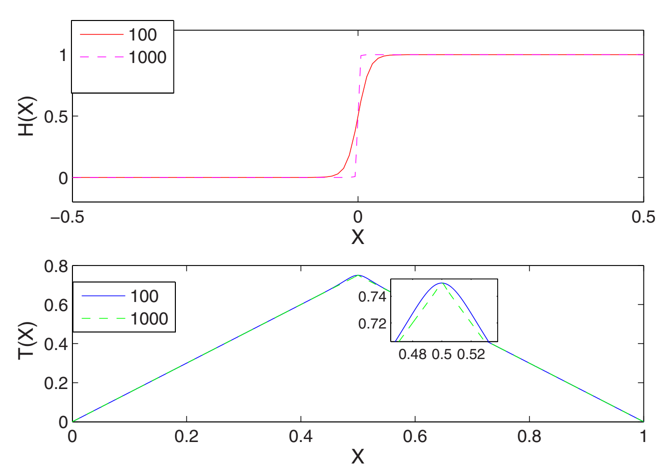

FIG. 2. (Color online) (a) The smoothing functions [Eq. (2)] for $\beta=100$ and 1000. (b) The resulted smoothed tent map.

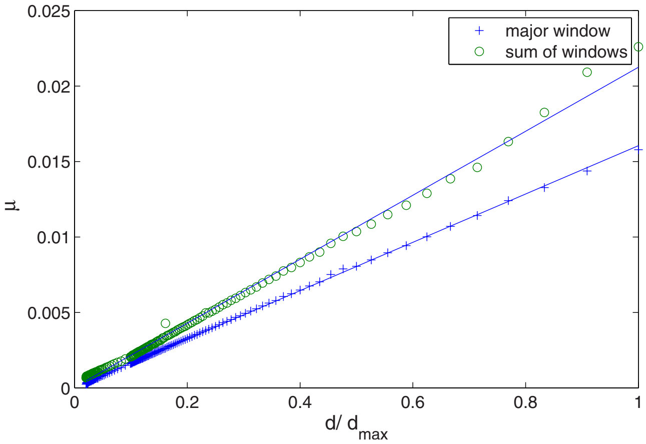

FIG. 3. (Color online) Measure of periodic windows versus the parameter of smoothing for system [Eq. (3)]. The upper trace corresponds to the case where all numerically detectable periodic windows are counted, whereas the lower trace is for the case where only the three largest periodic windows are counted.

per trace). This linear scaling appears to hold for major periodic windows as well, as shown in Fig. 3 (the lower trace), where only the three largest windows are counted.

The linear scaling behavior between μ and d does not appear to depend on the specific form of smoothing function. To verify this, we choose a different smoothing function,

$$H(x) = \frac{1}{1 + (x - 1.5)^{\beta}}, \tag{4}$$

where β is again a smoothing parameter and $d=1/\beta$ characterizes the parameter of smoothing. As shown in Fig. 4, the linear scaling between μ and d holds.

B. Two-dimensional map

We consider the system used previously to demonstrate robust chaos in two-dimensional maps [10]. The map models

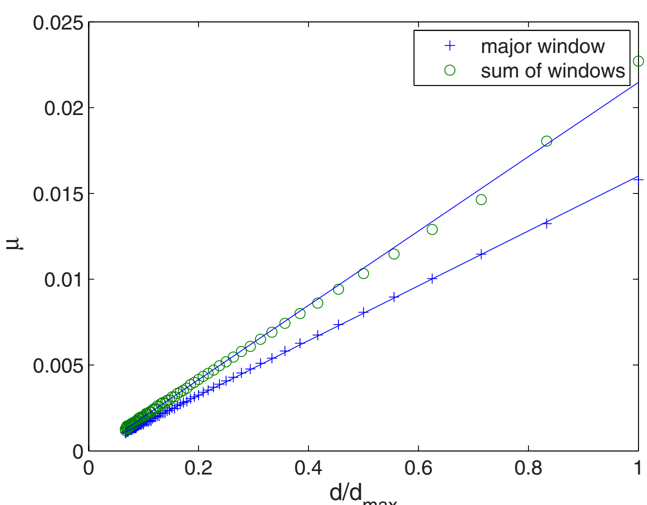

FIG. 4. (Color online) For a different smoothing function [Eq. (4)], measure of periodic windows versus the parameter of smoothing for the tent-map system [Eq. (3)]. Legends are the same as in Fig. 3.

