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Driving trajectories to a desirable attractor by using small control

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Abstract

Driving trajectories to a desirable attractor for dynamical systems with multiple coexisting attractors has been a challenging problem in the field of chaos control. We develop an algorithm to steer most trajectories to the desirable attractor by using only small feedback control. The idea is to build a hierarchy of paths to the desirable attractor and then stabilize trajectories around one of the paths in the hierarchy. A substantial improvement in the probability for a random trajectory to asymptote to the desirable attractor has been achieved when there are fractal basin boundaries in the phase space.

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1. Introduction

It is common for nonlinear dynamical systems to exhibit multiple coexisting attractors, each with its own basin of attraction. The basin of attraction of an attractor is the set of initial conditions in the phase space that asymptote to the attractor. In practical applications, when one of the attractors according to some criteria would yield superior systems performance over the others, it is important to be able to drive most trajectories to the desirable attractor in an efficient and economic way. That is, one wishes to drive trajectories to the desirable attractor rapidly by using only small feedback control to an accessible parameter or state of the system. Previous work has demonstrated that in periodically driven dynamical systems, multi-

ple basins of attraction can be eliminated by replacing the periodic driving by some appropriately chosen, but somewhat large-amplitude chaotic driving [1].

In this paper, we develop a method to drive most trajectories to a desirable attractor by using only small feedback control. We emphasize the need to use small feedback control [2] since (1) we do not wish to alter the system substantially, and (2) large perturbations to the system may be costly. As such, it is only possible to alter the fate of the trajectories resulting from initial conditions in the vicinity of basin boundaries because, for a trajectory deep in the basin of an undesirable attractor, small perturbations cannot change the attractor to which the trajectory is asymptoting. In this regard, it is necessary to distinguish between smooth and fractal basin boundaries [3–7]. Imagine there is an N -dimensional chaotic system. Consider a phase space region that contains part of the basin boundary. Assume that only small perturbations of magnitude ϵ

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($\epsilon \ll 1$) to an accessible system parameter or state are allowed. If the boundary is smooth, the dimension of the boundary is $D = N - 1$. Thus, the fraction of trajectories whose asymptotic attractors can be altered by small ϵ perturbation is on the order of magnitude of $\epsilon^{N-D} = \epsilon$, which is also very small. If, on the other hand, the basin boundary is fractal with box-counting dimension (capacity) D , where D is a fractional number that satisfies $N - 1 < D < N$, the fraction of trajectories whose fate can be manipulated using small perturbation is $f(\epsilon) \sim \epsilon^\alpha$, where $\alpha = N - D < 1$ is the uncertainty exponent² [3,4]. Thus, if $\alpha \ll 1$, $f(\epsilon)$ can be large. Fractal basin boundaries with $\alpha < 1$ are common in dynamical systems [3–7], and $\alpha \ll 1$ are particularly common in high-dimensional systems [9,10] or in systems with riddled basins³ [11–16]. Therefore, although the presence of fractal basin boundaries with $\alpha < 1$ poses a fundamental difficulty to predict the asymptotic attractor of the system because of the inevitable error in the specification of initial conditions or system parameters, these boundaries offer a possibility for us to greatly increase the probability that typical trajectories can be driven to the desirable attractor by using arbitrarily small perturbations, provided that we are able to harness the system in an intelligent way. The purpose of this paper is to present an algorithm and numerical examples to demonstrate that this is indeed possible.

This paper is organized as follows. In Section 2, we discuss the method of control. In Sections 3 and 4, we present numerical examples for a system with fractal basin boundaries and a system with riddled basins, respectively. In Section 5, we present a discussion.

2. Method of control

The setting of the problem is as follows. Let the dynamical system be described by an N -dimensional flow $dx/dt = F(x, p)$ or an N -dimensional map $x_{n+1} = M(x_n, p)$, where p is an accessible system parameter. For concreteness, assume there are two distinct attractors for the range of system parameter values of interest. Furthermore, assume that the coexistence of the two attractors is structurally stable, i.e., small change in the parameter changes the behavior of attractors and their basin structures only slightly. Denote the two distinct attractors by **A** and **B**. For a given region Σ in the phase space that contains part of the basin boundary, a fraction of initial conditions f_A will yield trajectories that asymptote to attractor **A**, and the remaining initial conditions, a fraction of $f_B \equiv 1 - f_A$, asymptote to attractor **B**. Without loss of generality, assume that f_A and f_B are of the same order of magnitude. Suppose that one of the two attractors yields much superior system performance than the other. We thus wish to increase f_A as much as possible so that most initial conditions asymptote to the attractor with better system performance. This will not occur if no external perturbations to the system are applied. Our goal is to devise an algorithm to increase substantially the fraction of initial conditions that asymptote to the desirable attractor, given that p can be adjusted finely around a nominal value p_0 : $p \in [p_0 + \Delta p, p_0 - \Delta p]$, where $\Delta p/p_0 \ll 1$.

