Algebraic decay and phase-space metamorphoses in microwave ionization of hydrogen Rydberg atoms

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We study classically the microwave ionization of hydrogen atoms using the standard one-dimensional model. We find that the survival probability of an electron decays algebraically for long exposure times. Furthermore, as the microwave field strength increases, we find that the asymptotic algebraic decay exponent can decrease due to phase-space metamorphoses in which new layers of Kolmogorov-Arnold-Moser (KAM) islands are exposed when KAM surfaces are destroyed. We also find that after such phase-space metamorphoses, the survival probability of an electron as a function of time can have a crossover region with different decay exponents. We argue that this phenomenon is typical for open Hamiltonian systems that exhibit nonhyperbolic chaotic scattering.

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Early experiments in microwave ionization of highly excited hydrogen atoms [1,2] have attracted theoretical work for more than a decade [3–11]. In such experiments, one typically measures the ionization probability of hydrogen atoms prepared in some highly excited state as a function of the intensity of an externally applied microwave field. It has been observed that, for fixed interaction time, the ionization probability can increase from zero to some finite value, as the microwave field strength increases through some critical value. Classical models [3–6] interpret the onset of ionization as due to the breakup of Kolmogorov-Arnold-Moser (KAM) surfaces as the field strength increases so that the electron can cross through the remnants of the KAM surfaces, the so-called Cantori [12].

For a typical Hamiltonian system, due to the presence of KAM islands in phase space, particles escape from some predefined phase-space region containing KAM islands according to the algebraic law [13,14] $N(t)\sim t^{-z}$ for large t, where $N(t)$ is the number of remaining particles in that predefined phase-space region at time t and z is the decay exponent [$N(t)$ is proportional to the survival probability of a particle]. The algebraic decay arises due to the island-around-island structure, where a particle can spend a long time. Since the hydrogen problem also gives a phase-space structure with KAM islands [5,10,15], we should expect the electrons to ionize according to an algebraic law in the limit of long exposure times. In fact, for the case where the hydrogen atoms are driven by narrow microwave pulses, it has been observed that electrons ionize algebraically [15]. In this Brief Report we show that this algebraic decay law also holds for the one-dimensional hydrogen-atom model with continuous microwave driving. Moreover, we show that the asymptotic algebraic decay exponent can even *decrease* as the microwave field strength *increases*. This fact is surprising because, naively, one might think that as the microwave field strength increases, it should be easier for electrons to escape and, consequently, the algebraic decay exponent z would keep increasing. We argue that such unexpected decrease in the decay exponent is due to *metamorphoses of the phase space* in which, when KAM surfaces are destroyed and are transformed into Cantori, new layers of KAM islands are exposed and become accessible to the electron. Another finding is that, after such phase-space metamorphoses, the survival probability of the electron as a function of time exhibits a crossover region with different decay exponents. We argue that the phase-space metamorphoses and the crossover behaviors are typical for open Hamiltonian systems that exhibit nonhyperbolic chaotic scattering.

We illustrate our results in the framework of the one-dimensional model of a hydrogen atom whose Hamiltonian is given by [5]

$$H(x,p,t)=\begin{cases} p^2/2-1/x+\varepsilon x\sin(\omega t)\ , & x>0 \\ \infty\ , & x\le 0\ . \end{cases} \quad (1)$$

Here ε and ω are the microwave field strength and frequency in atomic units, respectively. In order to numerically integrate the equations of motion, it is convenient to represent the dynamics in action-angle variables [3–11] (I,θ):

$$H(\theta,I,t)=-\frac{1}{2I^2}+\{2I^2\sin^2[\eta(\theta)]\}\varepsilon\sin(\omega t)\ , \quad (2)$$

where $\eta(\theta)$ is related to θ by $\theta=2\eta-\sin(2\eta)$. It should be noted that the Hamiltonian of Eq. (2) is equivalent to the Hamiltonian of Eq. (1) only within the bounded part of the phase space. Since we are interested in the ionization probability due to the presence of KAM islands in regions of the phase space which correspond to the bounded electron dynamics, we can set a threshold value I_c (large) for the action. Whenever $I(t) \geq I_c$, we consider the hydrogen atom to be ionized.