the dynamics of a boost converter, which is often used in regulated dc switch-mode power supplies. The major components of a boost converter are: a controlled switch S, an uncontrolled diode switch D, an inductor L, a capacitor C, and a load resistor R. In the logic operation called current-mode control, the switch is closed by clock pulses of time T apart. In this phase, the inductor current i increases until it reaches a pre-specified reference value I_{ref}. The switch opens when $i=I_{ref}$. A clock pulse can turn on, i.e., close the switch only when it has been in an open state. That is, for $i<I_{ref}$, the switch remains to be on (closed) regardless of the arrivals of clock pulses. If the on-time $T_{on}=L(I_{ref}-i_n)/V_{in}$, where V_{in} is a dc input voltage, is less than T, the operation in one clock cycle consists of an on phase and an off phase. Since the characteristic time of the RLC circuit is typically much larger than the clock cycle T, the dynamical evolution of the system in one clock cycle is linear. Using i and the capacitor (or resistor, which is in parallel with the capacitor) voltage v as dynamical variables, the system is described by the following two-dimensional linear map (for $T_{on}<T$):

$$i_{n+1} \equiv F_1(i_n,v_n) = I_{ref} + \left(\frac{1}{L}\right)\left(V_{in} - v_n + \frac{v_n T_{on}}{CR}\right)(T - T_{on}),$$

$$v_{n+1} \equiv G_1(i_n,v_n) = v_n - \frac{v_n T_{on}}{CR} + \left(\frac{I_{ref}}{C} - \frac{v_n}{CR} + \frac{v_n T_{on}}{C^2R^2}\right)(T - T_{on}). \tag{5}$$

For $T_{on}\geq T$, the map is given by

$$i_{n+1} \equiv F_2(i_n,v_n) = i_n + \frac{V_{in}}{L}T,$$

$$v_{n+1} \equiv G_2(i_n,v_n) = v_n - \frac{v_n}{CR}T. \tag{6}$$

The circuit system is thus described by a piecewise linear map. The system alternates between one form of the map to another when the current reaches I_{ref} exactly at the arrival of the next clock pulse. The borderline current is then determined by $I_{border}=I_{ref}-V_{in}T/L$. It has been demonstrated numerically and argued rigorously that robust chaos can arise in this system [10,12].

We now introduce the following smoothing function:

$$H(T_{on}) = \frac{e^{\beta(T_{on}-0.5)}}{1 + e^{\beta(T_{on}-0.5)}}, \tag{7}$$

where the parameter of smoothing is given by $d=1/\beta$. The circuit system is now described by

$$i_{n+1} = F_1(i_n,v_n) + [F_2(i_n,v_n) - F_1(i_n,v_n)]H(T_{on}),$$

$$v_{n+1} = G_1(i_n,v_n) + [G_2(i_n,v_n) - G_1(i_n,v_n)]H(T_{on}) \tag{8}$$

Figures 5(a) and 5(b) show the bifurcation diagrams for the original system and for the smoothed system, respectively. As for the one-dimensional tent map, a small amount of smoothing appears to cause the occurrence of periodic windows, destroying robust chaos. Figures 6(a) and 6(b) show the scaling of the measures of the largest periodic window

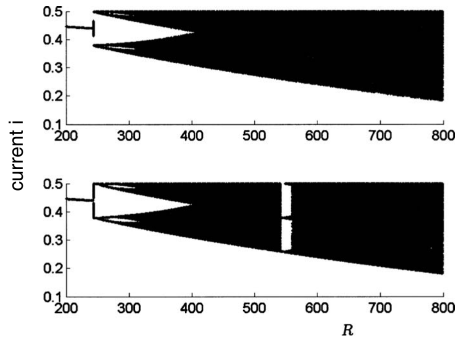

FIG. 5. For the two-dimensional map in Eqs. (5) and (6), bifurcation diagrams for $200 \leq R \leq 800$: (a) strictly piecewise-smooth case and (b) smoothed case where the smoothing function is given by Eq. (7). The ordinate is the current i.

