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ORIGINAL ARTICLE

Efficiency of carrier-phase integer ambiguity resolution for precise GPS positioning in noisy environments

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Abstract Precise GPS positioning relies on tracking the carrier-phase. The fractional part of carrier-phase can be measured directly using a standard phase-locked loop, but the integer part is ambiguous and the ambiguity must be resolved based on sequential carrier-phase measurements to ensure the required positioning precision. In the presence of large phase-measurement noise, as can be expected in a jamming environment for example, the amount of data required to resolve the integer ambiguity can be large, which requires a long time for any generic integer parameter estimation algorithm to converge. A key question of interest in significant applications of GPS where fast and accurate positioning is desired is then how the convergence time depends on the noise amplitude. Here we address this question by investigating integer least-sqaures estimation algorithms. Our theoretical derivation and numerical experiments indicate that the convergence time increases linearly with the noise variance, suggesting a less stringent requirement for the convergence time than intuitively expected, even in a jamming environment where the phase noise amplitude is large. This finding can be useful for practical design of GPS-based systems in a jamming environment, for which the ambiguity resolution time for precise positioning may be critical.

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1 Introduction

In general, centimeter-level GPS positioning accuracy requires precise tracking of the carrier phase that consists of two parts: a directly measured fractional part (with measurement error at millimeter level) and an unknown integer part, also called the *integer ambiguity*. The key to precise carrier-phase-based positioning is to resolve the integer ambiguity, which is an extremely challenging when large noise or jamming is present.

Existing ambiguity resolution techniques can be divided into several categories (e.g. Kim and Langley 2000).

– The first includes the simplest techniques that use C/A-code or P-code pseudoranges directly to determine the ambiguities of the corresponding carrier-phase observations. The precision of the code range is not good enough to determine the integer ambiguities and linear inter-frequency combinations are usually used for estimating the ambiguities (e.g. Hofmann-Wellenhof et al. 2001).
– The second category of algorithms employ the ambiguity function method (AFM) (Counselman and Gourevitch 1981). This technique uses only the fractional value of the instantaneous carrier-phase measurement and, hence, the ambiguity function

values are not affected by the whole cycle change of the carrier-phase or by cycle slips (also see Hofmann-Wellenhof et al. 2001; Xu 2003).

– The third category comprises the most abundant group of techniques, which are based on the theory of integer least squares (Hatch 1990; Frei and Beutler 1990; Teunissen 1993, 1994, 1995a,b, 1996; Chen and Lachapelle 1995; Xu et al. 1995; Xu 1998, 2001; Li and Gao 1998; Hassibi and Boyd 1998; Azimi-Sadjadi and Krishnaprasad 2001; Xu 2006). Parameter estimation, in theory, is carried out in three steps, the float solution, the integer ambiguity estimation, and the fixed solution. Each technique makes use of the variance–covariance matrix obtained from the float solution step and employs different ambiguity search processes at the integer ambiguity estimation step. Based on certain search criterion (Xu 2002), the search algorithm can utilize the traditional techniques of mathematical programming to guide the global optimization (Wei and Schwarz 1995; Xu et al. 1995; Xu 1998), and/or decorrelation techniques to reduce the search space (Teunissen 1995a,b; Hassibi and Boyd 1998; Xu 2001) and/or guided random searching techniques to combat nonlinearity (Li 1995; Azimi-Sadjadi and Krishnaprasad 2001). Note, however, that decorrelation would help speed up searching for the integer solution only if the dimension is not too large (Xu 2001).

For any method, a practical concern (especially for kinematic positioning) is that the resolution time of the integer ambiguity is sensitive to the carrier-phase measurement noise. In a noisy environment, e.g., a battle field with strong jamming signals, radio frequency interference (RFI) signals are spread in the frequency domain by the de-spreading process. These spectrally spread RFIs increase the effective noise floor in a GPS receiver, making carrier-phase measurements more noisy and hence the time to revolve integer ambiguity longer. Whenever a loss of track due to jamming or signal interruption occurs, the integer ambiguity has to be resolved again, during which no high-precision position information is available.

To reduce the phase-measurement noise and the integer ambiguity resolution time, various techniques such as those based on signal processing (e.g. time-frequency domain processing) (Amin 1997), subspace processing (Amin et al. 1999) and/or receiver antenna design (e.g. beam forming or null steering) were proposed (Fante and Vaccaro 2000; Tsoulos 2001) for antijamming. However, even with the application of these techniques, typically there are still residual errors in the carrier-phase measurement and greater efforts are required for fast resolution of the integer ambiguity. Given a minimal requirement for positioning accuracy and acquisition time, it is desirable to know the corresponding noise level of carrier-phase measurements, in order to employ only the necessary jamming mitigation techniques.

