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Persistence of supertransients of spatiotemporal chaotic dynamical systems in noisy environment

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Abstract

Superlong chaotic transients have been discovered in numerical simulations of model spatiotemporal chaotic dynamical systems. The presence of such transients poses a fundamental difficulty for observing the asymptotic state of the system. In this paper, we investigate the effect of small random noise on the lifetime of the chaotic transient. It is found that the averaged transient lifetime scales algebraically with the amplitude of the noise, with a near-zero exponent. This indicates that the transient lifetime is almost independent of the noise amplitude and, consequently, the presence of noise is not advantageous in attempts to reduce the transient lifetime. Therefore, we expect supertransients to be common in spatially-extended chaotic systems.

Transient chaos is ubiquitous in chaotic dynamical systems [1,2]. In such a case, trajectories starting from random initial conditions wander chaotically for some period of time before settling into a final nonchaotic attractor. Previous studies have established that transient chaos is due to the existence of chaotic saddles in phase space [1]. Chaotic saddles are nonattracting chaotic sets that have distinct dynamic properties [3,4]. When there is a chaotic saddle in the phase space, trajectories originating from random initial conditions usually wander in the vicinity of the chaotic saddle for a finite amount of time before escaping the chaotic saddle and settling into the asymptotic attractor, thereby giving rise to the phenomenon of transient chaos. Studies have also revealed that the averaged lifetime of the chaotic transient can be related to the dynamic characteristics, such as the fractal dimension and Lyapunov exponents, of the chaotic saddle [3]. A common feature of low-dimensional chaotic transients is that their lifetime is usually short.

Chaotic transients also occur in spatiotemporal dynamical systems which are high-dimensional. These transients differ from most low-dimensional chaotic transients in that the lifetime of chaotic transients in spatiotemporal systems is usually extremely long ("supertransients") [5–7]. Crutchfield and Kaneko first observed in numerical experiments that spatially extended systems exhibit chaotic transients: transients long enough so that the observation of the system's asymptotic attractor is practically impossible [5]. More recently, Hastings and Higgins demonstrated the existence of complex transient dynamics in simple discrete-time, spatially extended ecological models for a species with alternating reproduction and dispersal [7]. They observed that with sufficiently strong nonlinearity, the time required for the system to settle

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into the asymptotic attractor is usually very long, approaching thousands of generations. These results are consistent with the observed behavior in populations of certain biological species [7].

The presence of supertransients poses a fundamental difficulty for observing asymptotic dynamics of the system in reasonable time scales. Dynamically, supertransients mean that the corresponding chaotic saddle is extremely "sticky": trajectories wander in the neighborhood of the chaotic saddle for an extremely long time before escape. In practice, however, the existence of environmental noise is inevitable. Thus one might hope that under the influence of random noise, it would be easier for trajectories to escape the chaotic saddle and, consequently, the transient lifetime might be decreased as the noise amplitude increases. Motivated by this, in this paper we investigate, computationally, the effect of small-amplitude random noise on the lifetime of the supertransients in spatiotemporal systems. It is found that the transient lifetime τ scales with the noise amplitude σ in the following algebraic form,

$$\tau \sim \sigma^{-\eta}, \qquad (1)$$

where $\eta \geqslant 0$ is the algebraic scaling exponent. A surprising finding is that η is very close to zero, meaning that the transient lifetime decreases only incrementally even if the noise amplitude is increased over many orders of magnitude. The implication is that supertransients are a robust phenomenon in spatiotemporal chaotic systems and, consequently, it may never be possible to observe the "real" asymptotic state of such systems within practical time scales even when noise is present.

