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Aperiodic stochastic resonance and phase synchronization

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Abstract

Aperiodic stochastic resonance and phase synchronization have been considered as different phenomena in nonlinear physics which were discovered at about the same time. The former means enhancement of aperiodic signals by noise, while the latter characterizes a phase coherence in weakly coupled nonlinear oscillators. Here we show that aperiodic stochastic resonance is related to phase synchronization between the input and output signals. We introduce a measure to characterize the degree of phase synchronization and show its equivalence to a commonly used measure to quantify aperiodic stochastic resonance.

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The phenomenon of stochastic resonance (SR), since its discovery in 1981 [1], has continued to be an interesting topic in statistical and nonlinear physics [2–6]. Broadly speaking, SR means the enhancement and optimization by noise of a nonlinear system's response to signals. Early works focused on periodic signals for which the signal-to-noise ratio, defined with respect to the corresponding spectral peak in the frequency domain, characterizes the resonance in a natural way, but the phenomenon was demonstrated for aperiodic signals as well by Collins et al. [4]. For such a signal, there are no apparent peaks in its Fourier spectrum, so the signal-to-noise ratio is no longer meaningful. The proper measures to characterize a system's response to aperiodic signals include the correlation between the input and output signals [4], entropies and other quantities derived from information theory [5,6]. In this case, SR means the optimization of such a measure by noise. The common occurrence of aperiodic stochastic resonance (ASR) in excitable dynamical systems, particularly in arrays of such systems for which the optimization of the system's response can occur in a wide range of the noise level, leads to the speculation that ASR may be a fundamental mechanism for information processing in biological systems [4].

In about the same time of the discovery of ASR, another important phenomenon in nonlinear dynam-

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ics was uncovered, which is phase synchronization (PS) [7]. When slightly nonidentical oscillators are coupled, a coherence can arise where the motions of the oscillators tend to follow each other, even when the coupling is relatively weak. For an oscillatory motion, regular or chaotic, a phase variable can usually be defined [7,8]. Imagine then two oscillators whose phase variables are $\phi_{1,2}(t) = \omega_{1,2}t + h_{1,2}(t)$ in the absence of coupling, where $\omega_1 \approx \omega_2$ and $h_{1,2}(t)$ are small fluctuations. The phase difference thus grows linearly with time: $\Delta\phi(t) = |\phi_1(t) - \phi_2(t)| \sim |\omega_1 - \omega_2|t$. As the coupling is turned on, interaction between the two oscillators can lead to a slower growth in $\Delta\phi(t)$ and eventually locking of the two phases with respect to each other. PS is said to occur [7] when the phase difference is confined to within 2π: $\Delta\phi(t) < 2\pi$. If one measures two signals, one from each oscillator, one finds that the temporal variations of the signals approximately follow each other in that they tend to increase or decrease in roughly the same time intervals, regardless of their difference in amplitudes. Since its discovery, PS has been identified in many natural systems and it has generated a continuous interest [9].

Recently the relationship between SR and PS in noisy bistable systems with periodic and aperiodic signals [3,10] and in excitable systems with periodic signals [11] has been discussed. The purpose of this Letter is to show that ASR and PS in excitable systems with aperiodic signals are also fundamentally related. In particular, for a system that exhibits ASR, noise can induce a phase coherence between the input and output signals. Resonance occurs when the two are synchronized in phase. To place the connection between ASR and PS on a firm ground, we introduce a measure to characterize the phase coherence between the oscillatory motions of the input and output signals, and show that the measure can be optimized by noise. Further, we argue and demonstrate with numerical examples that ASR and PS are achieved at approximately the same optimal noise level. Our work builds up a "bridge" between the two most actively pursued phenomena in nonlinear science, which is useful for a deeper understanding of the interplay between nonlinearity and stochasticity.