The focus of this paper is on the performance of an integer least-squares ambiguity resolution algorithm in noisy environments. In particular, we investigate how the convergence (acquisition) time of the algorithm depends on the carrier-phase measurement noise. Naively, one may expect that the time and the noise variance σ^2 have a linear relationship, as suggested by the fact that the variance of n independent noisy measurements with variance σ^2 is σ^2/n. However, our theoretical analysis shows that the acquisition time depends linearly on the standard deviation of noise (or the noise amplitude) σ, not on σ^2, which is also verified by numerical simulations using a generic integer-parameter estimation algorithm (Hassibi and Boyd 1998).

Our main goal here is to provide a theoretical argument and numerical support for this linear relationship. Since the naive relationship $n \propto \sigma^2$ would require much more observation samples than the linear relation $n \propto \sigma$ (for large σ and n), the finding is important for the design of precise GPS positioning system in noisy environments with a tight constraint on time to ambiguity resolution, such as in real-time kinematic (RTK) GPS positioning. Our finding is particularly encouraging as it suggests the possibility of having an integer-parameter estimator to achieve significantly shorter convergence times for precise positioning.

The rest of the paper is organized as follows. A linear GPS positioning model is briefly reviewed in Sect. 2. A generic ambiguity resolution algorithm based on integer least squares is described in Sect. 3. Sections 4 and 5 study the performance of the algorithm in the presence of noise. In particular, the relationship between the convergence time and the noise amplitude is derived theoretically in Sect. 4 and verified by simulations in Sect. 5. Conclusions and ideas for future work are presented in Sect. 6.

We remark that the main objective of our work is to obtain a *scaling* relation between the convergence time and the noise strength, which is an order-of-magnitude type of estimate. The reason is that the convergence time depends on too many algorithmic and system details and it is not possible to give precise numbers to characterize it. As such, we have based our theoretical analysis on the upper and lower bounds in the probability of correct resolution of the integer ambiguity given in (Hassibi and Boyd 1998), as the required computational time is only polynomial. Note, however, that these bounds may be poor and much tighter bounds can be found in Xu (2006). We also wish to emphasize that the results obtained for the performance in the presence of noise are based on the integer least-squares principle in general and thus they should not be dependent on the specific search method used.

2 System model

GPS signals can usually be corrupted by jamming and several other forms of errors. These error sources can be partially cancelled by using the technique of double-differencing. For positioning based on the carrier-phase, we assume that jamming and the residual errors can be modeled as noise in the phase. Under this consideration, the distance between a satellite i and a receiver A can be modeled as

$$\rho_A^i(k) = \lambda[\Phi_A^i(k) + n_A^i] + v(k), \tag{1}$$

where k is the discrete sampling time, λ is the wave-length of the carrier-wave, Φ is the fractional part of the carrier-phase, n is the integer part of the initial carrier-phase (i.e., at $k = 0$), and v is the modeling error. The range ρ can be expressed in terms of the satellite position $[x^i(k), y^i(k), z^i(k)]$ and receiver position $[x_A(k), y_A(k), z_A(k)]$,

$$\rho_A^i(k) = \{[x^i(k) - x_A(k)]^2 + [y^i(k) - y_A(k)]^2 + [z^i(k) - z_A(k)]^2\}^{1/2}. \tag{2}$$

The goal is to calculate the receiver position $[x_A(k), y_A(k), z_A(k)]$ by using known positions of at least three satellites and the corresponding carrier-phase measurements Φ.

To linearize Eq. (2), we let $x_A = x_{A0} + \Delta x_A$, $y_A = y_{A0} + \Delta y_A$, $z_A = z_{A0} + \Delta z_A$, where $[x_{A0}, y_{A0}, z_{A0}]$ is a known reference position near the receiver, which can be estimated using code pseudoranging. Substituting the linearized version of Eq. (2) into Eq. (1) yields

$$\lambda\Phi_A^i(k) - \rho_{A0}^i(k) = a_x^i(k)\Delta x + a_y^i(k)\Delta y + a_z^i(k)\Delta z - \lambda n_A^i, \tag{3}$$

where

$$a_x^i(k) = -\frac{x^i(k) - x_{A0}}{\rho_{A0}^i(k)},$$
$$a_y^i(k) = -\frac{y^i(k) - y_{A0}}{\rho_{A0}^i(k)},$$
$$a_z^i(k) = -\frac{z^i(k) - z_{A0}}{\rho_{A0}^i(k)}.$$