In order to calculate the ionization probability for the different microwave field strength ε, we fix $\omega = 1$ and integrate the equations of motion corresponding to the Hamiltonian (2). We record the values of (I, θ) at the surface of section $\omega t = 2n\pi$, $n = 0, 1, 2, \ldots$. When $\varepsilon = 0$, the phase space consists of straight lines at $I = \text{const}$. In this case, no ionization occurs. As ε increases from zero, the phase space consists of both KAM islands and chaotic regions. Strictly speaking, there are KAM islands and two fundamentally different types of chaotic regions: open and confined. Except for a set of measure zero, electrons initialized in the open chaotic region will eventually escape (ionize). On the other hand, confined regions of the phase space are enclosed by shielding KAM surfaces. Therefore, electrons started in a confined area of the phase space will never escape.

To observe the decay of the survival probability of an electron, we choose a large number of initial conditions equidistributed in the angle θ at some constant I value in the open chaotic region and choose a critical action value I_c beyond which the electron is considered to have escaped. We then count the number of electrons $N(t)$ remaining below I_c at time t. Figures 1(a) and 1(b) show the $N(t)$ plots on a logarithmic scale for $\varepsilon = 0.015$ and 0.019, respectively, and $I_c = 2$. For both $\varepsilon = 0.015$ and 0.019, we choose 2000 initial conditions at $I_0 = 1.4$. From Figs. 1(a) and 1(b), we see that initially most of the electrons move in the open chaotic region and escape before they "feel" the existence of KAM islands. Hence, during this time, the decay of the electrons is faster than algebraic. Afterwards, the electrons decay algebraically. A least-squares fit gives an algebraic decay exponent of $z = 1.63 \pm 0.04$ for $\varepsilon = 0.015$. For $\varepsilon = 0.019$, there are two straight lines in different time regimes with a *crossover* at some time t_{cross}. The algebraic decay exponents for those two straight lines are $z_1 = 1.92 \pm 0.03$ and $z_2 = 1.00 \pm 0.03$, respectively. Observe that the slope in the short-time regime $(t < t_{\text{cross}})$ for the larger microwave field strength $(\varepsilon = 0.019)$ is larger than the slope $(z = 1.63 \pm 0.04)$ for the smaller microwave field strength $(\varepsilon = 0.015)$. However, the slope in the long-time asymptotic regime $(t > t_{\text{cross}})$ for $\varepsilon = 0.019$ is significantly smaller than the slope $(z = 1.63 \pm 0.04)$ for $\varepsilon = 0.015$. Physically, this means that, because of the newly accessible KAM islands [cf. Figs. 2(a) and 2(b)], it is possible for an electron to escape *more slowly* even though the microwave field strength is increased.

To understand how this phenomenon is related to KAM islands, we plot part of the phase-space structure for $\varepsilon = 0.015$ and 0.019, as shown in Figs. 2(a) and 2(b), respectively. In Fig. 2(a), besides the KAM surfaces, there are relatively large confined chaotic regions with

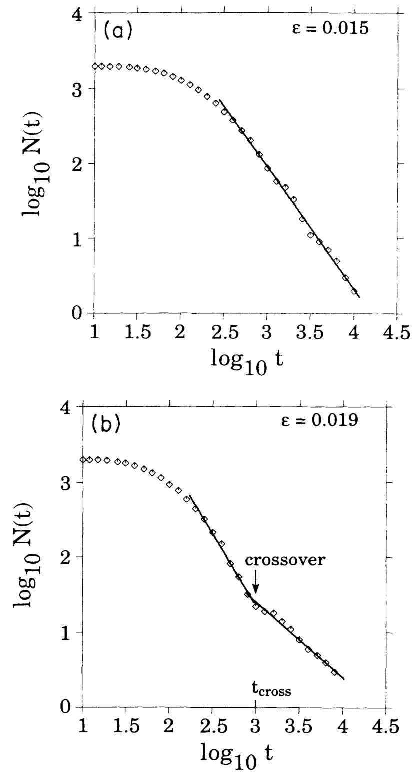

FIG. 1. Number $N(t)$ of surviving electrons as a function of exposure time t for $N(t=0) = 2000$ and (a) $\varepsilon = 0.015$, (b) $\varepsilon = 0.019$. For $\varepsilon = 0.019$ there is a crossover between two regimes of algebraic decay. Exposure time t is measured in microwave cycles.

