PHYSICAL REVIEW E VOLUME 62, NUMBER 4 OCTOBER 2000

Catastrophe of riddling

Ying-Cheng Lai
Department of Mathematics, Department of Electrical Engineering, and Department of Physics, Center for Systems Science and Engineering Research, Arizona State University, Tempe, Arizona 85287

(Received 25 May 2000)

Most existing works on riddling assume that the underlying dynamical system possesses an invariant subspace. We find that, under arbitrarily small, deterministic perturbations, a riddled basin is typically destroyed and replaced by fractal ones, a *catastrophe* of riddling. We elucidate, based on analyzing unstable periodic orbits, the dynamical mechanism of the catastrophe. Analysis of the critical behaviors leads to the finding of a transient chaotic behavior that is different from those reported previously.

PACS number(s): 05.45.Jn, 05.45.Ac

It is common for nonlinear dynamical systems to possess multiple coexisting attractors [1,2]. The basin boundaries between the attractors can be smooth or complicated. For such systems, an important question concerns whether the asymptotic attractor can be predicted if initial conditions are chosen in the vicinity of the boundary. For smooth boundaries, an improvement in the precision to specify the initial conditions results in an equal amount of improvement in the predictability of the asymptotic attractor. For fractal basin boundaries [1], a more precise specification of the initial conditions often results in a much smaller improvement in the probability to predict the attractor correctly. In the extreme case of riddled basins [3–9], a vast reduction in the uncertainty to specify the initial conditions typically results in almost no improvement in ability to predict the final attractor. As such, the phenomenon of riddled basins has received a lot of recent attention.

Riddling is first noticed by Pikovsky and Grassberger in their study of coupled map lattices [3] and is independently discovered, analyzed, and named for general chaotic systems by Alexander *et al.* [4]. The dynamical conditions for riddling to occur are described in Ref. [4], where it is shown that for systems with an invariant subspace $\mathcal{S}$: (a) if there is a chaotic attractor in $\mathcal{S}$, and (b) if a typical trajectory in the chaotic attractor is stable with respect to perturbations transverse to $\mathcal{S}$, then the basin of the chaotic attractor in $\mathcal{S}$ can be riddled with holes that belong to the basin of another attractor off $\mathcal{S}$, provided that such an attractor exists. Physically, the presence of a riddled basin means that, for every initial condition that asymptotes to the chaotic attractor in $\mathcal{S}$, there are initial conditions arbitrarily nearby that asymptote to the attractor off $\mathcal{S}$. As such, prediction of the asymptotic attractors becomes practically impossible because of the inevitable error in the specification of the initial conditions or parameters. Mathematically, a riddled basin is the complement of a *dense* open set belonging to the basin of the other attractor. Thus, a riddled basin contains no open sets, in contrast to fractal basins that do [1]. In most existing works on riddling, a common assumption is that the system possesses a perfect invariant subspace. Such an invariant subspace is usually caused by a simple symmetry in the system.

In this Rapid Communication, we address how symmetry breaking affects riddling in chaotic systems. Here we consider smooth deterministic perturbations, say, of magnitude ϵ, that destroy the existence of invariant subspace. That is, when $\epsilon=0$, the system possesses a perfect symmetry and consequently an invariant subspace $\mathcal{S}$. Assume that there are other coexisting attractors outside $\mathcal{S}$, as shown schematically in Fig. 1(a), let a be a system parameter whose variation perserves the system symmetry, and let a_c be the blowout bifurcation point [10]. Thus, if $\epsilon=0$, for $a \lesssim a_c$, the chaotic attractor in $\mathcal{S}$ is transversely stable and its basin is riddled and, for $a \gtrsim a_c$, the chaotic attractor is transversely unstable and it is no longer an attractor in the full phase space. Our principal results are as follows: (1) In the subcritical case $(a<a_c)$, the riddled basin of the chaotic attractor in $\mathcal{S}$ is immediately destroyed and replaced by fractal ones in the presence of a symmetry-breaking perturbation, no matter how small it is. Accompanying this catastrophe of riddling is