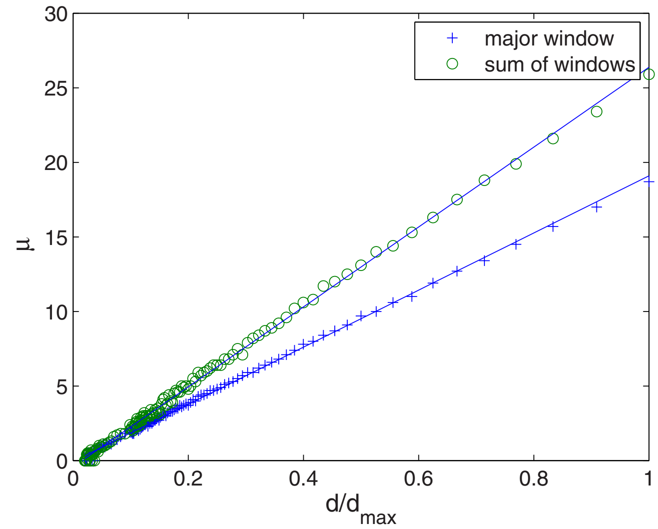

FIG. 6. (Color online) For the two-dimensional map Eq. (8), (a) measure of the largest periodic window versus d, the parameter of smoothing for $R \in [535,565]$, (b) measure of four largest periodic windows versus d, which exist in the parameter regions $R \in [360,370]$, [450,465], [535,565], and [775, 790], respectively.

and of the four largest windows with d, respectively. We observe again approximately linear scaling relations.

III. HEURISTIC THEORY

To gain theoretical insights, we consider the tent map and its smoothed version, where the originally discontinuous point in the derivative is now replaced by a quadratic extremum. Nonrobustness in the chaotic attractor then occurs through the mechanism of tangent bifurcation that creates periodic windows of various sizes. Note that one of the period 3 phase points, say x_1, is located near 1/2. While the other two points, x_2 and x_3, are relatively far from this point. According to the stability criterion, the period-3 orbit is stable if and only if the inequality

$$-1 \leq \Pi_{i=1}^{3} f'(x_i, a) \leq 1$$

holds. At the two bifurcation points, we thus have

$$\Pi_{i=1}^{3} f'(x_i, a_{\min}) = 1 \quad \text{and} \quad \Pi_{i=1}^{3} f'(x_i, a_{\max}) = -1, \qquad (9)$$

where the values of $f'(x_{2,3}, a)$ can be well approximated by $\pm a$, respectively. That is

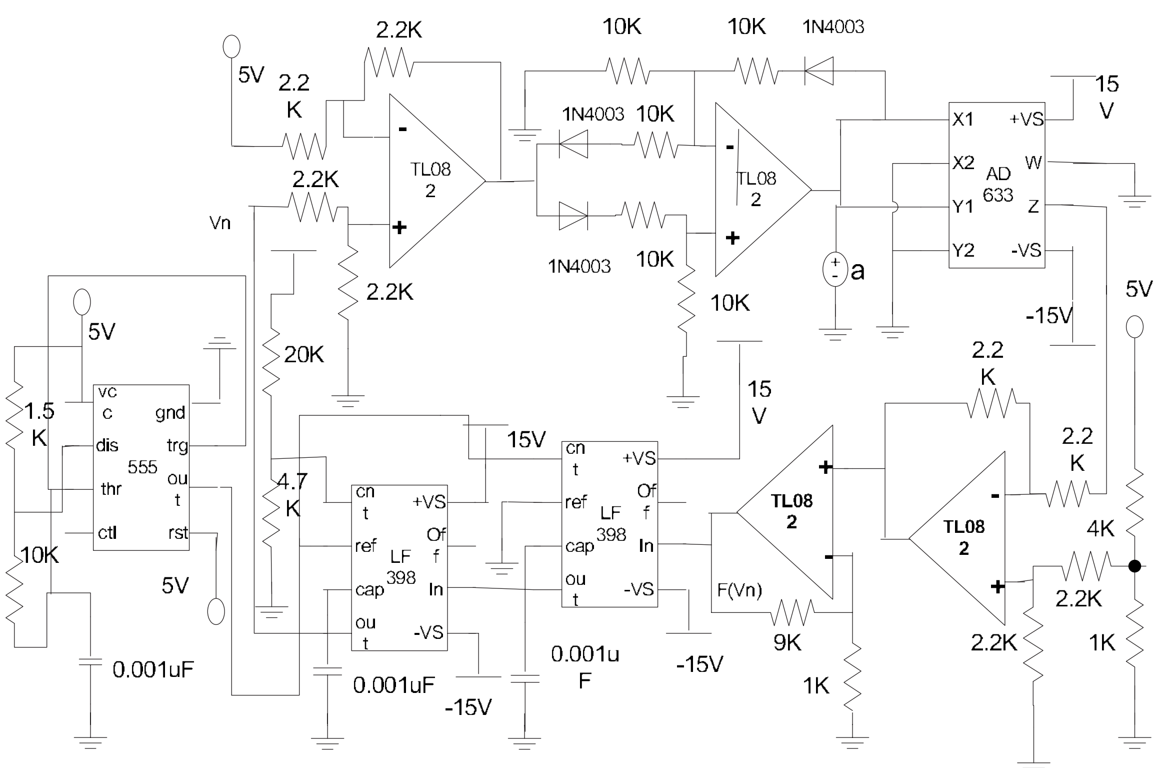

FIG. 7. Schematic of the circuit realization of Eq. (14). The voltage V_n corresponding to the dynamical variable x is indicated.