A paradigm to address ASR is excitable dynamical systems. Here we consider an array of N such oscillators. To be as general as possible, we assume the input signal $S(t)$ is contaminated by noise, which occurs, for instance, when jamming is present in a practical application. To each oscillator, which is a threshold device, an independent, controllable noise is provided. If $S(t)$ is a subthreshold signal, the combination of the jamming and the controllable noise effectively lowers the threshold of the oscillator so that it is capable of responding to the input. The output signal from each oscillator is spike trains, and the information associated with the input signal is typically encoded in the instantaneous firing rate $R(t)$. For an array of oscillators, the output can be chosen conveniently as the ensemble-averaged firing rate, denoted by $R_E(t)$. To characterize the phases in a simple way, we make use of the idea for describing phase in discrete-time dynamical systems [12] and define two discrete-value signals: $\phi_S(t) = \dot{S}(t)/|\dot{S}(t)|$ and $\phi_R(t) = \dot{R}_E(t)/|\dot{R}_E(t)|$, respectively, for the input and output signals, where $\dot{S}(t)$ is the time-derivative of $S(t)$. Thus, if $S(t)$ is increasing in a time interval, $\phi_S(t) = 1$, and we say the phase of $S(t)$ in this time interval is *up*. Likewise, $\phi_S(t) = -1$ signifies a *down* phase. PS between the input and output signals can thus be conveniently quantified by the following cross-correlation measure between $\phi_R(t)$ and $\phi_S(t)$:

$$\beta \equiv \left| \overline{\phi_S(t)\phi_R(t)} \right| = \left| \overline{\frac{\dot{S}(t)\dot{R}_E(t)}{|\dot{S}(t)||\dot{R}_E(t)|}} \right|, \tag{1}$$

where the overbar denotes the time average. If $S(t)$ and $R_E(t)$ are phase-matched in a perfect way, i.e., they increase or decrease simultaneously (in phase) or one increases and another decreases (anti-phase) in exactly the same time intervals, we have $\beta = 1$. Complete lack of phase coherence between $S(t)$ and $R_E(t)$ gives $\beta = 0$.

Our main objective is to show that β can be optimized by noise. In particular, we will argue for a representative class of excitable systems that β typically increases with noise and reaches maximum at some optimal noise level. The system we consider is an array of FitzHugh–Nagumo (FHN) oscillators [13], which is perhaps the most commonly studied excitable system for SR,

$$\begin{aligned} \epsilon\dot{x}_i &= x_i(x_i - 1/2)(1 - x_i) - y_i + S(t) + D_J\eta(t), \\ \dot{y}_i &= x_i - y_i - b + D\xi_i(t), \quad i = 1, \ldots, N \end{aligned} \tag{2}$$

where $\epsilon \ll 1$ and $0 < b < 1/2$ are parameters, D_J is the strength of the jamming noise associated with the input signal $S(t)$, D is the amplitude of the controllable noise, $\eta(t)$ and $\xi_i(t)$ $(i = 1, \ldots, N)$ are independent Gaussian random processes of zero mean and unit variance: $\langle \eta(t)\eta(t')\rangle = \delta(t - t')$ and $\langle \xi_i(t)\xi_i(t')\rangle = \delta(t - t')$. We start our analysis by considering a single oscillator. Utilizing the change of variables: $x \to x + 1/2$ and $y \to y - b + 1/2$, a single FHN oscillator can be written as: $\epsilon\dot{x} = -x(x^2 - 1/4) - y + A + S(t) + D_J\eta(t)$ and $\dot{y} = x - y + D\xi(t)$, where $A \equiv b - 1/2$. For $\epsilon \ll 1$, the time rate of change of $x(t)$ is much greater than that of $y(t)$ and, hence, $x(t)$ and $y(t)$ can be regarded as fast and slow variables, respectively. Using the approximation $\dot{y} \approx 0$ or $y(t) \approx x_f(t) + D\xi(t)$, where $x_f(t)$ is the fixed point of the FHN system in the presence of signal $S(t)$ but in the absence of noise, the x-equation becomes $\epsilon\dot{x} = -x(x^2 - \frac{1}{4}) - x_f(t) + A + S(t) + D_T\zeta(t)$, where $D_T\zeta(t) \equiv D_J\eta(t) - D\xi(t)$ represents the combined noise and $D_T = \sqrt{D_J^2 + D^2}$. This equation can thus be written as $\epsilon\dot{x} = -\partial U(x,t)/\partial x + D_T\zeta(t)$, where $U(x,t) = x^4/4 - x^2/8 + [x_f(t) - A - S(t)]x$ is a tilted double-well potential. The solution of the single FHN equations then corresponds to, approximately, the motion of a heavily damped particle in the potential, with time-dependent slope of tilting. A firing event in the FHN system is equivalent to a crossing of the particle through the barrier. The ensemble-averaged firing rate $\langle R(t)\rangle$ is thus given by the well-known Kramer's formula in statistical physics [14]. Utilizing a perturbation analysis for $x_f(t)$ and for finding the locations of the minima of the potential well as well as the top of the barrier, one can obtain the following ensemble-averaged firing rate [4]:

$$\langle R(t)\rangle \sim \exp\left\{-\frac{2}{3}\sqrt{3}\left[B^3 - 3B^2 S(t)\right]\epsilon/D_T\right\}, \tag{3}$$

where B is a constant which is the "distance" of the system's excitation level to the threshold [4].

For the array of FHN oscillators in Eq. (2), the mean firing rate is the ensemble-averaged firing rate of a single FHN oscillator. Fluctuations of the firing rate are determined by noise of the following form: $D_J\eta(t) + (D/N)\sum_{i=1}^{N}\xi_i(t)$, which can be written as $D_S\zeta'(t)$, where $D_S = \sqrt{D_J^2 + D^2/N}$ and $\zeta'(t)$ is a Gaussian random signal of zero mean and unit variance. Taking into account random fluctuations in the ensemble-averaged firing rate, we have $R_E(t) = \langle R(t)\rangle + \sigma(D_S)\kappa(t)$, where $\sigma(D_S)$ is positive and proportional to D_S, and $\kappa(t)$ is a Gaussian random signal. Taking time derivative of $R_E(t)$ and substituting it into Eq. (1), we obtain

$$\beta \sim \frac{\overline{2\sqrt{3}B^2\epsilon[\dot{S}(t)]^2\langle R(t)\rangle/D_S + \sigma(D_S)\dot{S}(t)\dot{\kappa}(t)}}{|\dot{S}(t)||\dot{R}_E(t)|}. \tag{4}$$

Since $\overline{\dot{S}(t)\dot{\kappa}(t)} \approx 0$, Eq. (4) can be written as

$$\beta \sim \overline{\frac{\chi_0(N, D, D_J)}{|\dot{S}(t)||\dot{R}_E(t)|}}, \tag{5}$$

where $\chi_0(N, D, D_J) = 2\sqrt{3}B^2\epsilon[\dot{S}(t)]^2\langle R(t)\rangle/D_S$. From Eq. (5) we see that the main factor contributing to β is the numerator $\chi_0(N, D, D_J)$. For instance, as N is increased, $\chi_0(N, D, D_J)$ increases but if N is large enough, $\chi_0(N, D, D_J)$ is essentially independent of N. Consequently, we expect β to increase with N and then reach a plateau for large N. On the other hand, $\chi_0(N, D, D_J)$ versus D_J exhibits a resonant behavior in that it increases with D_J, reaches maximum, and then decreases, so does $\chi_0(N, D, D_J)$ versus D. The conclusion is then that β can be optimized by noise.

The relation between PS and ASR becomes thus apparent. At the optimal noise level where the measure β is maximized, we expect the cross-correlation measure between the input $S(t)$ and the output firing rate $R_E(t)$

$$C = \frac{\overline{S(t)R_E(t)}}{[\,\overline{S^2(t)}\,]^{1/2}\{\overline{[R_E(t) - \overline{R_E(t)}\,]^2}\}^{1/2}}, \tag{6}$$

which characterizes ASR [4], to reach maximum as well. This is so because intuitively, if $R_E(t)$ is synchronized perfectly in phase with $S(t)$, we have both $\beta = 1$ and $C = 1$. On the other hand, if the phases of $R_E(t)$ and $S(t)$ are completely independent of each other, we have both $\beta = 0$ and $C = 0$. In what follows we will show by numerical examples that β and C exhibit similar behaviors versus the noise level and the system size, and in particular, they reach maximum at approximately the same noise strength.

