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Desynchronization and on-off intermittency in complex networks

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Abstract – Most existing works on synchronization in complex networks concern the synchronizability and its dependence on network topology. While there has also been work on desynchronization wave patterns in networks that are regular or nearly regular, little is known about the dynamics of synchronous patterns in complex networks. We find that, when a complex network becomes desynchronized, a giant cluster of a vast majority of synchronous nodes can form. A striking phenomenon is that the size of the giant cluster can exhibit an extreme type of intermittent behavior: on-off intermittency. We articulate a physical theory to explain this behavior. This phenomenon may have implications to the evolution of real-world systems.

When a regular oscillator network becomes desynchronized, either wave patterns are generated [1–8] or stationary synchronization clusters are formed [9]. Consider, for instance, a one-dimensional ring of identical oscillators, each coupled with its nearest neighbors. For nonlinear oscillator dynamics, it is typical that synchronization occurs only when the coupling parameter, say ε, lies in a finite range: $\varepsilon_1 \leqslant \varepsilon \leqslant \varepsilon_2$ [10]. When ε is decreased through ε_1, long-wave bifurcation occurs in the sense that the desynchronization wave patterns generated for $\varepsilon \lesssim \varepsilon_1$ have wavelengths of the order of the system size [1,2]. As ε is increased through ε_2, wave patterns with wavelength much smaller than the system size are generated, henceforth the term short-wave bifurcation [2–4]. For $\varepsilon \gtrsim \varepsilon_2$, even when the actual coupling strengths are randomized (in this case ε is a nominal coupling parameter), robust regular wave patterns can arise [7]. Considering that in the synchronization regime, noise and small system mismatch can induce desynchronization bursts, the occurrence of stable wave patterns in the desynchronization parameter regime is quite remarkable. A more recent work reveals that regular wave patterns can persist in small-world networks that deviate slightly in topology from a regular network, but the patterns can be destroyed if there are too many random links in the network [8]. Previous works have also revealed the situation where, when desynchronization occurs, a regular network breaks into a finite number of synchronous clusters. That is, oscillators in each cluster are synchronized but the synchronized dynamics differ from cluster to cluster. Such clusters are usually *stationary* due to the regularity of the underlying network.

While wave patterns associated with desynchronization in regular networks, *e.g.*, lattice and globally coupled systems, have been relatively well understood, little has been done to address the problem in complex networks, for example random [11] and scale-free networks [12]. Intuitively, since the underlying network does not possess a regular topology, no wave patterns can be formed. However, we find that, in the desynchronization regime, synchronization clusters can occur commonly. The remarkable phenomenon that we wish to report in this paper is that the evolution of the clusters can exhibit an extreme form of intermittency. In particular, there can be a giant cluster containing a substantial fraction of synchronized nodes most of the time, but there can

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also be times when there are many small synchronous clusters. If one focuses on the giant cluster, its size can vary in an extremely intermittent fashion, exhibiting characteristics that are typical of on-off intermittency observed commonly in nonlinear dynamical systems [13]. As we shall demonstrate, the intermittency is due completely to the underlying complex topology of the network, and can occur with or without random noise. To understand the origin of the *size intermittency*, we have developed a theory based on the concept of snapshot attractors in nonlinear dynamics [14,15]. Our finding reveals that a complex network can be extremely dynamic: nodes can spontaneously form different groups of synchronous clusters at different times. This may have broader implications, since our result suggests the necessity to view synchronization as a nonstationary and dynamic property: at different times different groups of nodes (*e.g.*, neurons) can be synchronized.

We consider the following model of coupled-map network: $\mathbf{x}_i(t+1)=\mathbf{F}[\mathbf{x}_i(t)]-\varepsilon\sum_j C_{i,j}\mathbf{H}[\mathbf{f}(\mathbf{x}_j(t))]$, where $\mathbf{x}_i(t+1)=\mathbf{F}[\mathbf{x}_i(t)]$ is the d-dimensional map representing the local dynamics of node i, ε is the global coupling parameter, C is the coupling matrix determined by both the network topology and properties such as the weight of links and the directionality of interaction between nodes, and $\mathbf{H}$ is the coupling function. To preserve the generality of our finding, we use the following coupling scheme [16]: $C_{i,j}=-A_{i,j}k_j^{\beta}/\sum_{j=1}^{N}A_{i,j}k_j^{\beta}$, for $j\neq i$ and $C_{i,i}=1$, where k_i is the degree of node i, $\mathbf{A}$ is the adjacent matrix of the network, and β is a parameter that can be adjusted to model different coupling schemes and interaction patterns in the network [16]. To facilitate numerical computation while taking into account the possibility of complicated oscillatory node dynamics, we use the one-dimensional chaotic logistic map $F(x)=4x(1-x)$ and choose $H(x)=x$. The linear stability of of the global synchronization state $\{x_i(t)=s(t),\forall i\}$ is determined by the corresponding variational equations of the model equation, which can be diagonalized into N blocks of the form $y(t+1)=[4(1-2s(t))+\sigma)]y(t)$, where possible values of σ are $\sigma(i)=\varepsilon\lambda_i$ and $0=\lambda_1<\lambda_2\leqslant\ldots\leqslant\lambda_N$ are the eigenvalues of the coupling matrix C. For $\sigma\neq 0$, the largest Lyapunov exponent $\Lambda(\sigma)$ from the evolution of the infinitesimal variation $y(t)$, or the master stability function [10], can be calculated. Synchronization can be physically or numerically observed if $\Lambda(\sigma)<0$. For typical nonlinear oscillatory dynamics, the function $\Lambda(\sigma)$ can be negative only for $\sigma\in(\sigma_1,\sigma_2)$. For the chaotic logistic-map dynamics, we find $\sigma_1\approx 0.5$ and $\sigma_2\approx 1.5$. Thus, two necessary conditions for synchronization are $R\equiv\lambda_N/\lambda_2<\sigma_2/\sigma_1\approx 3\equiv R_c$ and $\varepsilon_1<\varepsilon<\varepsilon_2$, where R is the eigenratio, $\varepsilon_1=\sigma_1/\lambda_2$, and $\varepsilon_2=\sigma_2/\lambda_N$.