KAM islands immersed in them (e.g., at $I \simeq 1$) and an open chaotic region for $I > 1.1$. Both regions are separated from each other by KAM surfaces around $I = 1.1$. Thus, for $\varepsilon = 0.015$, only the KAM islands immersed in the open chaotic region determine the asymptotic decay exponent of electrons. As ε increases, KAM islands in the open chaotic regions are destroyed, causing an increase in the decay exponent in the short-time regime of Fig. 1(b). On the other hand, as ε increases from 0.015 to 0.019, all the KAM surfaces that separate the open and the relatively large confined chaotic regions at $I \simeq 1$ are destroyed and transformed into Cantori. Now the originally confined chaotic regions with the corresponding island structures (for $\varepsilon = 0.015$) become accessible to the electron trajectories for $\varepsilon = 0.019$, as shown in Fig. 2(b). We call such a change in the structure of the phase space a *phase-space metamorphosis.* After a phase-space metamorphosis, the previously disconnected regions are connected via "leaky" Cantori and, as a result, electrons can now be transported between both regions and can escape from both. In particular, since the gaps in the Cantori can be arbitrarily small, if some electrons penetrate the Cantori and get into the previously confined chaotic region (or, if they are in this region initially), it is then likely for these electrons to wander in this region for a

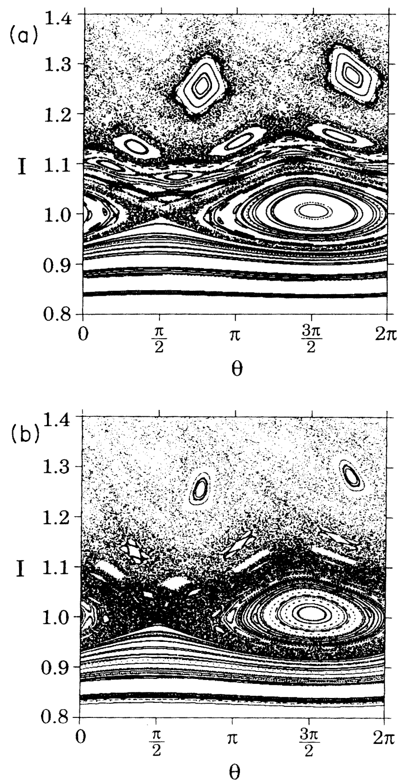

FIG. 2. The phase space for (a) $\varepsilon=0.015$ and (b) $\varepsilon=0.019$ in the one-dimensional hydrogen model.

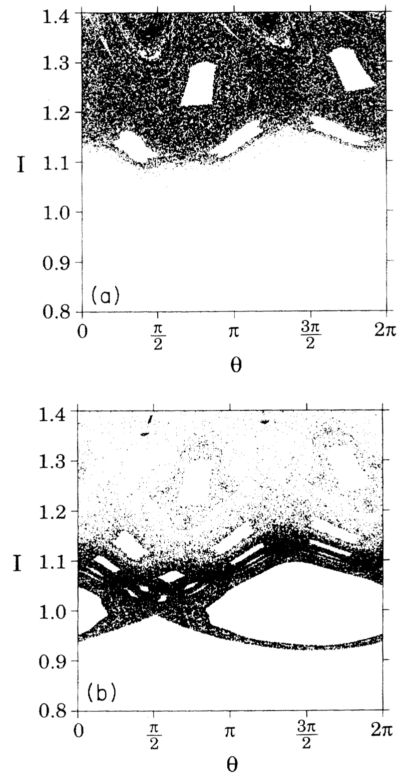

FIG. 3. Regions of phase space visited by electrons that escape in a time (a) $t_{\rm ion} \le t_{\rm cross}$ and (b) $t_{\rm ion} > t_{\rm cross}$.

very long time before they cross back the Cantori and escape. This causes a significant drop in the asymptotic algebraic decay exponent z, as shown in Figs. 1(a) and 1(b).

The above analysis implies that for $\varepsilon=0.019$, if we initialize a large number of electrons in the previously open chaotic region, some of the electrons only wander in this region and escape in a shorter time, while others cross the Cantori and wander in the previously confined chaotic region. These electrons take a longer time to exit since they have to make their way back through the Cantori. Hence, for *small time t,* only those electrons that wander in the previously open chaotic region contribute to the escape of electrons, while for *large time t,* electrons that penetrate the Cantori and wander in the previously confined chaotic region contribute to the escape of electrons. Therefore, the $N(t)$ versus t plot gives a straight line on a logarithmic scale with larger slope for small time t and another straight line with smaller slope for large time t. At some critical time $t_{\rm cross}$, we see a crossover from one slope to another. This situation is shown in Fig. 1(b). In this case, the asymptotic algebraic decay exponent is the absolute value of the slope of the straight line for $t \ge t_{\rm cross}$.