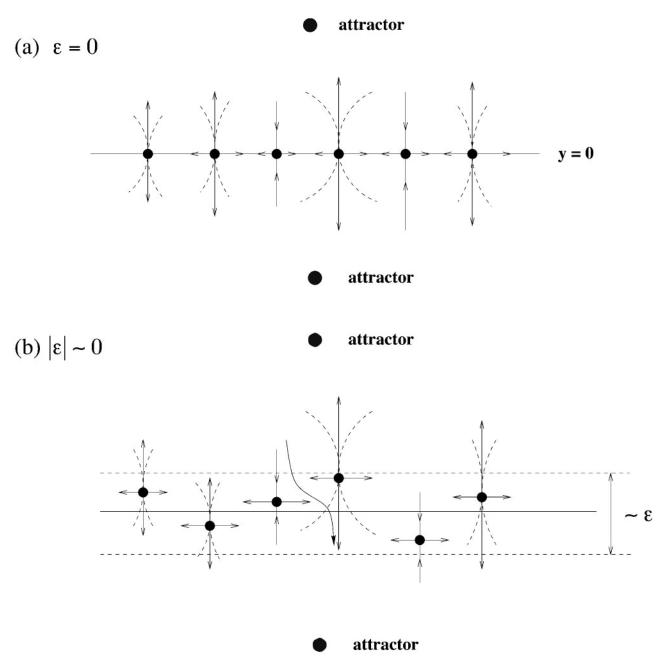

FIG. 1. Schematic illustrations of the dynamics of unstable periodic orbits: (a) for $\epsilon=0$, $y=0$ is invariant and the roots of the tongues are dense in $y=0$, creating a riddled basin; (b) for $\epsilon\neq 0$, $y=0$ is no longer invariant, the locations of the periodic orbits shifted about $y=0$, and the roots of the tongues are no longer dense, leading to fractal basins.

the birth of long chaotic transients whose lifetimes scale with the parameter variation as

$$\tau \sim \exp[C(\epsilon)(a_c - a)|\ln \epsilon|], \quad a < a_c, \tag{1}$$

where $C(\epsilon) > 0$ is a factor that depends on ϵ. (2) In the supercritical case ($a > a_c$), a smooth-to-fractal basin boundary metamorphosis arises as ϵ is increased from zero. In this case, the chaotic transient lifetime is much shorter than that in the subcritical case. We establish the above results by analyzing the behavior of unstable periodic orbits, by deriving scaling laws based on a physical theory, and by numerical support.

A riddled basin contains no open sets and it is full of holes (in the measure-theoretic sense), yet it must have a positive Lebesgue measure [4]. Thus, to argue that the basin of attractor A (in the Milnor's sense [11]) is riddled, the following two conditions must be established: (i) a set of positive measure is attracted to the attractor A; and (ii) the complement of the basin of A is a *dense* open set, i.e., sufficiently many points near A are repelled from it. In contrast, a fractal basin is open and it is defined with respect to the basin boundary: a basin is fractal if its boundary is a fractal set. These are the key differences between a riddled basin and a fractal one. In view of the mathematical difficulty to establish such an existence, it is suggested in Ref. [7] that the existence of riddling can be argued by focusing on its dynamical mechanism. In particular, Ref. [7] describes how riddling can arise as a system parameter changes. The key point is that the chaotic attractor A in the invariant subspace $\mathcal{S}$ has embedded within itself an infinite number of unstable periodic orbits. Depending on the parameter, these periodic orbits can be stable or unstable with respect to perturbations *transverse* to $\mathcal{S}$. Riddling occurs when an unstable periodic orbit, typically of low period, first becomes transversely unstable [7]. When this occurs, a set consisting of an infinite number of tongue-like structures is open at the location of the periodic orbit and the locations of all its preimages. The ''roots'' of these structures are thus *dense* in $\mathcal{S}$ and have Lebesgue measure zero. The complement of the set of these roots thus assumes the full measure in $\mathcal{S}$. By continuity, in the vicinity of $\mathcal{S}$, the complement of the set of tongues, which is the basin of A in $\mathcal{S}$, must have a positive measure, thereby establishing condition (i) for riddling. Away from $\mathcal{S}$, the set of infinite number of tongues intersect with a hyperplane of the same dimension as that of $\mathcal{S}$ in a set of positive measure. The tongues thus constitute an open set in the transverse subspace. The open dense set of tongues constitutes the basin of another attractor (if it exists). This establishes condition (ii) for riddling.