Our main idea is to build a hierarchy or “tree-like” structure of paths to the desirable attractor [17]. Specifically, let **A** be the desirable attractor. We first randomly choose an initial condition in Σ such that it generates a trajectory to **A**. Call this trajectory the “root” path 1 to **A** and denote it by X_0, X_1, \dots, X_A , where X_A is a point on **A** (or a point in the vicinity of **A**). We then choose a second trajectory to **A** from an arbitrary initial condition Y_0 in Σ . But for this second path, we examine if it approaches to **A** directly without coming close to root path 1, in which case we call it root path 2. It is also likely that a point on this trajectory Y_n can fall into a suitably small neighborhood of some point along root path 1 before it comes close to **A**. In this case, we store Y_n together with the path of $n - 1$ points leading to Y_n . We call Y_0, Y_1, \dots, Y_n the secondary path of the root path 1. This procedure can be repeated for initial conditions chosen on a uniform

² The relation between the uncertainty exponent and the box-counting dimension, $\alpha = N - D$, was rigorously proven for Axiom-A systems [8]. It was conjectured that the same relation holds for more general dynamical systems [3,4].

³ It should be noted that a riddled basin has a dimension equal to N and its basin boundary is the basin itself. Thus, the boundary also has a dimension equal to N . However, its uncertainty exponent is not equal to zero, although in most cases it is close to zero. Therefore, strictly speaking, the relation between the basin boundary dimension and the uncertainty exponent $D = N - \alpha$ does not hold for riddled basins.

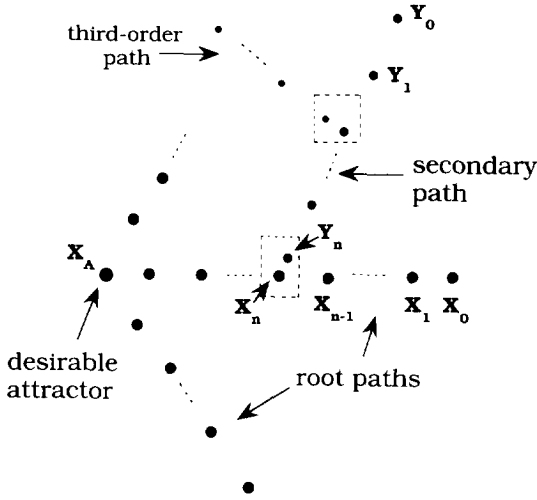


Fig. 1. A schematic illustration of the hierarchy of paths (bush) to the desirable attractor in the phase space.

grid of size δ in Σ . Of course, if a trajectory goes to an undesirable attractor, we simply disregard this trajectory in the tree-building process. Finally, with suitably chosen δ , a hierarchy of paths to A in Σ can be built with, say, N_R root paths. On each root path i , there can be some secondary paths, and on each secondary path there can be third-order paths, etc. We therefore obtain a tree of paths to A in a region that contains the basin boundary, as schematically shown in Fig. 1. In fact, since there can be many root paths, this is more like a “bush” of paths leading to the desirable attractor⁴. A remaining question is how fine the grid from which initial conditions are chosen should be. Clearly, the size of the grid δ should be comparable to the magnitude of the allowed parameter perturbation Δp , which is approximately the size of controlling neighborhood around each point on the bush of paths. If $\delta \gg \Delta p$, most trajectories that originally go to the undesirable attractors will not come close to a bush of paths and therefore will not be controlled. If $\delta \ll \Delta p$, the bush of paths may have contained too many details and therefore may have used too much computer memory that is unnecessary for realizing the control.

⁴ This bush-like structure of paths is somewhat different from the tree structure of paths used in the targeting procedure in Ref. [17]. In that case, there is only one root path to the target point on the attractor, and all paths to the target live on the same chaotic attractor:

To control a trajectory to direct it to the desirable attractor after it comes close to a path on the bush, we employ a simple feedback scheme. For simplicity we consider the N -dimensional map $x_{n+1} = M(x_n, p)$. Suppose a trajectory originated from a random initial condition x_0 falls into an ϵ -neighborhood of a point y_n on the bush at some later time n , i.e., $|x_n - y_n| \leq \epsilon$. Let y_n, y_{n+1}, \dots, y_A be the path on the bush that starts at y_n and ends at y_A which is in the ϵ -neighborhood of the desirable attractor. In the vicinity of y_n , we have the following linearized dynamics: $\Delta x_{n+1} = DM(x_n, p) \Delta x_n + (\partial M / \partial p) \Delta p_n$, where $\Delta x_n = x_n - y_n$, $\Delta p_n = p_n - p_0$, and the Jacobian matrix $DM(x_n, p)$ and the vector $\partial M / \partial p$ are evaluated at $x_n = y_n$ and $p_n = p_0$. Choosing a unit vector u in the phase space and letting $u \cdot \Delta x_{n+1} = 0$, we obtain for the required parameter perturbation,

$$\Delta p_n = \frac{-u \cdot DM(x_n, p) \cdot \Delta x_n}{u \cdot \partial M / \partial p}. \tag{1}$$

In principle, the unit vector u can be chosen arbitrarily provided that (1) it is not orthogonal to x_{n+1} , and (2) the denominator in Eq. (1) is not close to zero. In practice, we define a maximum allowed magnitude for the parameter perturbation $\Delta p_{\max} \sim \epsilon$. If the computed $|\Delta p_n|$ exceeds Δp_{\max} , we set $\delta p_n = 0$. Doing this would cause loss of control occasionally. But we find in our numerical experiments that robust control can still be achieved since setting $\Delta p_n = 0$ is done only rarely. Because Δx_n is small, Δp_n is also small. In the sequel, we present two numerical examples to illustrate our control method.

3. Example 1: controlling fractal basin boundaries

We consider the following two-dimensional map [4],

$$\begin{aligned} \theta_{n+1} &= \theta_n + a \sin(2\theta_n) - b \sin(4\theta_n) - x_n \sin(\theta_n), \\ x_{n+1} &= -J_0 \cos(\theta_n), \end{aligned} \tag{2}$$

where x can be regarded as the radial distance from the center of an annulus, θ is an angle variable so that θ and $\theta + 2\pi$ are equivalent, and a, b and J_0 are parameters. The system is invariant under the symmetry $\theta \rightarrow 2\pi - \theta$. The determinant of the Jacobian matrix

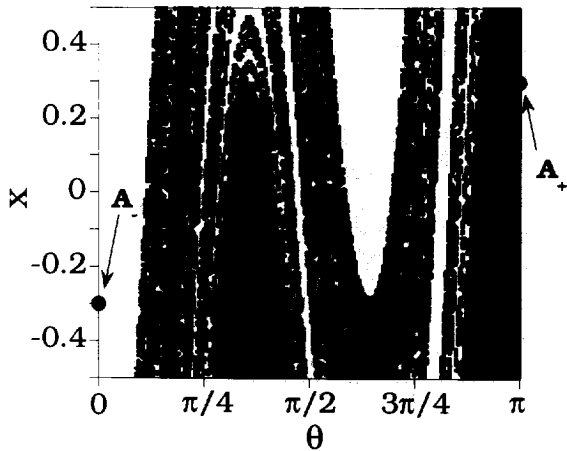


Fig. 2. Basins of attraction for Eq. (2). Black and blank regions are the basins of the A_+ and A_- attractors, respectively. Parameter setting is: $a = 1.32$, $b = 0.9$, and $J_0 = 0.3$.

is $J_0 \sin^2(\theta) < 1$ (for $J_0 < 1$). At the following parameter setting, $a = 1.32$, $b = 0.9$, $J_0 = 0.3$, there are two attractors, located at $x = -0.3$, $\theta = 0$ (denoted by A_-) and $x = 0.3$, $\theta = \pi$ (denoted by A_+), respectively. The boundaries between basins of the two attractors are fractal, as shown in Fig. 2, where black dots represent the basin of the A_+ attractor. The dimension of the basin boundary is approximately 1.8, corresponding to an uncertainty exponent of $\alpha \approx 0.2$ [4].

Now assume that the attractor A_+ corresponds to a better system performance so that it is the desirable attractor. Without control, the fraction of initial conditions that asymptote to A_+ is about 50% for the phase-space region in Fig. 2. Assume a is an accessible parameter which can be perturbed slightly around its nominal value $a_0 = 1.32$. We first build a bush of paths to A_+ by using a grid of 100×100 initial conditions in the region ($0 \leq \theta \leq \pi$, $-0.5 \leq x \leq 0.5$) (corresponding to grid size $\delta \approx 3.3 \times 10^{-2}$). We arbitrarily choose $u = (1/\sqrt{2})(1, 1)$ to compute the parameter perturbation Δa_n from Eq. (1). Fig. 3a shows a controlled trajectory (solid line) to the desirable attractor A_+ , where both the size of the controlling neighborhood ϵ and the maximal allowed parameter perturbation Δp_{\max} are set to be 10^{-2} . The trajectory would asymptote to the undesirable attractor without control, as shown by the dotted line in Fig. 3a. When control is applied using these values of ϵ