Suppose that the receiver is static and three satellites are continuously tracked from epoch $k = 0$ to epoch $k = n$, the linearized observation equations can be expressed in matrix form as

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{z} + \mathbf{v}, \tag{4}$$

where $\mathbf{y} = [\lambda\Phi_A^1(0) - \rho_{A0}^1(0), \lambda\Phi_A^2(0) - \rho_{A0}^2(0), \ldots]^{\mathrm{T}}$, $\mathbf{x} = [\Delta x, \Delta y, \Delta z]^{\mathrm{T}}$, $\mathbf{z} = [n_A^1, n_A^2, n_A^3]^{\mathrm{T}}$, $\mathbf{v}$ is the measurement noise vector, and the matrices $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = \begin{bmatrix} a_x^1(0) & a_y^1(0) & a_z^1(0) \\ a_x^2(0) & a_y^2(0) & a_z^2(0) \\ a_x^3(0) & a_y^3(0) & a_z^3(0) \\ a_x^1(1) & a_y^1(1) & a_z^1(1) \\ a_x^2(1) & a_y^2(1) & a_z^2(1) \\ a_x^3(1) & a_y^3(1) & a_z^3(1) \\ \vdots & \vdots & \vdots \end{bmatrix}_{3(n+1)\times 3} \tag{5}$$

and

$$\mathbf{B} = \begin{bmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \\ -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \\ \vdots & \vdots & \vdots \end{bmatrix}_{3(n+1)\times 3} . \tag{6}$$

3 Integer least squares

In the linearized observation equation (Eq. 4), $\mathbf{x}$ and $\mathbf{z}$ are unknown real and integer vectors, respectively, which are to be estimated from the satellite data by using the maximum likelihood (ML) estimator,

$$(\mathbf{x}_{\mathrm{ML}}, \mathbf{z}_{\mathrm{ML}}) = \arg\max_{\mathbf{x},\mathbf{z}} P_{\mathbf{y}|\mathbf{x},\mathbf{z}}(\mathbf{y}|\mathbf{x},\mathbf{z}), \tag{7}$$

where $(\mathbf{x}, \mathbf{z}) \in R^p \times Z^q$, and $P_{\mathbf{y}|\mathbf{x},\mathbf{z}}(\mathbf{y}|\mathbf{x},\mathbf{z})$ is the probability of observing $\mathbf{y}$ given $\mathbf{x}$ and $\mathbf{z}$. Assuming the stochastic process $\mathbf{v}$ is Gaussian with zero mean and covariance $\mathbf{\Sigma}$, we have:

$$(\mathbf{x}_{\mathrm{ML}}, \mathbf{z}_{\mathrm{ML}}) = \arg\min_{\mathbf{x},\mathbf{z}} (\mathbf{y} - \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{z})^{\mathrm{T}}\mathbf{\Sigma}^{-1}(\mathbf{y} - \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{z}). \tag{8}$$

If we consider a block matrix $[\mathbf{A}\,\mathbf{B}]$ and the unknown vector $[\mathbf{x}\,\mathbf{z}]^{\mathrm{T}}$, the floating solution $[\hat{\mathbf{x}}\,\hat{\mathbf{z}}]^{\mathrm{T}}$ can be obtained by solving

$$[\mathbf{A}\,\mathbf{B}]^{\mathrm{T}}\mathbf{\Sigma}^{-1}[\mathbf{A}\,\mathbf{B}][\hat{\mathbf{x}}\,\hat{\mathbf{z}}]^{\mathrm{T}} = [\mathbf{A}\,\mathbf{B}]^{\mathrm{T}}\mathbf{\Sigma}^{-1}\mathbf{y}, \tag{9}$$

Eliminating $\hat{\mathbf{x}}$ leads to the least-squares solution of $\hat{\mathbf{z}}$:

$$\hat{\mathbf{z}} = \mathbf{\Gamma}\mathbf{B}^{\mathrm{T}}(\mathbf{I} - \mathbf{A}(\mathbf{A}^{\mathrm{T}}\mathbf{\Sigma}^{-1}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}})\mathbf{y}, \tag{10}$$

where $\mathbf{\Gamma} = (\mathbf{B}^{\mathrm{T}}\mathbf{C}'\mathbf{B})^{-1}$ is the covariance matrix for $\hat{\mathbf{z}}$, and $\mathbf{C}' = \mathbf{\Sigma}^{-1} - \mathbf{\Sigma}^{-1}\mathbf{A}(\mathbf{A}^{\mathrm{T}}\mathbf{\Sigma}^{-1}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{\Sigma}^{-1}$.