We first present results with desynchronization in random networks in the regime $\varepsilon\lesssim\varepsilon_1$ (see footnote [1]). We generate a random network of $N=1000$ nodes and average degree $\langle k\rangle=20$. The smallest nontrivial and the largest eigenvalues are $\lambda_2\approx 0.528$ and $\lambda_N\approx 1.44$ and, hence, $\varepsilon_1=\sigma_1/\lambda_2\approx 0.946$. The possible values of the largest Lyapunov exponents associated with various transverse subspaces are $\Lambda(\varepsilon\lambda_i)$ for $i=2,\ldots,N$. For ε slightly below ε_1, only a small fraction of the exponents is slightly positive while the vast majority of them remain negative. For example, for $0.90\lesssim\varepsilon\lesssim\epsilon_1$, we have $\Lambda(\varepsilon\lambda_2)\gtrsim 0$ while $\Lambda(\varepsilon\lambda_i)<0$ $(i=3,\ldots,N)$. Since the asymptotic value of $\Lambda(\varepsilon\lambda_2)$ is only slightly positive, in any finite time interval the exponent can assume both positive and negative values and, it fluctuates randomly among these values in the course of time evolution. This is in fact a necessary condition for on-off intermittency [13]. For instance, we have examined the evolution of $\Delta X(t)\equiv\sqrt{\sum(x_i(t)-\bar{x}(t))^2/N}$, where $\bar{x}(t)\equiv\sum x_i(t)/N$ and found an apparent on-off intermittent behavior with power law distribution of laminar phases that has the well-known $-3/2$ exponent over several orders of magnitude of the laminar-phase length. However, the quantity $\Delta x(t)\equiv x_i(t)-\bar{x}(t)$ characterizes only the temporal deviation of the node dynamics from their mean field. It does not reveal any spatial organization of the desynchronization dynamics.

To reliably detect any spatial pattern associated with desynchronization, we introduce a symbolic approach. Given a long time series $x_i(t)$ from node i, we define a symbolic sequence $\theta_i(t)$ where $\theta_i(t)=0$ for $x_i(t)<0.5$ and $\theta_i(t)=1$ for $x_i(t)>0.5$. We then divide $\theta_i(t)$ into segments of equal length $n\gg 1$. If, for time t', we have $\theta_i(t)=\theta_j(t)$ for all $t=t'-n,\ldots,t'-1$, we say that node i is synchronized with node j at time t'. Insofar as n is reasonably large, the synchronization state so defined is robust with exceedingly small numerical uncertainty, as it requires a match of n bits between two chaotic symbolic sequences. A synchronization cluster at time t' is identified as all nodes whose dynamics satisfy this matching condition. In generating the symbolic sequence, the decomposition threshold can be any value from the system attractor, which will not affect the dynamical features of pattern evolutions. However, with a good threshold, *e.g.* the value 0.5 in logistic map, the pattern phenomena will be more distinct and the analysis becomes efficient.

A typical example of the desynchronization dynamics in complex networks, organized in terms of temporally synchronous clusters, is shown in fig. 1 for the random network for $\varepsilon=0.945$, where fig. 1(a) shows the evolution of n_c, the number of synchronized clusters. We observe that most of time n_c assumes small values, indicating the existence of only a few synchronous clusters. In this case, it is possible to have some giant cluster that contains a large number of synchronized nodes. Occasionally, n_c can be large, which corresponds to the occurrence of a relatively large number of small clusters. Figure 1(a) reveals that large and small values of n_c occurs in a highly

[1] Similar results have been obtained in the regime $\varepsilon\gtrsim\varepsilon_2$.