and p_{\max} , about 70% of the initial conditions in the region ($0 \leq \theta \leq \pi$, $-0.5 \leq x \leq 0.5$) asymptote to A_+ , increased by 20% as compared with the case without control. Let δ be the size of the covering when the bush is built. Clearly, the fraction of initial conditions that asymptote to A_+ depends on both δ and the size of the controlling neighborhood ϵ . Fig. 3b shows, with the same bush of paths to A_+ as in Fig. 1a, f^+ versus ϵ ($\Delta p_{\max} = \epsilon$) for δ fixed at about 3.3×10^{-2} and $10^{-4} \leq \epsilon \leq 10^{-1}$. In the figure, for each value of ϵ , $N_0 = 90000$ (300×300) initial conditions uniformly distributed in the region ($0 \leq \theta \leq \pi$, $-0.5 \leq x \leq 0.5$) are chosen and the number of controllable initial conditions N_+ , i.e., those asymptote to A_+ via control, are recorded. The fraction f^+ is approximated by N_+/N_0 . Since the grid size for building the bush is $\delta \sim 10^{-2}$, we see that when $\epsilon \ll \delta$, essentially no improvement in f^+ is achieved because it is unlikely for initial conditions originally asymptoting to A_- to fall in the vicinity of points along the bush. When $\epsilon \sim 10^{-2}$, a maximum increase in f^+ is achieved because in this case, it is easy for trajectories to come close to the bush and to be controlled. However, if $\epsilon > \delta$, control may be lost for some initial conditions because the linearized dynamics used to derive the parameter perturbation equation (1) no longer holds at large values of ϵ , although in this case it is easy for trajectories to fall in the ϵ -neighborhood of the bush. Thus, we see that f^+ starts to decrease as ϵ increases above 10^{-2} . To more clearly see the dependence of f^+ on both δ and ϵ , we compute f^+ for systematically chosen δ and ϵ values. Fig. 3c shows the three-dimensional plot of f^+ versus δ and ϵ for $10^{-2} < \delta < 10^{-0.4}$ and $10^{-4} < \epsilon < 10^{-1}$. Because of the two-dimensional phase space region used to construct the bush, decreasing δ to values below 10^{-2} leads to a huge number of points on the bush and thus to numerical difficulties. Nonetheless, it is clear from Fig. 3c that for the range of δ values chosen, the maximum improvement in f^+ occurs at $\epsilon \sim 10^{-2}$.

To understand why the optimal improvement in f^+ occurs at $\epsilon \sim 10^{-2}$, we note that the maximum value f_{\max}^+ of f^+ , as ϵ changes, depends on δ . Generally, f_{\max}^+ is small if δ is too large because in this case the basin boundaries are not adequately covered. As δ decreases, f_{\max}^+ increases. But if δ becomes so small that the entire basin boundaries are covered by the bush, decreasing δ further does not help to increase f_{\max}^+ .