In general, $\hat{\mathbf{z}}$ is a real vector and the ML solution can be found in the following least-squares sense (Teunissen 1993; Xu et al. 1995):

$$\hat{\mathbf{z}}_{\mathrm{ML}} = \arg\min_{\mathbf{z}}(\mathbf{z} - \hat{\mathbf{z}})^{\mathrm{T}}\mathbf{\Gamma}^{-1}(\mathbf{z} - \hat{\mathbf{z}}) \quad \text{for } \mathbf{z} \in Z^q. \tag{11}$$

If $\mathbf{\Gamma}$ is diagonal, the solution for $\mathbf{z}$ can be found by rounding each component of $\hat{\mathbf{z}}$ to its nearest integer. This simple approach, however, does not work in realistic situations where the matrix $\mathbf{\Gamma}$ is typically not diagonal. As a result, simple rounding off will not give the correct estimate for $\mathbf{z}$ and the search for the true integer vector has to be performed over the entire integer space by using some efficient searching algorithm (Hatch 1990; Frei and Beutler 1990; Chen and Lachapelle 1995; Teunissen 1994; Hassibi and Boyd 1998; Xu 2006).

Once $\hat{\mathbf{z}}_{\mathrm{ML}}$ is found, the estimate of the real position vector $\mathbf{x}$ can be obtained by substituting $\hat{\mathbf{z}}_{\mathrm{ML}}$ into the least-squares equation (8). This yields

$$\hat{\mathbf{x}}_{\mathrm{ML}|\mathbf{z}} = (\mathbf{A}^{\mathrm{T}}\Sigma^{-1}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\Sigma^{-1}(\mathbf{y} - \mathbf{B}\hat{\mathbf{z}}_{\mathrm{ML}}). \tag{12}$$

It has to be noted that the ambiguity fixing process (Eq. 11) breaks down Eq. (9), resulting in the above least-squares ambiguity search (LSAS) algorithm no longer being an optimal one, i.e., it is not guaranteed that $(\hat{\mathbf{x}}_{\mathrm{ML}|\mathbf{z}}, \hat{\mathbf{z}}_{\mathrm{ML}}) = (\mathbf{x}_{\mathrm{ML}}, \mathbf{z}_{\mathrm{ML}})$. It has been shown that $(\hat{\mathbf{x}}_{\mathrm{ML}|\mathbf{z}}, \hat{\mathbf{z}}_{\mathrm{ML}})$ can be offset from $(\mathbf{x}_{\mathrm{ML}}, \mathbf{z}_{\mathrm{ML}})$ and a more general searching criterion has to be used (Xu 2002, 2003, 2004).

4 Scaling relation between convergence time and noise amplitude

In practice, we are most interested in cases when the positioning accuracy is high, or the probability of a correct estimate of the integer ambiguity P_{c}, is close to one. Particularly, we want to know, given certain noisy environment, how long it will take us to achieve a desired value of P_{c}, where P_{c} is close to one. This can be addressed by investigating the influence of noise on P_{c}. In this study, P_{c} obtained from LSAS is used and $(\hat{\mathbf{x}}_{\mathrm{ML}|\mathbf{z}}, \hat{\mathbf{z}}_{\mathrm{ML}}) = (\mathbf{x}_{\mathrm{ML}}, \mathbf{z}_{\mathrm{ML}})$ is assumed for simplicity, since we are only concerned with large P_{c} and LSAS algorithm generally produces good results in this situation.

Assume $\mathbf{y}$ to be a Gaussian variable, then $\hat{\mathbf{z}}$—a linear function of $\mathbf{y}$—is Gaussian too. We write $\hat{\mathbf{z}} = \mathbf{z} + \mathbf{u}$, where $\mathbf{u}$ is Gaussian with zero mean and covariance matrix $\mathbf{\Gamma}$. Multiplying both sides of the above equation by $\mathbf{G} \equiv \mathbf{\Gamma}^{-1/2}$ and by defining $\hat{\mathbf{y}} = \mathbf{G}\hat{\mathbf{z}}$, we get $\hat{\mathbf{y}} = \mathbf{G}\mathbf{z} + \hat{\mathbf{u}}$, where $\hat{\mathbf{u}}$ is a Gaussian random variable with zero mean and unit variance. Equation (11) can then be written in an equivalent form

$$\mathbf{z}_{\mathrm{ML}} = \arg\min||\hat{\mathbf{y}} - \mathbf{G}z||^2 \quad \text{for } \mathbf{z} \in Z^q, \tag{13}$$

where the set $\{\mathbf{Gz}|\mathbf{z} \in Z^q\}$ constitutes a lattice in R^q.