We have tested the correspondence between PS and ASR using Eq. (2) with different aperiodic signal including amplitude-modulated (AM), frequency-modulated (FM), and chaotic signals under strong jamming. The parameters in the FHN equations are

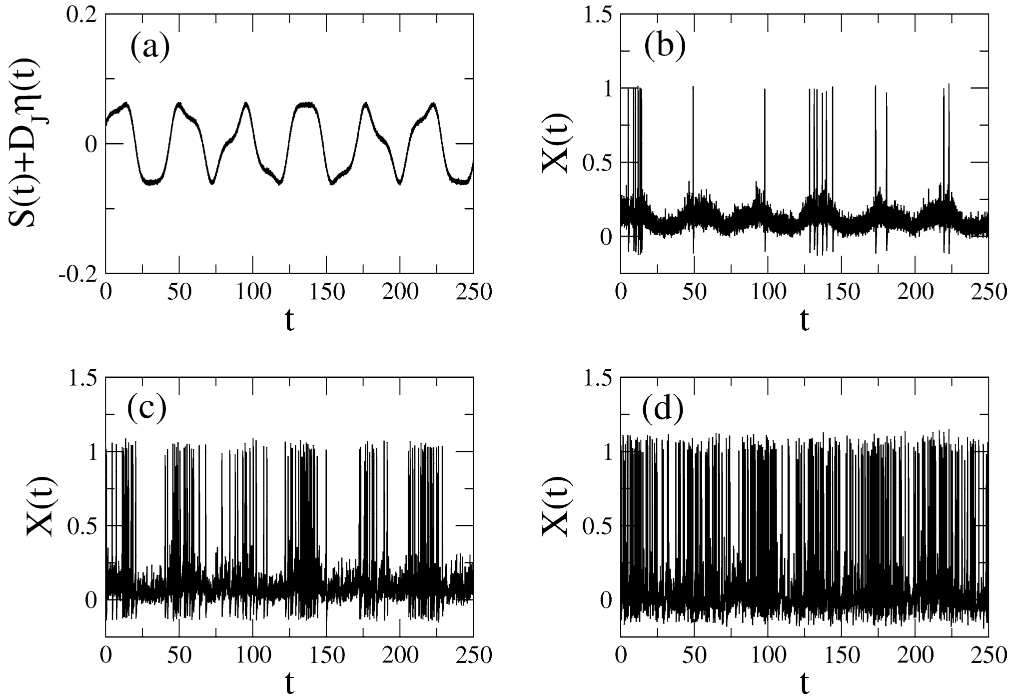

Fig. 1. (a) Noisy input FM signal, (b)–(d) output spike trains $X(t)$ for $D = 0$, 0.035, and 0.1, respectively. A "match" between $S(t)$ and the instantaneous firing rate $R_E(t)$ is apparently achieved for moderate level of controllable noise (c).

chosen to be $\epsilon = 0.005$ and $b = 0.15$ so that each FHN oscillator is subthreshold. For illustrative purpose here we present a case with the following FM signal: $S(t) = 0.06\sin[\omega_0 t + 0.5\cos(\omega_1 t)]$, where $\omega_0 = 0.15$ and $\omega_1 = \omega_0(1+\sqrt{5})/2$, the jamming strength is fixed at $D_J = 0.0015$, as shown in Fig. 1(a), the noisy input signal $S(t) + D_J\eta(t)$. Fig. 1(b)–(d) show, for $N = 1$, the output spike trains $X(t) = \sum_i^N x_i(t)$ for $D = 0$, $D = 0.035$, and $D = 0.1$, respectively. We see that without the controllable noise [Fig. 1(b)], the output instantaneous firing rate $R_E(t)$ is low and it cannot represent the input signal $S(t)$. For strong controllable noise [Fig. 1(d)], the firing rate is high but its temporal variation does not appear to match that of $S(t)$ either. A reasonably good match between $R_E(t)$ and $S(t)$ is apparently achieved for moderate noise [Fig. 1(c)], where the firing behavior is significantly stronger in time intervals where the input signal is near its maxima.