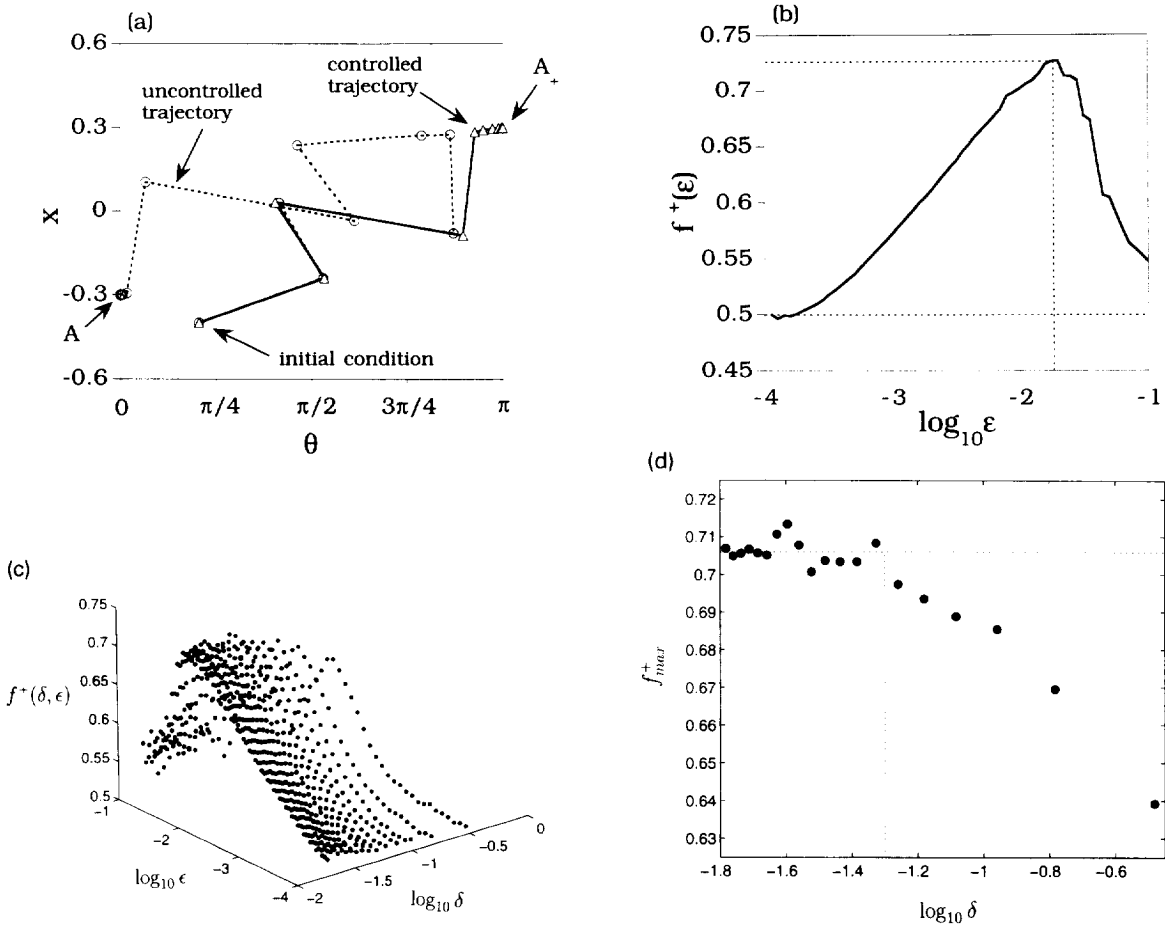


Fig. 3. (a) A controlled trajectory to the desirable attractor A_+ , where the size of the controlling neighborhood ϵ and the maximum allowed parameter perturbation Δp_{max} are both 10^{-2} . The trajectory would asymptote to the undesirable attractor A_- without control. (b) At $\delta \sim 10^{-2}$, the fraction f^+ of initial conditions that asymptote to A_+ versus $\log_{10} \epsilon$. (c) The three-dimensional plot of f^+ versus $\log_{10} \delta$ and $\log_{10} \epsilon$. We see that the optimal ϵ value for which f^+ reaches maximum is about 10^{-2} . (d) The maximum value f_{max}^+ versus $\log_{10} \delta$.

Thus, f_{max}^+ saturates as δ decreases through a critical value. This behavior is shown in Fig. 3d, where we see that f_{max}^+ saturates at $\delta \approx 10^{-1.26}$. The saturated value of f_{max}^+ is about 0.706. At this δ value, there are approximately $N_c = 17300$ points on the bush. These are the required points to cover the basin boundaries adequately. We ask, how many of these points can be influenced by perturbations of magnitude ϵ ? The answer is $\sim \epsilon^\alpha N_c$ because the fraction of basin boundary points that are uncertain with respect to perturbation ϵ scales like ϵ^α . Since these N_c points on the bush provides a good covering of the basin boundaries, we have $\epsilon_{optimal}^\alpha N_c \epsilon_{optimal}^2 \sim 1$, which gives $\epsilon_{optimal} \sim$

$N_c^{-1/(2+\alpha)} \approx 1.2 \times 10^{-2}$. This agrees with the numerical observation in Figs. 3b and 3c.

4. Example 2: controlling riddled basins

We first briefly review the concept of riddled basins. Riddled basins usually occur in dynamical systems with a simple type of symmetry. The existence of symmetry often leads to invariant subspace in the phase space. The description of riddled basins was introduced in Ref. [11], where it was shown that for a certain class of dynamical systems with an invariant subspace: (i) if there is a chaotic attractor in the in-

variant subspace; (ii) if there is another attractor in the phase space; and (iii) if the Lyapunov exponent transverse to the subspace is negative, then the basin of the chaotic attractor in the invariant subspace can be riddled with holes belonging to the basin of the other attractor. That is, for every initial condition that asymptotes to the chaotic attractor in the invariant subspace, there are initial conditions arbitrarily nearby that asymptote to the other attractor. Rigorous results on the dynamics of riddled basins for discrete maps were presented in Refs. [11] and [12]. The dynamics of riddled basins was subsequently investigated in Ref. [13] using a more realistic physical model. A more extreme type of basin structure referred to as “intermingled basins” in which the basins of more than one chaotic attractors are riddled, was also studied using both discrete maps [11] and a more realistic physical system [14]. Riddled basins have been verified in experiments conducted using coupled electrical oscillators [15,16]. The mechanism for riddling to occur, and the basin structure associated with the riddling, were investigated by Ashwin et al. [16].