Equation (13) suggests that the ML value of $\mathbf{z}$ can be found by computing the nearest lattice point to the vector $\hat{\mathbf{y}}$. The probability P_{c} that $\mathbf{z}_{\mathrm{ML}}$ is true is completely determined by the Voronoi cell. By definition of the lattice, a lower packing distance between two neighboring points can be computed by using

$$d \propto |\mathbf{G}|^{1/q}, \tag{14}$$

where $|\mathbf{G}|$ is the determinant of the matrix $\mathbf{G}$ and q, the number of unknown integers, is the dimension of $\mathbf{G}$.

As a rough estimate, the distance d can be used to compute the lower bound of the probability P_{c}. Alternatively, given a desired value of P_{c}, there is a corresponding value of d that depends on factors such as the noise amplitude and the number of data samples. We emphasize, however, that such use of the rough estimate of the lower bound of P_{c} is only for the purpose of obtaining a scaling dependence of the convergence time on the noise strength, which by its nature is not precise. In fact, some precise estimates of the lower and upper bounds have been obtained recently (Xu 2006).

To be able to find a closed-form solution for $|\mathbf{G}|$, here we consider a 2D GPS setup as shown in Fig. 1. All results based on this 2D model can, in principle, be extended to the real 3D GPS setup.

Assume that there are q satellites tracked and the noise terms for the corresponding carrier-phase measurements are independent of each other with covariance matrix $\mathbf{\Sigma} = \sigma^2\mathbf{I}$. The inverse of $\mathbf{\Gamma}$ can be simplified as

$$\begin{aligned}\mathbf{\Gamma}^{-1} &= \frac{1}{\sigma^2}[\mathbf{B}^{\mathrm{T}}\mathbf{B} - \mathbf{B}^{\mathrm{T}}\mathbf{A}(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{B}] \\ &= \frac{\lambda^2}{\sigma^2}\left[n\mathbf{I} - \frac{\mathbf{Q}}{d_1 d_3 - d_2^2}\right],\end{aligned} \tag{15}$$

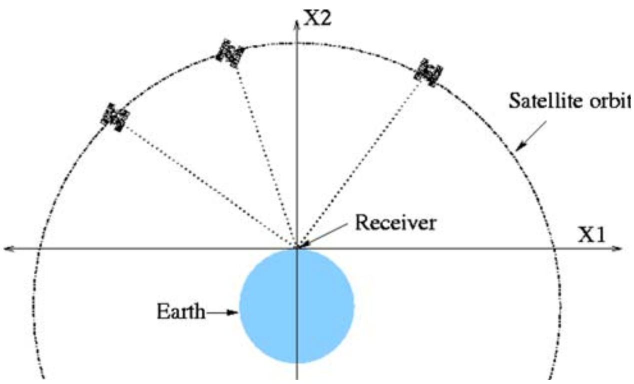

Fig. 1 Simulated 2D GPS setup

where

$$d_1 = \sum_{k=0}^{n}\sum_{i=1}^{q}[a_x^i(k)]^2, \tag{16}$$

$$d_2 = \sum_{k=0}^{n}\sum_{i=1}^{q}a_x^i(k)a_y^i(k), \tag{17}$$

$$d_3 = \sum_{k=0}^{n}\sum_{i=1}^{q}[a_y^i(k)]^2, \tag{18}$$

and $\mathbf{Q}$ is a $q \times q$ matrix with elements given by

$$Q_{ij} = d_3 \sum_{k=0}^{n} a_x^j(k) \sum_{k=0}^{n} a_x^i(k) - 2d_2 \sum_{k=0}^{n} a_x^j(k) \sum_{k=0}^{n} a_y^i(k) + d_1 \sum_{k=0}^{n} a_y^j(k) \sum_{k=0}^{n} a_y^i(k).$$

Apparently, $|\mathbf{G}|^{1/q} = |\mathbf{\Gamma}^{-1}|^{1/2q}$ depends on both σ and n.