We can now describe what happens when there is a symmetry-breaking. The key observation here is that unstable periodic orbits are *structurally stable* in the sense that they cannot be destroyed by smooth perturbations. If the symmetry-breaking perturbation is small, we expect the stabilities of most of the unstable periodic orbits in $\mathcal{S}$ to be unchanged. That is, most orbits remain transversely stable (unstable) under a small perturbation if they are transversely stable (unstable) in the absence of the perturbation. We expect, however, the locations of these orbits to shift under the perturbation. Specifically, say when there is no symmetry-breaking ($\epsilon = 0$), the invariant subspace $\mathcal{S}$ is located at $\mathbf{y} = \mathbf{0}$, where $\mathbf{y}$ is a vector in the transverse subspace. In this case, all unstable periodic orbits in $\mathcal{S}$ are localized in $\mathbf{y} = \mathbf{0}$ and the set of all their preimages, including themselves, is dense in $\mathbf{y} = \mathbf{0}$, as shown schematically in Fig. 1(a). For $\epsilon \gtrsim 0$, these orbits no longer have $\mathbf{y} = \mathbf{0}$. Instead, their locations will be shifted and most of them will be confined, approximately, in the ϵ-neighborhood of $\mathbf{y} = \mathbf{0}$, as shown in Fig. 1(b). Because of the redistribution of the unstable periodic orbits in the vicinity of $\mathbf{y} = 0$, typically the set of all their preimages (including themselves) *is no longer dense*, leaving its complement an open set. Riddling is thus destroyed and is replaced by fractal basins, no matter how small ϵ is.

To be illustrative, we consider the following analyzable two-dimensional map:

$$x_{n+1} = T(x_n) = \begin{cases} 2x_n, & 0 \leqslant x < 1/2 \\ 2(1 - x_n), & 1/2 \leqslant x \leqslant 1, \end{cases}$$
$$y_{n+1} = \begin{cases} a x_n y_n - \epsilon, & 0 \leqslant y < 1 \\ \lambda y_n, & y \geqslant 1, \end{cases} \tag{2}$$

where $T(x)$ is the tent map; a, $\epsilon > 0$, and $\lambda > 1$ are parameters; and the phase-space region of interest is $\{0 \leqslant x \leqslant 1, -\infty < y < \infty\}$. When $\epsilon = 0$, the system possesses the one-dimensional invariant subspace: $y = 0$, which is caused by the reflecting symmetry: $y \to -y$. The symmetry is broken when $\epsilon \neq 0$. Because $\lambda > 1$, Eq. (2) has two other attractors: $y = \pm\infty$. For $\epsilon = 0$, the transverse Lyapunov exponent of the chaotic attractor in $y = 0$ can be defined as follows: $h_T = \int_0^1 \ln(ax)\rho(x)dx = \ln a - 1$, where $\rho(x) = 1$ (for $0 \leqslant x \leqslant 1$) is the natural invariant density of the chaotic attractor in the tent map. We see that a blowout bifurcation occurs at $a_c = e$, where $h_T \leqslant 0$ for $a \leqslant a_c$ and $h_T > 0$ for $a > a_c$. The existence of riddling for $a \leqslant a_c$ can be established through the following theorem.

Theorem. For $\epsilon = 0$, let $\mathcal{A}$ be the chaotic attractor in the invariant subspace $y = 0$. The basin of $\mathcal{A}$ is riddled for $a \lesssim a_c$.

The proof of the theorem proceeds in two steps [12]: (1) Since $h_T \lesssim 0$, by Lemma 2 in Ref. [4], there is a set of positive measure that asymptotes to $\mathcal{A}$; and (2) explicit mathematical estimates can be obtained which show that there is a dense open set asymptoting to the attractors at infinity.

When ϵ is increased from zero, unstable periodic orbits embedded in the original chaotic attractor in $y = 0$ are perturbed. In particular, for a periodic orbit of period p: $(x_1^{(p)},0),(x_2^{(p)},0),\ldots,(x_p^{(p)},0)$, where $T(x_i^{(p)}) = x_{i+1}^{(p)}$ ($i = 1,\ldots,p-1$) and $T(x_p^{(p)}) = x_1^{(p)}$, for $|\epsilon| \neq 0$, the y locations of the orbit points are given by

$$y_i^{(p)} = -\epsilon\left[1 + a x_{i-1}^{(p)} + a^2 x_{i-1}^{(p)} x_{i-2}^{(p)} + \ldots + a^{p-1} \prod_{l=1, l\neq i}^{p} x_l^{(p)}\right] \Bigg/ \left[1 - a^p \prod_{m=1}^{p} x_m^{(p)}\right]. \tag{3}$$