We consider the following two-dimensional map,

$$x_{n+1} = g(x_n) + by_n^2, \quad y_{n+1} = ax_n y_n + y_n^3, \quad (3)$$

where $g(x)$ is a chaotic map, b and a are parameters. The invariant subspace is the one-dimensional line defined by $y = 0$ since if $y_0 = 0$, then $y_n = 0$ for $n \geq 1$. For simplicity we choose $g(x)$ to be the logistic map $g(x) = rx(1-x)$ with a chaotic attractor. The transverse Lyapunov exponent is given by

$$A_{\perp} = \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{n=0}^L \ln \left| \frac{\partial y_{n+1}}{\partial y_n} \right|_{y_n=0} = \int ax\rho(x) dx, \quad (4)$$

where $\rho(x)$ is the invariant density of the chaotic attractor produced by the logistic map. We choose $r = 3.8$ and obtain $a_c \approx 1.725$ where $A_{\perp} \geq 0$ for $a \geq a_c$ and $A_{\perp} < 0$ for $a < a_c$. For $a < a_c$, there are two attractors, one is $y = 0$ and the other is $|y| = \infty$. The basin of the $y = 0$ attractor is riddled. Fig. 4a shows part of the basin of the $y = 0$ attractor (black dots) for $a = 1.7 < a_c$ and $b = 0.1$, where a grid of 600×600 initial conditions is chosen in the region $0 \leq x \leq 1$ and $0 < y \leq 0.2$. Examination of the figure on finer and finer scales reveals that there are white regions

(basin of the $y = \infty$ attractor) near every black dot, a typical feature of riddling.

Now assume that the $y = 0$ chaotic attractor is the desirable attractor and the $y = \infty$ attractor is the undesirable one. To facilitate numerical computation, we choose to control trajectories starting from initial conditions on a line, say, $y = 0.1$. Without control, about 56% of the initial conditions on this line go to the desirable attractor. We build a bush of trajectories starting from $y = 0.1$ with size δ , i.e., we use δ^{-1} points on $y = 0.1$ to determine the points that asymptote to the desirable attractor. A difficulty here is that it typically takes many iterations for a trajectory starting at $y = 0.1$ to reach the desirable attractor (numerically a trajectory is regarded as having $y = 0$ if it stays within 10^{-12} of $y = 0$ for certain prescribed number of iterations). Thus, for small grid size the number of points on the bush can be very large. It then becomes computationally difficult to determine whether a trajectory point is close to the bush. To make the computation feasible, we adopt the following strategy. For a random initial condition chosen from the line at $y = 0.1$, we examine whether it falls in an ϵ -neighborhood of a starting point of a path on the bush. If yes, we control it. Otherwise we let it evolve without control. Assume r in the logistic map is the accessible parameter to be perturbed. The parameter perturbations can be computed from Eq. (1). Fig. 4b shows f^+ versus $\log_{10} \epsilon$ for $\delta = 10^{-5}$, where f^+ is the fraction of initial conditions that asymptote to the desirable attractor. The plot exhibits similar feature to that of Fig. 3b. We see that no improvement in f^+ is achieved if ϵ is too small because there are almost no points that come close to the bush. If ϵ is too large, although many trajectories would fall in the ϵ -neighborhood of the bush, control can get lost because Eq. (1) is only a linear control law. The optimal ϵ value for which f^+ reaches maximum is about $10^{-3.5}$, and the maximum possible value of f^+ is about 0.86, a substantial improvement in f^+ compared with the case of no control. Fig. 4c shows the three-dimensional plot of f^+ versus δ and ϵ . We see that for $\delta < 10^{-2}$, f^+ reaches a maximum at $\epsilon_{\text{optimal}} \approx 10^{-3.5}$. Fig. 4d shows f_{max}^+ versus $\log_{10} \delta$. For $10^{-2} < \delta < 10^{-1}$, f_{max}^+ is about the same as if there were no control. It then increases rapidly as δ is decreased from 10^{-2} and starts to increase slowly as δ decreases through 10^{-4} . Compared with the example of controlling fractal basin boundaries, we see