To find the relationship between σ and n for a given P_c, or equivalently, a constant $|\mathbf{\Gamma}^{-1}|$, we assume $[x_{A0}, y_{A0}] = [0, 0]$, the initial angle of satellite i is α, the orbital speed of the satellite is $\dot{\alpha}$, and the sampling period is T. We then have

$$\begin{aligned}\sum_{k=0}^{n} a_x^i(k) &= -\sum \cos(\alpha + \dot{\alpha}kT) \\ &\approx -\sum \cos\alpha + \sin\alpha \sum \dot{\alpha}kT \\ &= \frac{1}{2}n^2 T\dot{\alpha}\sin\alpha - (n+1)\cos\alpha,\end{aligned} \tag{19}$$

where higher order terms of $\dot{\alpha}kT$ in the Taylor series expansion are neglected. The approximation error is generally small since $\dot{\alpha}kT$ is relatively small. Similarly, we have

$$\sum_{k=0}^{n} a_y^i(k) \approx \frac{1}{2}n^2 T\dot{\alpha}\cos\alpha - (n+1)\sin\alpha \tag{20}$$

$$\sum_{k=0}^{n}[a_x^i(k)]^2 \approx \frac{1}{6}n(n+1)(2n+1)T^2\dot{\alpha}^2\sin^2\alpha - 2n^2 T\dot{\alpha}\sin\alpha\cos\alpha + (n+1)\cos^2\alpha \tag{21}$$

$$\sum_{k=0}^{n}[a_y^i(k)]^2 \approx \frac{1}{6}n(n+1)(2n+1)T^2\dot{\alpha}^2\cos^2\alpha - 2n^2 t\dot{\alpha}\sin\alpha\cos\alpha + (n+1)\sin^2\alpha \tag{22}$$

$$\sum_{k=0}^{n} a_x^i(k)a_y^i(k) \approx \frac{1}{6}n(n+1)(2n+1)T^2\dot{\alpha}^2\sin\alpha\cos\alpha - \frac{1}{2}n^2 T\dot{\alpha} + (n+1)\sin\alpha\cos\alpha. \tag{23}$$

Substituting Eqs. (19) and (20) into Eq. (15), one can see that each entry in $\mathbf{\Gamma}^{-1}$ can be expressed as a second-order polynomial of n divided by σ^2. Thus, for increased σ, n has to be increased proportionally in order to maintain a desired positioning accuracy (or a constant $|\mathbf{\Gamma}^{-1}|$). This implies

$$n \propto \sigma, \tag{24}$$

for $n \gg 1$.

Equation (24) is our main result. While larger noise requires more data samples from the satellites, the linear relation indicates that to achieve a desired positioning accuracy, the requirement is not as stringent as one would naively expect from $n \propto \sigma^2$. Our derivation suggests that if the satellites were kept static with respect to the static GPS receiver [i.e., $\dot{\alpha} = 0$ in Eq. (19)], then the scaling relation $n \propto \sigma^2$ would hold. It is the relative movement between GPS satellites and Earth surface that introduces another degree of dependence on n, resulting in a shorter estimation time than the static case. Note that the movement of the receiver will have similar effect if the velocity of the receiver is known exactly. This is quite encouraging as it suggests that the integer parameter estimation time for kinematic GPS is likely to be shorter.

5 Simulation results

Since the purpose of the simulations is to verify the theoretically derived dependence of the convergence time on the noise amplitude, it is necessary to be able to vary the noise variance in a systematic way. A synthetic 2D GPS setup, as illustrated in Fig. 1, is suitable for this purpose. The setup is similar to the one used by Hassibi and Boyd (1998).

We assume that the position of the (GPS) receiver **x**, which is to be determined, can be modeled as a zero-mean Gaussian random variable with certain variance in each dimension. The coordinate axes are chosen such that the origin is a point on the surface of the Earth (a point on the periphery of a circle of radius equal to that of the Earth $R_e = 6,357$ km). We further suppose that there are three visible satellites orbiting the Earth at the altitude of 20, 200 km and with a period of 12 h (angular velocity of $1/120\,\mathrm{s}^{-1}$). The satellites transmit a carrier signal of wavelength $\lambda = 19$ cm each, and their coordinates are known to the receiver.

The receiver, which is assumed to be completely synchronized with the satellites (meaning that it can generate the transmitted carrier signals), measures the phase of the received carrier signals every $T = 2$ s and unwraps them as time goes by. By multiplying these (unwrapped) phase measurements by the wavelength divided by 2π, the receiver can measure its distance (or range) to each satellite up to some additive noise, which is assumed to be $N(0, \sigma^2)$ and, of course, up to an integer multiple of the wavelength. (This integer multiple can be thought as the number of carrier signal cycles between the receiver and the satellite when the carrier signal is initially phase-locked.)