To compute β, we use the following algorithm. We first filter the noisy input signal by using a low-pass filter to obtain $S'(t)$, motivated by the general consideration that the input signal $S(t)$ itself is usually not directly available. We then filter the output spike train $X(t)$ by using a unit-area, symmetric, moving Hanning window of width $\delta t = 6$, whose output is the instantaneous firing rate $R_E(t)$. Fig. 2(a), (b) show $S(t)$ and $R_E(t)$, respectively, for $N = 50$ and $D = 0.005$. To compute β, the continuous-time signals $S'(t)$ and $R_E(t)$ are sampled to yield two discrete-time signals $G_S(t_i)$ and $G_R(t_i)$, where $t_i = i\Delta t$ ($i = 1, \ldots, K$), and Δt is two orders of magnitude smaller than the average periods of both $S'(t)$ and $R_E(t)$. The instantaneous phase variable $\phi_S(t_i)$ is thus taken to be 1 (or -1) if $G_S(t_{i+1}) > G_S(t_i)$ [$G_S(t_{i+1}) < G_S(t_i)$], and the same for $\phi_R(t_i)$. We have $\beta \approx (1/K)\sum_{i=1}^K \phi_S(t_i)\phi_R(t_i)$. Fig. 2(c) shows the dependence of β on D for $N = 10$ (upper trace), $N = 5$ (middle trace), and $N = 1$ (lower trace). We see that in all cases, β is small for small D, increases with D, reaches maximum for $D_{\text{opt}} \approx 0.035$, and decreases as D is increased further. Prominent PS between the input and output thus occurs for $D \approx D_{\text{opt}}$. Fig. 2(d) shows C, the cross-correlation measure char-