We consider the following diffusively coupled logistic map lattice [8],

$$\begin{aligned} x_{n+1}^i &= [1-(\delta+\sigma h_n)] f(x_n^i) \\ &+ \tfrac{1}{2}(\delta+\sigma h_n)[f(x_n^{i+1}) + f(x_n^{i-1})], \\ i &= 1,\dots,N, \end{aligned} \qquad (2)$$

where $f(x) = ax(1-x)$ is the one-dimensional logistic equation, i and n denote discrete spatial site and time, N is the number of coupled maps, and δ is the coupling strength. To model the influence of noise, we add a term σh_n to the coupling strength, where σ is the noise amplitude, and h_n ($|h_n| \leqslant 1$) is a random number with uniform probability distribution. The coupling exists only among nearest neighbors (diffusively coupling). Periodic boundary conditions, i.e., $x^1 = x^{N+1}$, are assumed. The noiseless form ($\sigma = 0$) of Eq. (2) was first proposed by Kaneko [8] as a simple model for investigating the phenomenology of spatiotemporal chaos. It is perhaps the most extensively studied model spatiotemporal dynamical system in the literature so far. We choose the following set of parameter values: $a = 4$, $\delta = 0.8$ and $N = 20$. Fig. 1 shows, when $\sigma = 0$, a time series obtained at site 8 resulting from an arbitrary initial condition. The trajectory exhibits very long chaotic behavior (about 10^5 iterations) before settling into a final nonchaotic attractor. In general, for different initial conditions, the length of the chaotic transient is different. The average transient lifetime τ can be defined as follows. Suppose that at $t = 0$ we choose N_0 initial conditions, where N_0 is large. Evolve these N_0 initial conditions under the dynamics. Let $N(t)$ be the number of trajectories that are still chaotic at time t. Then due to the ergodic nature of the chaotic saddle, $N(t)$ decays exponentially with time [1],

$$N(t) = N_0 \exp(-t/\tau). \qquad (3)$$

For the parameter setting of Fig. 1, Eq. (2) exhibits a long transient with $\tau \approx 51800$ when $\sigma = 0$ [9].

To assess a different type of coexisting dynamical invariant sets under the influence of noise, we compute the maximum Lyapunov exponent λ_1 for a large number of uniformly chosen initial conditions. Using finite time steps, exponents computed using different

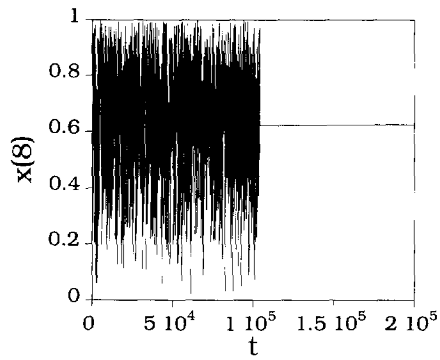

Fig. 1. Time series from a random initial condition for the diffusively coupled logistic map (2) for $N = 20$, $a = 4$, and $\delta = 0.8$. The trajectory behaves chaotically for an extreme long time (over 10^5 iterations) before settling into the final attractor.