As we have described, by linearizing the range equations, the problem becomes one of estimating a real parameter **x** (the coordinates of the receiver) and an integer parameter **z** (the integer multiples of the wavelengths) in a linear model.

In the simulation that follows, the actual location of the receiver is $\mathbf{x} = [50; 100]^{\mathrm{T}}$, which will be estimated using the carrier-phase measurements. We assume that the standard deviation of **x** is 100 m along each coordinate axis. The satellites make angles of 100°, 130° and 50° with the axis initially, and the direction of rotation for all of them is clockwise. The standard deviation of carrier-phase measurement noise in units of length is assumed to be in the centimeter range. Using the carrier-phase measurements, the receiver tries to find its own position **x** (as well as the ambiguous integer multiples of the wavelengths) as a function of time by solving for the ML estimates.

Figures 2 and 3 show the results of the algorithm for $\sigma = 1$ cm in terms of P_c and receiver positioning, respectively. The exact value of the P_c is computed by Monte-Carlo simulation of 1,500 runs. It can be seen that the position estimation error reduces as the probability of correct integer estimation approaches unity. When the integer ambiguity is resolved correctly, the [simulated] position estimation error is of the order of millimeters.

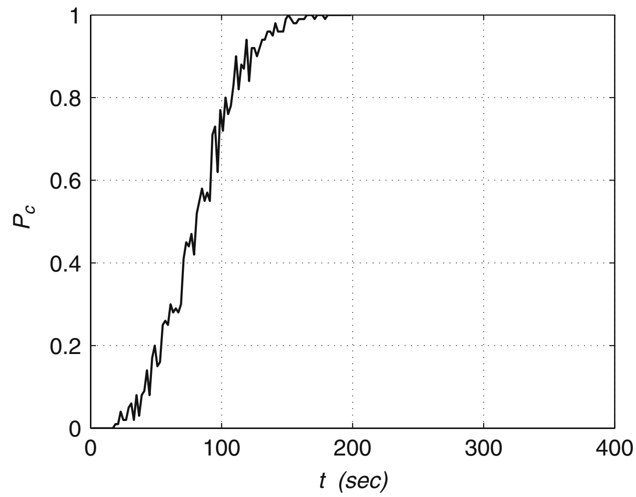

Fig. 2 P_c versus time for $\sigma = 1$ cm. Note that P_c after $t = 200$ s is almost equal to one in the simulation

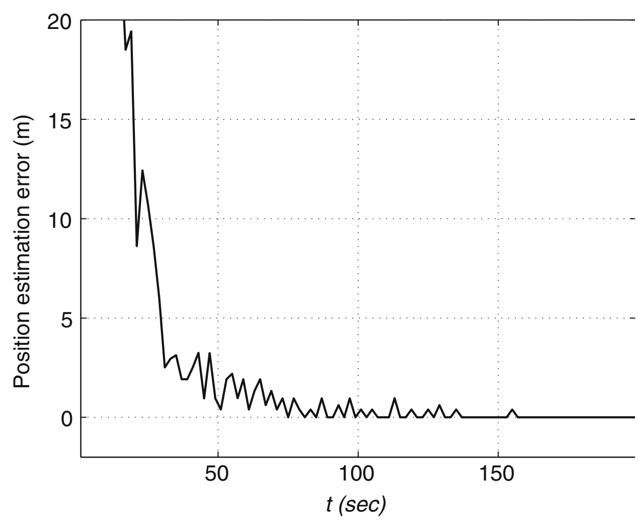

Fig. 3 Position estimation error (meters) versus time (seconds), where the initial errors ($t < 20$ s) are large and not shown

Having established the accuracy of the integer least-squares algorithm for static GPS positioning, we wish to evaluate its performance in the presence of noise. For this purpose, the algorithm was provided with various values of carrier-phase measurement noise variance as input. The range of σ varied from 0.2 to 2 cm with increment of 0.2 cm. The values of P_c for all cases were calculated using 1,500 Monte-Carlo simulations.