From Eq. (3), the following can be seen: (1) if the original periodic orbit is a repeller, i.e., $a^p \prod_{m=1}^{p} x_m^{(p)} > 1$, it remains a repeller but its location is shifted upward, i.e., $y_i^{(p)} > 0$; (2) if the original orbit is a saddle, i.e., $a^p \prod_{m=1}^{p} x_m^{(p)} < 1$, it is still a saddle but its position is shifted downward: $y_i^{(p)} < 0$. Thus, under the symmetry-breaking perturbation, all repellers are located in $y > 0$ and all saddles are located in $y < 0$. As a consequence, a trajectory starting in $y > 0$ can move across the x axis and asymptote to the $y = -\infty$ attractor. That is, due to the symmetry breaking, the $y = -\infty$ attractor acquires basins in $y > 0$. Insofar as the original symmetric system possesses two distinct classes of unstable periodic orbits (repellers and saddles), the basin of the $y = -\infty$ attractor has a component in $y > 0$, regardless of whether $a \lesssim a_c$ or $a \gtrsim a_c$. It can be argued that there is still an open set in $0 \leqslant (x,y) \leqslant 1$ that asymptotes to the $y = +\infty$ attractor, but this set is no longer dense [12]. The boundary between the basins of the $y = \pm\infty$ attractors is thus fractal. We remark, however, that under the perturbation, the basin of the $y = -\infty$ attractor may appear undistinguishable, say, in numerical experiments, from that of a riddled basin because most unstable periodic orbits in the original invariant subspace are perturbed only slightly.

The critical behaviors associated with the catastrophe of riddling can be analyzed using the model of random walk introduced in Ref. [5]. By solving the Fokker-Planck (FP) equation for the probability distribution of walker in space and time, adopting various initial and boundary conditions, we are able to derive the scaling laws [12] for the following quantities that can be measured in numerical or physical experiments: (1) the fraction of fractal basins after the catastrophe, (2) the lifetime of the chaotic transient, and (3) the dimension of the fractal basin boundary. Here we briefly describe results (2) and (3). Say we distribute a large number N_0 of initial conditions near $y = 0$. Then the number of trajectories that still remain in the vicinity of $y = 0$ at time t is $N(t) = N_0 \exp(-t/\tau)$, where τ is the lifetime of the chaotic transient. Solution of the FP equation, under appropriate initial and boundary conditions, yields Eq. (1). We see that τ increases *exponentially* as a is decreased from a_c. To get a rough idea about how long the transient lifetime can be, say we have $C = 1$ and $|\epsilon| = 10^{-10}$. Thus, if $a_c - a = 0.1$, we have $\tau \sim 10^3$. But if $a_c - a = 1.5$, then $\tau \sim 10^{15}$. We note that the exponential scaling is different from both the typical algebraic scaling for low-dimensional chaotic transients [13] and that associated with superpersistent chaotic transient [14,7]. For (3), the dimension of the fractal basin boundary, the result is that the dimension is fairly close to that of the phase-space dimension. This indicates that the basins after the catastrophe of riddling, while being fractal, lead to a similar type of extreme sensitivity of the asymptotic attractor on initial conditions to that resulted from a riddled basin.

We now present numerical support with the following two-dimensional map: $(x_n, y_n) \rightarrow (f(x_n) + 0.1 y_n^2, a x_n y_n + y_n^3 - \epsilon)$, where $f(x_n) = 3.8 x_n (1 - x_n)$ is the logistic map that apparently possesses a chaotic attractor, a, b, r, and $\epsilon \geqslant 0$ (characterizing the symmetry-breaking) are parameters. For $\epsilon = 0$, $y = 0$ is the one-dimensional invariant subspace. The y^3 term stipulates that trajectories with large values of y asymptote to the attractors at $|y| = \infty$ rapidly. There is a blowout bifurcation at $a_c \approx 1.726$, where $h_T \lesssim 0$ for $a \lesssim a_c$ and $h_T \gtrsim 0$ for $a \gtrsim a_c$. Thus, for $a \lesssim a_c$, the basin of the