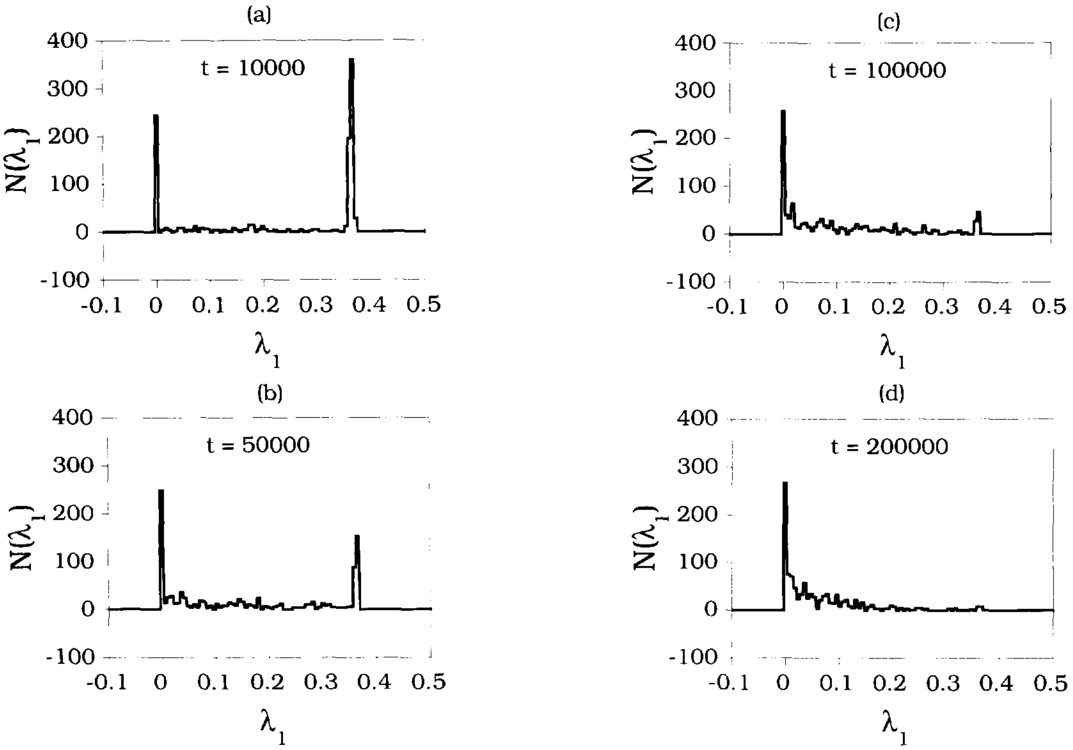

Fig. 2. Snapshots of histograms of the maximum Lyapunov exponent λ_1 at (a) $t = 10^4$, (b) $t = 5\times10^4$, (c) $t = 10^5$ and (d) $t = 2\times10^5$ when the noise amplitude is $\sigma = 10^{-8}$. These histograms are computed using a uniform grid of 32×32 initial conditions chosen on a two-dimensional section ($x(8)$ and $x(9)$) of the 20-dimensional phase space. There are two peaks, one at $\lambda_1 \approx 0.36$ and the other at $\lambda_1 = 0$. The former corresponds to a chaotic transient which decays as time progresses, and the latter represents a quasiperiodic attractor.

initial conditions are different. A histogram of these exponents can then reveal the existence of a different type of dynamical invariant sets. In particular, a peak at $\lambda_1 > 0$ indicates a chaotic set, which can be either a chaotic attractor or a chaotic saddle. As time progresses, a peak at a positive value of λ_1 would either sustain, which indicates a chaotic attractor, or decay. A decaying peak at $\lambda_1 > 0$ implies the existence of a chaotic saddle. There can also be peaks at $\lambda_1 = 0$ or $\lambda_1 < 0$, which usually correspond to quasiperiodic or periodic attractors, respectively. To compute the average transient lifetime for the chaotic saddle, it is necessary to construct snaphsots of histograms of λ_1 and count the number of trajectories that are still chaotic at time t. In our numerical experiments, a 32×32 grid of initial conditions was chosen in the two-dimensional region defined by $0 \leqslant x(8) \leqslant 1$ and $0 \leqslant x(9) \leqslant 1$, while values of $x(j)$ ($j = 1,\dots,N, j \neq 8,9$) of these initial conditions are fixed. Values of λ_1 for these 1024 initial conditions were then computed with 10000 pre-iterations. Histograms of λ_1 at successive time steps $t = 10000n$ ($n = 1,2,\dots,20$) were constructed. In general, these computations are very intensive, and we have utilized the massively parallel connection machines CM5 to compute λ_1 for many initial conditions in a parallel fashion.

Figs. 2a-2d show, for $\sigma = 10^{-8}$, histograms obtained at $t = 10^4, 5\times10^4, 10^5$ and 2×10^5, respectively. At $t = 10^4$, there are two peaks: One at $\lambda_1 = 0$ and another at $\lambda_1 \approx 0.36$. The height of the peak at $\lambda_1 \approx 0.36$ decreases with time. This indicates that the peak at $\lambda_1 \approx 0.36$ corresponds to a chaotic saddle. The peak at $\lambda_1 = 0$ represents a quasiperiodic attractor. As time progresses, trajectories escape the chaotic saddle and approach asymptotically to the quasisperiodic attractor. It is found that the quasiperiodic attractor contains several components, and its extent in phase space is quite small. Thus the value of $x(8)$ in the time series