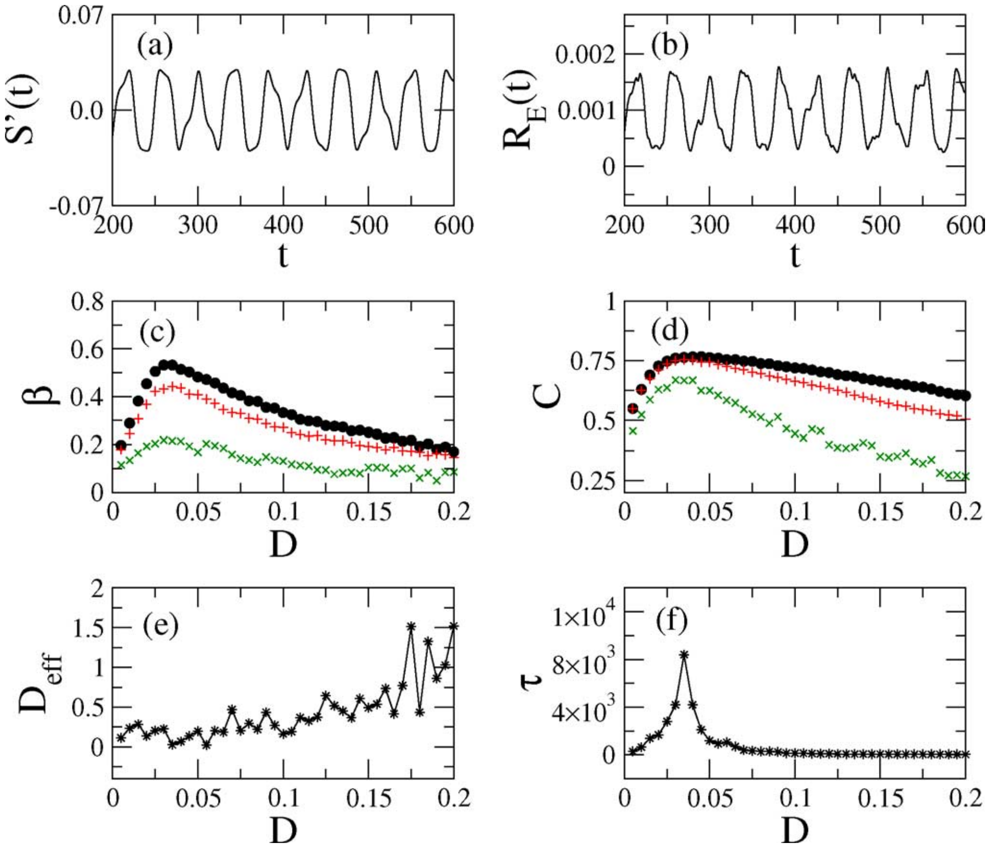

Fig. 2. (a), (b) For $N = 50$ and $D = 0.005$, continuous-time signals $S'(t)$ and $R_E(t)$, respectively. (c) For $N = 10$ (upper trace), $N = 5$ (middle trace), and $N = 1$ (lower trace), β versus D, where it can be seen that β is optimized for $D \approx 0.035$, indicating a strong degree of PS. (d) For $N = 10$ (upper trace), $N = 5$ (middle trace), and $N = 1$ (lower trace), the cross-correlation measure C versus D. (e) The effective diffusion constant D_{eff} versus D for $N = 5$. (f) Average time τ for which phase synchronization is maintained versus D for $N = 5$. It can be seen that τ exhibits a relatively sharp maximum at $D \approx D_{\text{opt}} \approx 0.035$.

acterizing ASR, versus D for the three system sizes in Fig. 2(c). We see that C has a maximum value for $D \approx 0.035$.

We now discuss two alternative quantities for characterizing the correspondence between phase synchronization and stochastic resonance: the effective diffusion constant and the average time for phase synchronization to be maintained. To compute the diffusion constant, we note that the instantaneous phase $\Phi_{M(t)}(t)$ ($M(t) = X(t)$ or $S'(t)$) can be defined as $\Phi_{M(t)}(t) = \arctan[Z(t)/M(t)]$, where $Z(t)$ is the Hilbert transform of the original process $M(t)$ [3]. The phase difference between the input and the output is $\Phi(t) = \Phi_{X(t)}(t) - \Phi_{S'(t)}(t)$. The effective diffusion constant is then

$$D_{\text{eff}} = \frac{1}{2}\frac{d}{dt}\Big[\big\langle \Phi^2(t)\big\rangle - \big\langle \Phi(t)\big\rangle^2\Big]. \quad (7)$$

Fig. 2(e) shows D_{eff} versus D. We see that the behavior of D_{eff} does not appear to indicate stochastic resonance. On the other hand, we find that the average time τ for temporal synchronization, which is the mean time interval between successive 2π-phase slips, exhibits a relatively sharp maximum as a function of the noise amplitude, as shown in Fig. 2(f). We see that the location of the maximum is approximately the optimal noise amplitude for aperiodic stochastic resonance. The behavior of τ in Fig. 2(f) thus strengthens the connection between phase synchronization and aperiodic stochastic resonance.

In summary, we have shown that aperiodic stochastic resonance can be understood naturally as a phase-synchronization phenomenon. We propose a measure, which is experimentally accessible, to characterize the phase synchronization between the input and output signals. Our heuristic theory and extensive numerical computations with various input signals and system configurations (although we show only one type of input signal here) indicate that when aperiodic stochastic resonance is achieved, phase synchronization is optimized, and vice versa. Further, the phase synchronization appears to be robust against noise associated with input signals. Our finding may be particularly useful for understanding the mechanisms of information processing in biological systems, as both aperiodic stochastic resonance and phase synchronization are speculated to be ubiquitous [4,9].

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