In order to further support our argument, we demonstrate that the two straight lines in Fig. 1(b) correspond to electrons that wander in different phase-space regions connected by the Cantori before they escape. To do so, we take several initial conditions that escape in a time either less than $t_{\rm cross}$ or larger than $t_{\rm cross}$ and then plot their phase-space trajectories before they escape. Figures 3(a) and 3(b) show the trajectories of electrons that ionize in a time $t_{\rm ion}$ less than $t_{\rm cross}$ and larger than $t_{\rm cross}$, respectively. Clearly, the two classes of electrons wander in *distinct* phase space regions before they escape. In particular, electrons that escape in $t_{\rm ion} \le t_{\rm cross}$ only wander in the

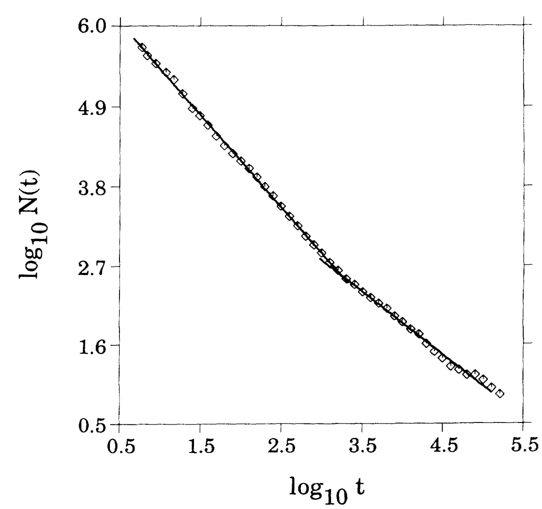

FIG. 4. Crossover between two regimes of algebraic decay in a Hamiltonian system that exhibits nonhyperbolic chaotic scattering.

previously open chaotic region, while electrons that escape in $t_{\rm ion} > t_{\rm cross}$ spend a large amount of time in the previously confined chaotic region (see the high density of electron trajectories in the phase-space region around $I \approx 1$).

Since a crossover of the two decay exponents results from a phase-space metamorphosis, such crossover behavior is expected to be typical in open nonhyperbolic Hamiltonian systems. In fact, we find the same phenomena in a simple typical Hamiltonain system [16–18] that exhibits nonhyperbolic chaotic scattering [19,20]. In this system, the particles are scattered from an infinite array of nonoverlapping elastic scatterers in the plane [16–18,20]. These scatterers are placed at constant intervals D along a straight line (say, the y axis) and each scatterer is represented by a circular potential $V(r)$ (attractive) that vanishes for $r > R$, where $R < D/2$. Choosing $V(r) = -V_0[1-(r/R)^2]$, where $V_0 = 0.2$ and $R = 1$, this system exhibits nonhyperbolic chaotic scattering. We can then examine the escape of particles from the scattering region containing KAM islands. Figure 4 shows a crossover of two straight lines on a $\log_{10}N(t)$ versus $\log_{10}t$ plot for $D = 3.72$. Our analysis shows that particle trajectories on the two straight lines in Fig. 4 correspond to two distinct phase-space regions separated by Cantori [20]. These two regions are separated from each other by KAM surfaces at D values slightly smaller than $D = 3.72$.

Since quantum mechanics is known [9] to mimic the classical time evolution of observables for times $t < t^*$ where $t^* \to \infty$ for $\hbar \to 0$, our results are of immediate relevance for the quantum thoery of microwave ionization. It is, however, difficult to estimate whether state-of-the-art microwave experiments can penetrate deeply enough into the semiclassical regime for $t^* > 10^4$ microwave cycles. Thus, in order to observe the phenomena addressed in this Brief Report [namely, (1) the algebraic decay, (2) the decrease of decay exponents as the microwave field strength increases, and (3) the crossover between two decay exponents], it may be necessary to ionize Rydberg atoms prepared in a quantum state with very large principal quantum number or search for phase-space metamorphoses which result from $t_{\rm cross} \ll 1000$ microwave cycles.

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