$$\Pi_{i=2}^{3} f'(x_i,a) \approx -a^2. \tag{10}$$

After some algebra, $f'(x_1,a)$ can be obtained as

$$f'(x_1,a) = a\left[-1 + 2\frac{1+e^{\beta c} - \beta c e^{\beta c}}{(1+e^{\beta c})^2}\right],$$

where $c=x_1-1/2$. It is straightforward to numerically check that $\beta c \ll 1$. Using this property, to first order in βc, we can rewrite $f'(x_1,a)$ as

$$f'(x_1,a) \approx -a\beta c. \tag{11}$$

Substituting Eqs. (10) and (11) into Eq. (9), we obtain

$$a_{\min}^3 \beta c_{\min} = 1 \quad \text{and} \quad a_{\max}^3 \beta c_{\max} = -1, \tag{12}$$

where $c_{\min}$ (or $c_{\max}$) denotes the value of c at $a=a_{\min}$(or at $a=a_{\max}$). The window size Δa can then be estimated as

$$\Delta a \equiv a_{\max} - a_{\min} = \frac{1}{\beta}\left(\frac{-1}{c_{\max}} + \frac{-1}{c_{\min}}\right)\frac{1}{a_{\max}^2 + a_{\max}a_{\min} + a_{\min}^2} \sim \frac{1}{\beta}, \tag{13}$$

where we have used the fact that $\Delta a \ll a_{\min}$. This relation suggests a linear scaling between Δa, the measure of the period-3 window, and $1/\beta$. Since similar argument can be carried out for other periodic windows, we conclude that the measure of the periodic windows, or any subset of windows, scales linearly with the parameter of smoothing.

In smoothed two-dimensional maps, periodic windows are generated by saddle-node bifurcations. In addition, homoclinic or heteroclinic tangencies between the stable and unstable manifolds can occur and the subsequent crossings generate horseshoe-type of dynamics and nonattracting chaotic sets that are responsible for transient chaos in periodic windows. In this sense, nonrobustness of chaotic attractors is connected with homoclinic or heteroclinic tangencies, analogous to one-dimensional maps where the nonrobustness is connected with quadratic extrema.

IV. EXPERIMENTAL EVIDENCE OF PERIODIC WINDOWS

We have constructed an electronic circuit [17,18] that generates the following one-dimensional, piecewise linear, discrete dynamical system,

$$x_{n+1} = 1 - a\cdot|x_n - 1/2| \quad \forall\, x \in [0,1]. \tag{14}$$

The dynamics of the circuit is described by

$$V_{n+1} = 10 - a\cdot|V_n - 5| \quad \forall\, V \in [0,10]. \tag{15}$$

The circuit diagram is shown in Fig. 7. The building blocks of the circuit are a voltage-subtraction circuit, a precision absolute-value circuit, an analog multiplier, a gain circuit, a sample-and-hold circuit. In order to observe and measure the signal better, the variable x_n is scaled by 10, as in Eq. (15).

The output of the sample-and-hold circuit holds the current state of the system as represented by V_n. This voltage is fed back to the subtraction-circuit block and the voltage V_n-5 is obtained. This voltage is used as an input to the precision absolute-value circuit, yielding the voltage $|V_n-5|$. It is necessary to multiply this voltage by the bifurcation parameter a, which is accomplished by using AD633 that multiplies two voltages and divides the output by 10. The bifurcation parameter a is generated as a 80 Hz ramp signal using a signal generator. The output of the multiplier is $a\cdot|V_n-5|/10$. This voltage is then subtracted from 1 V and multiplied by 10 to obtain $10-a\cdot|V_n-5|$. The resulted voltage is sampled, giving rise to the dynamical variable V_n in Eq. (15), and it is held for the next iteration. The sampling time is controlled by a 555-timer, which is set to complete an iteration in 20 μs. The 1 V voltage reference is generated by a voltage-divider circuit. The 5 and 15 V voltage references are generated by a DC power supply.