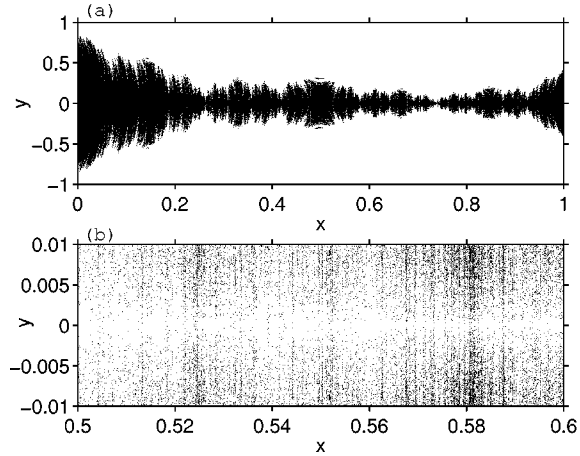

FIG. 2. For $\epsilon = 0$ in the numerical model, (a) the basin of the $y = 0$ chaotic attractor (black regions), and (b) the basin of the $y = \pm\infty$ attractors (black dots) near $y = 0$.

chaotic attractor in $y = 0$ is riddled, as shown in Fig. 2(a), where $a = 1.7$, a grid of 1000×1000 initial conditions are chosen in $(0 < x < 1, -1 < y < 1)$, and black dots denote initial conditions whose trajectories stay within 10^{-10} of $y = 0$ for successive 1000 iterations (which are numerically considered as having approached the $y = 0$ chaotic attractor). Figure 2(b) illustrates that arbitrarily near $y = 0$, there are initial conditions that asymptote to the $y = \pm\infty$ attractors, where black dots in $y > 0$ $(y < 0)$ denote initial conditions to the $y = +\infty$ $(y = -\infty)$ attractor.

When $\epsilon \neq 0$, $y = 0$ is no longer invariant and, the riddling observed in Figs. 2(a) and 2(b) is destroyed and is replaced by fractal basins, no matter how small ϵ is. For $\epsilon \gtrsim 0$, the $y < 0$ region still belongs to the basin of the $y = -\infty$ attractor but it now has a basin component in $y > 0$, due to the symmetry-breaking. The basin boundary between the $y = +\infty$ and the $y = -\infty$ attractors in $y > 0$ is a fractal, as shown in Fig. 3(a) for $a = 1.72 \lesssim a_c(\epsilon = 0)$, where black dots denote initial conditions that asymptote to the $y = -\infty$ attractor. Although, as we have argued, the basins are fractal, they mimic riddled basins, which is demonstrated by Fig. 3(b), where a small region near $y = 0$ in Fig. 3(a) is magnified but now

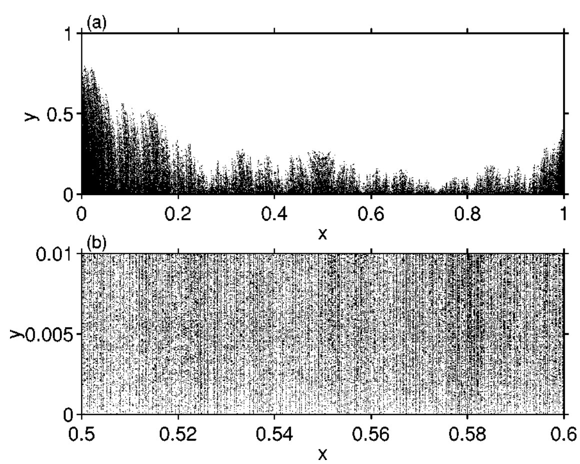

FIG. 3. For $\epsilon = 10^{-10}$ and $a = 1.72 < a_c(\epsilon = 0)$ in the numerical model, (a) the basin of the $y = -\infty$ chaotic attractor (black dots), and (b) the basin of the $y = +\infty$ attractors (black dots) near $y = 0$.

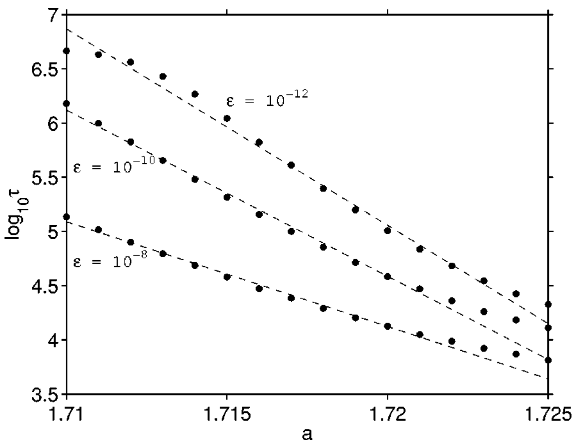

FIG. 4. Exponential scaling of the chaotic transient lifetime under small symmetry-breaking perturbations.