Figure 4 shows P_c versus time for various σ values. It is observed that as σ increases, the time it takes to achieve

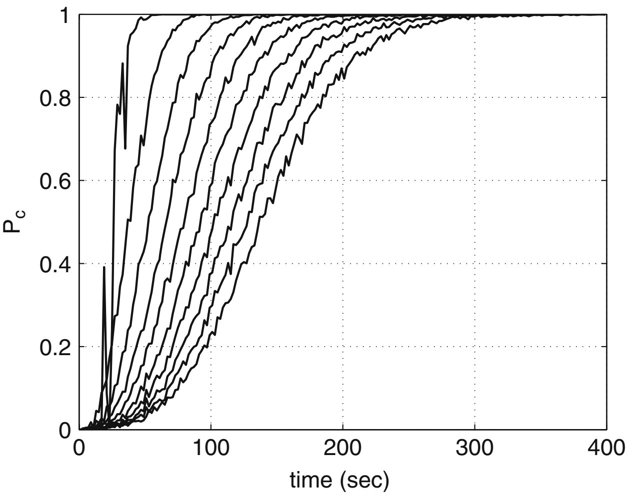

Fig. 4 P_c versus time for σ values ranging from 0.2 to 2 cm (from *left* to *right*). Maximum allowable noise variance for given values of time and P_c can be determined accordingly

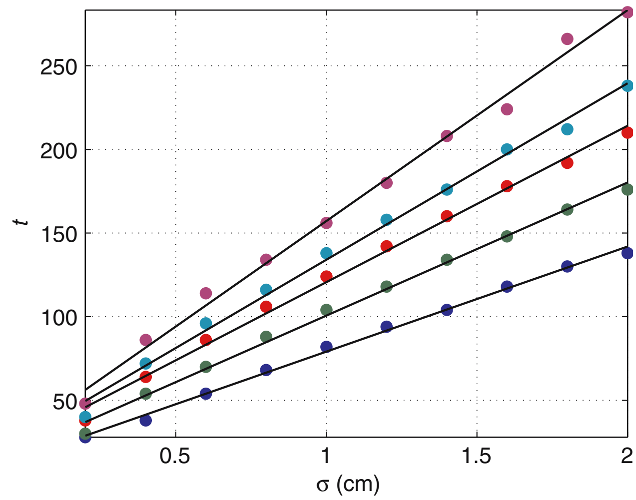

Fig. 5 Sensitivity of performance to measurement noise variance. From *top* to *bottom*: $P_c = 0.99, 0.95, 0.9, 0.75, 0.5$ respectively

a certain level of P_c also increases. The family of curves in Fig. 4 can be used to calculate the maximum allowable noise variance for a given amount of observation time and required value of P_c. For example in Fig. 4, if the observation time is 150 s and the integer ambiguity estimate is to be reliable with 90% accuracy ($P_c = 0.9$), the maximum allowable noise variance is about 1.2 cm.

Figure 5 shows the linear relation characterizing the sensitivity of performance to noise amplitude, as predicted by our theoretical analysis. A use of this result is that the performance of integer least-squares algorithm can be roughly predicted for a given noise level, remembering the earlier disclaimers on the exactness of the method.

6 Conclusion

This paper has addressed the performance of integer least-squares algorithm for GPS signals in noisy environments. Mathematically, integer ambiguity resolution is equivalent to searching for the integer vector closest to a given real vector on an integer lattice. In a noisy environment, the probability of error can be significantly large. We find that the observation time required to achieve a fixed value of a lower bound of P_c and thus P_c itself is directly proportional to the standard deviation of phase measurement noise, in contrast to the naive expectation that the time is proportional to the variance of the phase noise. This suggests the possibility of achieving short convergence time even if large noise is present.

It should be noted that we have assumed an ideal system model in this study. For example, systematic errors are assumed to be eliminated by using double differencing, and the resulted system errors are approximately Gaussian with zero mean. This may not be true in practice, especially when long baselines are utilized. In this situation, additional modeling of large residual errors has to be employed, e.g., the means of residual ionospheric and tropospheric errors have to be estimated together with **x** and **z**. If the mean values of these residual errors remain constant or change slowly during the process of ambiguity resolution, the convergence time of ambiguity resolution will only be delayed approximately by a constant, the overall linear relationship between the time and the noise amplitude should still hold. It will be interesting to see how it works in different practical environments with well controlled strength of noise or jammming signals.

For future directions, parallel algorithms taking advantage of multiple satellite signals to reduce the ambiguity resolution time should be pursued. In addition, the study for the effect on noise on GPS positioning should be extended to kinematic GPS positioning algorithms. It is also recommended to develop a particle-filter-based algorithm and to compare its sensitivity to noise with those of integer least-squares algorithms for both static and kinematic positioning.

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