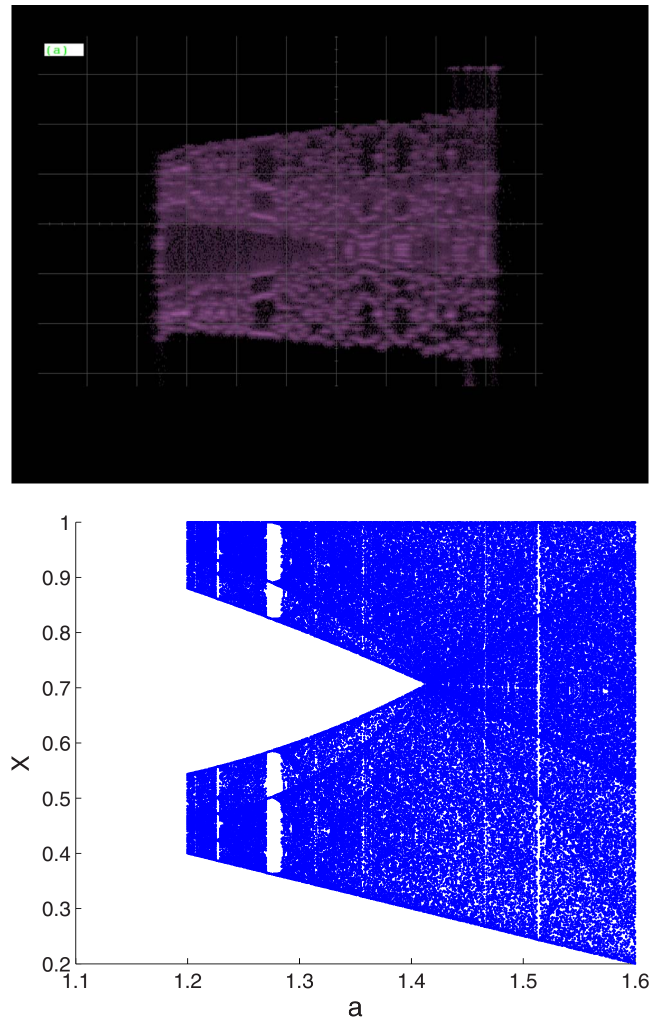

FIG. 8. (Color online) (a) Experimentally obtained bifurcation diagram of the circuit. The full screen sweep is 100 ms, which corresponds to sweeping through the parameter a in Eq. (14) in the range [1.2,1.6]. (b) Numerically computed bifurcation diagram of the smoothed tent map for $\beta=100$.

In our experiments, the bifurcation parameter a is varied from 1.2 to 1.6. Figure 8(a) shows the experimentally obtained bifurcation diagram. We observe periodic windows, indicating lack of robust chaos, despite our effort to make the circuit as nonsmooth as possible. Figure 8(b) shows a numerically obtained bifurcation diagram of the tent map smoothed around the point $x=0.5$ by the smoothing function Eq. (2), where the smoothing parameter is $\beta=100$. A strong similarity can be found between the experimental bifurcation diagram from the circuit and the numerical diagram from the smoothed tent map, indicating that the circuit might contain a small amount of smoothing, preventing robust chaos. It is important to note that, despite the strong similarity, the origin of periodic windows cannot be pinned down exactly for this realistic physical system. As described in Sec. I, both smoothing and nonlinearity in pieces can cause periodic windows. In the circuit implemented using various linear/nonlinear elements, both factors can be present and it is difficult to separate them experimentally.

V. CONCLUSION AND DISCUSSIONS

We have studied the effect of smoothing on robust chaos in piecewise-smooth dynamical systems. Our main result is that an arbitrarily small amount of smoothness around the border region can destroy robust chaos in the sense that periodic windows will arise. The measure of periodic windows in the parameter space is found to scale linearly with the parameter of smoothing. In certain realistic physical systems, it is difficult to achieve strictly piecewise smoothness. For example, in electronic circuits, piecewise smoothness is implemented by using nonlinear elements such as diodes whose current-voltage (*I*-*V*) relations cannot be absolutely nonsmooth. There is always a small degree of smoothness at every point in the *I*-*V* relation. Our study suggests that in these cases, periodic windows can always be expected to arise, as we have observed frequently in experiments that involve piecewise-smooth systems such as the Chua's circuit.

While our paper has clarified the role of smoothing in generating periodic windows, the inverse problem of identifying the origin of periodic windows in piecewise-smooth dynamical systems remains unsolved, as strong nonlinearity in different pieces of the system can also induce periodic windows [15,16]. We also remark that, while we have demonstrated that robust chaotic attractors are not possible in some realistic dynamical systems, there are physical situations where such attractors can arise. For example, uniformly hyperbolic attractors have been a paradigm of rigorous mathematical study in nonlinear dynamical systems, which are known to be structurally stable and thus robust [19]. Recently, hyperbolic attractors have been suggested and experimentally realized in physical systems [20–22].

ACKNOWLEDGMENTS

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