black dots denote initial conditions that go to the $y=+\infty$ attractor. We see that there are initial conditions arbitrarily near $y=0$ that asymptote to the $y=+\infty$ attractor, similar to the behavior depicted in Fig. 2(b). Comparing Figs. 3(a) and 3(b) with Figs. 2(a) and 2(b), we observe that the symmetry-breaking induced fractal basins are visually similar to riddled basins. In fact, similar fractal basins exist for $a \gtrsim a_c(\epsilon=0)$. The exponential scaling of the transient lifetime is shown in Fig. 4 for $\epsilon=10^{-8}$, 10^{-10}, and 10^{-12}, where we see that for $a<a_c(\epsilon=0)$, the lifetime can easily reach 10^6 iterations even when a is slightly below $a_c(\epsilon=0)$. Extensive numerical computations also establish support for the scaling of the fraction and dimension of the fractal basins after the catastrophe of riddling [12].

In summary, we establish in this Rapid Communication that, while riddling is robust against perturbations that preserve the symmetry and invariance of the system, it is structurally unstable under perturbations that destroy the symmetry. Such perturbations remove riddling, create fractal basins with physical properties similar to those of a riddled one, and induce long chaotic transients that scale exponentially with parameter variations. An implication of this work is that riddled basins may not actually be observable in physical experiments, say, in systems of coupled, slightly nonidentical chaotic oscillators. What can be observed is fractal basins that appear like riddled ones.

This work is supported by AFOSR under Grant No. F49620-98-1-0400 and by NSF under Grant No. PHY-9722156.

[1] For example, C. Grebogi, S. W. McDonald, E. Ott, and J. A. Yorke, Phys. Lett. A **99**, 415 (1983); S. W. McDonald, C. Grebogi, E. Ott, and J. A. Yorke, Physica D **17**, 125 (1985).

[2] For example, F. T. Arecchi and A. Califano, Phys. Lett. **101A**, 443 (1986); F. T. Arecchi, R. Badii, and A. Politi, Phys. Rev. A **29**, 1006 (1984).

[3] A. Pikovsky and P. Grassberger, J. Phys. A **24**, 4587 (1991).

[4] J. C. Alexander, J. A. Yorke, Z. You, and I. Kan, Int. J. Bifurcation Chaos Appl. Sci. Eng. **2**, 795 (1992).

[5] E. Ott, J. C. Alexander, I. Kan, J. C. Sommerer, and J. A. Yorke, Physica D **76**, 384 (1994).

[6] J. F. Heagy, T. L. Carroll, and L. M. Pecora, Phys. Rev. Lett. **73**, 3528 (1994).

[7] Y.-C. Lai, C. Grebogi, J. A. Yorke, and S. C. Venkataramani, Phys. Rev. Lett. **77**, 55 (1996).

[8] Y.-C. Lai, Phys. Rev. E **56**, 3897 (1997); Y.-C. Lai and C. Grebogi, Phys. Rev. Lett. **77**, 5047 (1996).

[9] H. Nakajima and Y. Ueda, Physica D **99**, 35 (1996); L. Billings, J. H. Curry, and E. Phipps, Phys. Rev. Lett. **79**, 1018 (1997); K. Kaneko, *ibid.* **78**, 2736 (1997); Y. L. Maistrenko, V. L. Maistrenko, A. Popovich, and E. Mosekilde, Phys. Rev. E **57**, 2713 (1998); M. Woltering and M. Markus, Phys. Rev. Lett. **84**, 630 (2000).

[10] E. Ott and J. C. Sommerer, Phys. Lett. A **188**, 39 (1994).

[11] J. Milnor, Commun. Math. Phys. **99**, 177 (1985); **102**, 517 (1985).

[12] Y.-C. Lai and V. Andrade (unpublished). This paper contains detailed mathematical arguments for catastrophe of riddling, physical analyses of critical behaviors associated with the catastrophe, and extensive numerical results including those from coupled nonidentical chaotic electronic circuits.

[13] C. Grebogi, E. Ott, and J. A. Yorke, Physica D **7**, 181 (1983).

[14] C. Grebogi, E. Ott, and J. A. Yorke, Phys. Rev. Lett. **50**, 935 (1983); Erg. Th. Dyn. Sys. **5**, 341 (1985).