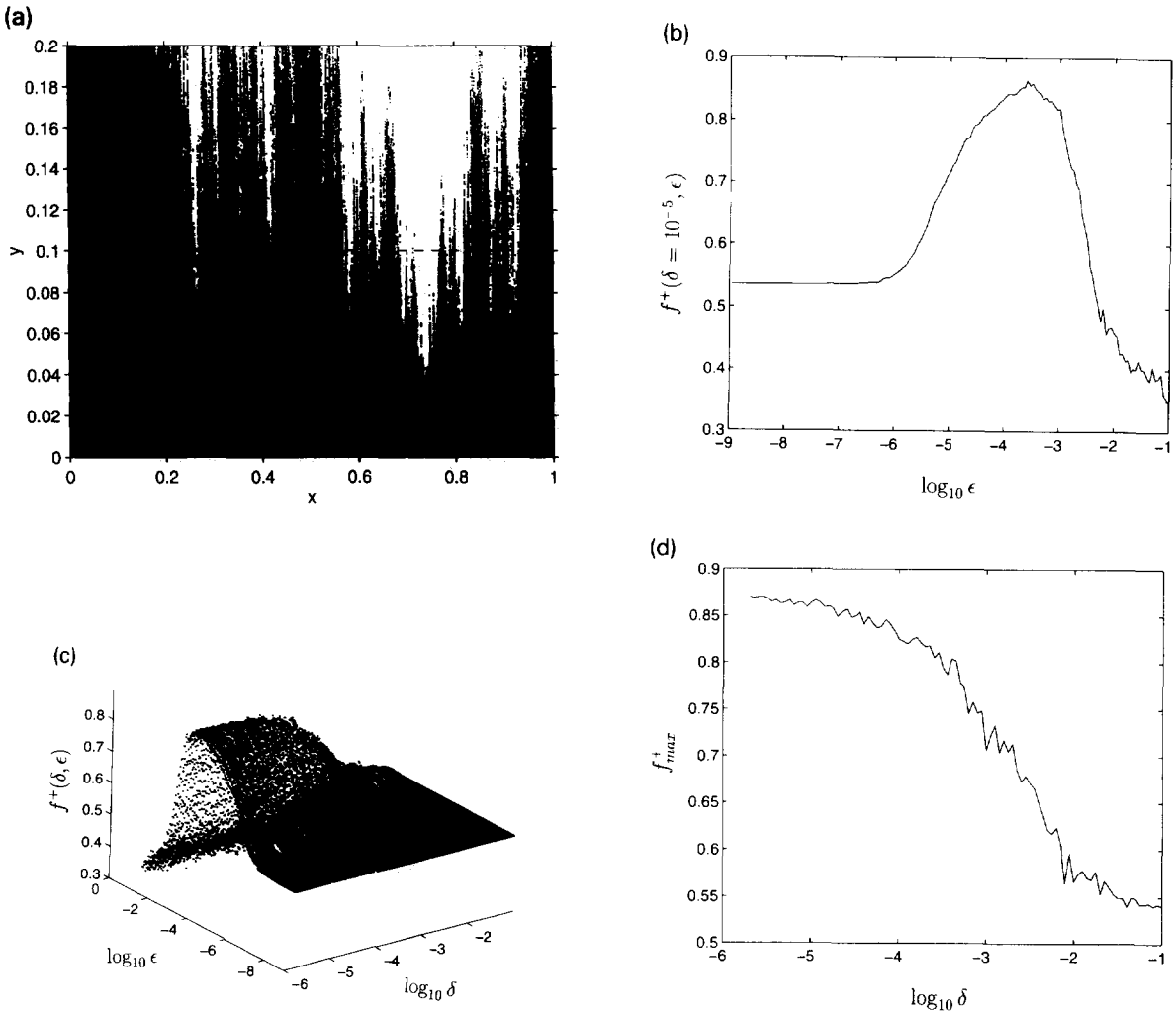


Fig. 4. (a) Riddled basin of the $y = 0$ attractor for Eq. (3). The parameter setting is $r = 3.8$, $a = 1.7$ and $b = 0.1$. (b) For $\delta = 10^{-5}$, f^+ versus $\log_{10} \epsilon$. (c) The three-dimensional plot of f^+ versus $\log_{10} \delta$ and $\log_{10} \epsilon$. (d) The maximum value f_{\max}^+ versus $\log_{10} \delta$.

that the maximum value of f_{\max}^+ can be higher. This is due to the feature of riddled basins where the uncertainty exponent α is close to zero. The reason that we obtain $\epsilon_{\text{optimal}} \approx 10^{-3.5}$ can be understood by a similar argument in Section 3. At $\delta \approx 10^{-4}$, there are about 5560 points on $y = 0.1$ that belong to the bush. Thus we have $\epsilon_{\text{optimal}}^{1+\alpha} N \sim 1$. Since $\alpha \approx 0$, this gives $\epsilon_{\text{optimal}} \sim 10^{-3.7}$.