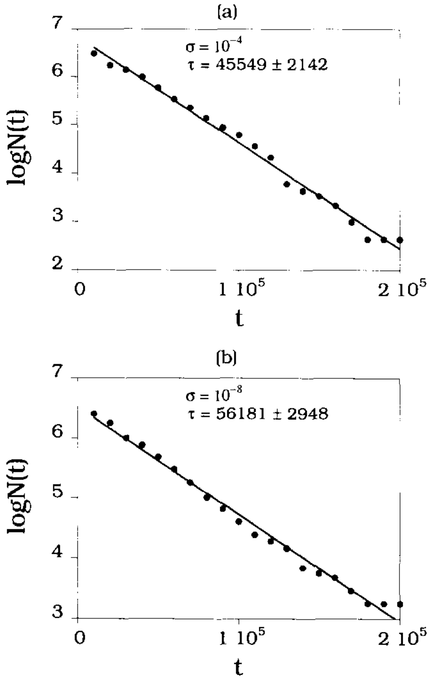

Fig. 3. Plots of $N(t)$, number of chaotic trajectories with $\lambda_1 > 0.3$ at time t, on a semi-log plot for $\sigma = 10^{-4}$ (a) and $\sigma = 10^{-8}$ (b). In both cases, the decay of $N(t)$ is exponential. The average transient lifetimes are 45549 ± 2142 for $\sigma = 10^{-4}$ and 56181 ± 2948 for $\sigma = 10^{-8}$.

on the quasiperiodic attractor appears to be constant, as shown in Fig. 1. Figs. 3a and 3b show the number of chaotic trajectories $N(t)$ versus t in a semi-log plot for $\sigma = 10^{-4}$ and $\sigma = 10^{-8}$, respectively, where a trajectory is counted as chaotic at time t if $\lambda_1 > 0.3$ at t. Both plots can be fitted by a straight line, the slopes of which determine τ. It is found that $\tau = 45549 \pm 2142$ for $\sigma = 10^{-4}$ (Fig. 3a) and $\tau = 56181 \pm 2948$ for $\sigma = 10^{-8}$ (Fig. 3b). Fig. 4 shows a plot of τ versus σ on a logarithmic scale, where σ varies over eight decades. This plot can roughly be fitted by a straight line, indicating the scaling relation (1). The scaling exponent is found to be $\eta = 0.015 \pm 0.010$, a value that is very close to zero.

A near-zero scaling exponent η indicates that the length of supertransients will not decrease substantially even if the amplitude of the noise increases by several orders of magnitude. To appreciate this, assume $\eta = 0.015$. Then in order to reduce the transient lifetime by a factor of two, it is necessary to amplify

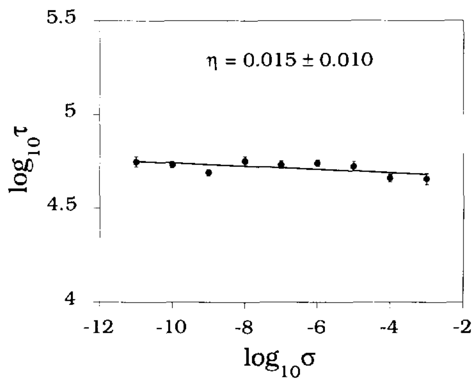

Fig. 4. Plot of the transient lifetime τ versus the noise amplitude σ on a logarithmic scale. The plot can be fitted by the scaling relation (1), with a scaling exponent $\eta = 0.015 \pm 0.010$. The closeness of η to zero suggests that the transient lifetime will not decrease substantially even if the amplitude of the noise increases over many orders of magnitude.

the noise amplitude by a factor of $10^{(\log_{10} 2)/0.015} \approx 10^{20}$ – an increase over 20 orders of magnitude! Therefore, we expect supertransients to be persistent in practical environments where noise is inevitable.

The origin of noise-independent supertransients can be understood by examining dynamical properties of the chaotic saddle which is responsible for the supertransients. We find that the stable manifold measure of such a chaotic saddle has a fractal dimension that is arbitrarily close to the phase-space dimension [9]. As a consequence, random perturbations, regardless of their magnitude, have almost equal probability of "kicking" out a trajectory on the chaotic saddle, thereby causing η be to near zero [9].

In conclusion, our computational studies on simple, but well accepted model spatiotemporal chaotic systems suggest that superlong chaotic transients sustain when random noise with amplitude varying through many orders of magnitude is present. Therefore, we expect supertransients to be common in spatiotemporal dynamical systems which are highly relevant models in fields such as fluid mechanics, biology, ecology and many others as well.

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