5. Discussions

In conclusion, we have presented an algorithm to drive trajectories to a desirable attractor by using small feedback control for dynamical systems with multiple coexisting attractors. The basic idea is to build a bush-like structure of paths to the target attractor and to stabilize a trajectory around one of the many paths on the bush so that the trajectory asymptotes to the desirable attractor. Such a structure of paths, in principle, can be built up even in more realistic applications. For instance, an experimentalist could run the system

first, measure time series resulting from many initial conditions, and build the bush of paths to the desirable attractor in the reconstructed phase space by using the delay-coordinate embedding technique. One can then use techniques such as the direct proportional feedback control [18] to compute the required parameter perturbations as it may be a formidable task to apply Eq. (1) in practice. But of course, at present there is no guarantee that our method can be applied to practical applications. We stress, however, that from a more general theoretical point of view, the success of the method relies on the region in the phase space to which the bush extends. As such, the method is particularly effective when there are fractal basin boundaries with large values of fractal dimension (or small values of the uncertainty exponent α) in the phase space region of interest. In contrast, there is no appreciable increase in the probability for a trajectory to be driven to the desirable attractor if the basin boundaries are smooth. *One could, therefore, deliberately build into the system fractal basin boundaries or riddled basins in order to drive most initial conditions to the desirable attractor.* While there is a great uncertainty in determining the asymptotic attractor for individual initial conditions when there are fractal basin boundaries or riddled basins, the uncertainty is greatly reduced for a path that consists of a large number of points in the phase space. Therefore, insofar as a trajectory can be stabilized around a path on the bush, the fate of the trajectory is almost certain, i.e., the desirable attractor.

Theoretically, there is no reason for restricting the control to a bush. For instance, one may obtain more optimal results in the following way⁵. Assume that for a map M we have basin of attraction D for a desirable attractor A . For perturbation of magnitude ϵ , let $B_\epsilon(D)$ be the union of all ϵ -balls around all points in D (the ϵ -parallel body of D). Consider the union of all the preimages of the $B_\epsilon(D)$. This union would give the largest possible domain (always an open set) for which there exist ϵ -pseudo orbits of the map that eventually asymptote to the desirable attractor A . If a practical method can be devised to cover this union and to drive trajectories in this union to A , we would expect to achieve an absolute maximum size for the basin of attraction of A under arbitrarily small perturbations. At present, how to cover such a union and how to

devise a control algorithm to achieve this theoretical maximum remains unknown.

The central problem in controlling dynamical systems with multiple coexisting attractors is how to maximize the probability of being able to control an arbitrary initial condition. Thus, it is important to assess how this probability varies with the maximum allowed perturbation and the dimension of the basin boundaries. To obtain this information, we now imagine an “ideal” controller. We restrict to situations where only the initial conditions near basin boundaries are accessible to control. For a given initial condition, the ideal controller would evolve the system to see if the asymptotic attractor is the desirable one. If not, a small parameter perturbation ϵ is applied and the system is evolved from the same initial condition. The controller would then check if the initial condition yields the desirable attractor. It could repeat this procedure for a given number of time, insofar as the asymptotic attractor is not the desirable one. In this case, the probability for driving an arbitrary initial condition to the desirable attractor is proportional to the fraction of uncertain initial conditions, which scales with the perturbation as ϵ^α . Thus, we see that for fixed α , increasing ϵ would increase the desired probability. For fixed $\epsilon < 1$, increasing the dimension of the basin boundary, which is equivalent to decreasing the uncertainty exponent α , would increase the desired probability. This, of course, holds only for the ideal controller. In a more practical situation, we see that there exists an optimal ϵ value for achieving the desired probability (see Figs. 3 and 4). This optimal ϵ value depends on many factors including the dimension of the basin boundaries. Nonetheless, high desired probability can be achieved if the dimension of the basin boundaries is large (or α is small). In cases where the basin-boundary dimension is close to the phase-space dimension (or α is close to zero, such as in riddled basins), one expects to achieve higher desired probability as in such a case, most initial conditions are near the basin boundaries.

Finally, we emphasize that the control method proposed in this paper represents *only one possible* approach to solve the general problem of controlling dynamical systems with multiple basins of attraction. There will undoubtedly be better methods that await for future investigation.

⁵ This is due to an anonymous referee.

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