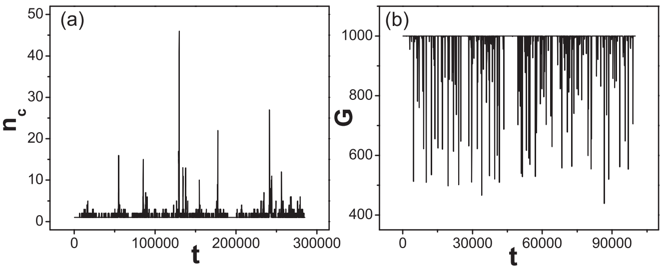

Fig. 1: For a random network of 1000 nodes and average degree 20, with parameters $\beta = 0$ and $n = 100$, (a) evolution of the number of distinct synchronous clusters, and (b) evolution of the size G of the largest synchronous cluster for coupling parameter slightly below the desynchronization/synchronization transition point (see text for parameter values). There is apparently on-off intermittency.

intermittent fashion[2]. The remarkable result is shown in fig. 1(b), the time evolution of the size G of the largest synchronous cluster, or the giant cluster, where G is the number of nodes contained in the cluster. We observe that G is close to the system size ($N = 1000$) most of the time, but the number can decrease dramatically in an extremely intermittent fashion. This indicates that a complex network can be extremely dynamic in the desynchronization regime: the vast majority of nodes in the network stay synchronized most of the time, but there are brief moments where the synchronization state is destroyed, leaving the network in a scattered state of mostly desynchronized motion. The striking phenomenon is that such a desynchronized state lasts only for a short time, after which the network quickly reorganizes itself to restore a highly coherent state. This on-off intermittent synchronization phenomenon is purely deterministic: it does not require any random perturbation to the network. As we will analyze, the phenomenon is due to the interplay between the nonlinear node dynamics and the complex topology of the network. The phenomenon is also robust: it persists even when there are external random perturbations to the network.

We now provide a physical theory to explain the on-off intermittency in the size of the giant synchronous cluster. Focusing on the individual node dynamics, we see that any node interacts with a number of nodes according to the complex-network topology. To analyze the detailed interactions among nodes is practically infeasible and may not be necessary. Theoretically, the mean-field approach is thus appealing. In particular, at every step of time evolution, the mean field $\bar{x}(t)$ can be defined, which acts uniformly on every node in the network. In the mean-field approximation, complicated node interactions are replaced by the driving field $\bar{x}(t)$, common to all nodes at any given instant of time. For densely connected random networks, the couplings are mostly uniform with respect to nodes in the network, so the mean-field treatment is appropriate. Due to the chaotic node dynamics, the mean field can be regarded effectively as random and, its influence on each node is identical. Since the node dynamics are also identical, we can imagine a single dynamical system, where dynamics at different nodes correspond to trajectories from different initial conditions, all evolving under the same system. The mean field is effectively a random perturbation uniformly applied to all trajectories. We have thus mapped our network synchronization problem to a problem in random dynamical systems: distribute a cloud of initial conditions (particles), evolve them under the same system equations and common noise, and examine the attractors formed by the particles at given instants of time —snapshot attractors [14,15]. Complete synchronization of all nodes corresponds to total collapse of the particles onto a single point that moves randomly (or chaotically) in time in the phase space. Partial synchronization, characterized by a number of synchronous clusters, corresponds to a few localized clouds of particles. A giant synchronous cluster is represented by a localized set of almost all particles. The numerically observed on-off intermittency in the size of the giant synchronous cluster is reflected on a similar behavior in the size of the snapshot attractor in the phase space, a problem that has been treated previously in nonlinear dynamics [15].

To establish a quantitative relation between G, the size of the giant cluster in the network, and S, the actual size of the snapshot attractor in the phase space, we consider typical times where most of the nodes in the network are embedded in the giant cluster, and let x_G denote the dynamical variable of this cluster. The remaining set of desynchronized nodes can be distributed in small synchronous clusters of various sizes, and their dynamical variables can be scattered in the phase space. Let x_D denote the "center of mass" of the dynamical variables of all the nodes that are not contained in the giant synchronous cluster. The size of the snapshot attractor is [15] $S \equiv \sqrt{(1/N)\sum_{i=1}^{N}(x_i - \bar{x})^2}$, where x_i ($i = 1, \ldots, N$) is the dynamical variable of node i and $\bar{x} = (1/N)[Gx_G + (N-G)x_D]$. We have

$$S = \sqrt{\frac{G(N-G)}{N^2}}|x_G - x_D|. \tag{1}$$

In the giant-synchronous-cluster case, we have $G \lesssim N$ and, hence, $S \sim \sqrt{1 - G/N}$. In a nearly global synchronization state where G approaches N, we have $S \to 0$, which corresponds to the "off" state in figs. 1(a) and (b).

The physical theory of fractals in random dynamical systems [15] predicts that, slightly above the transition to chaotic attractor, the size of the snapshot attractor exhibits on-off intermittency. Replacing chaotic motion

[2]In the simulation of coupled maps, spurious synchronization might arise due to the finite computation precision. That is, after a long transient period, the maps can always be synchronized whatever the synchronization conditions [6]. This spurious synchronization, however, is practically excluded in our model, as the transient period of a large and sparse network is extremely long. In our simulations, spurious synchronization is not found up to 10^{10} iterations.

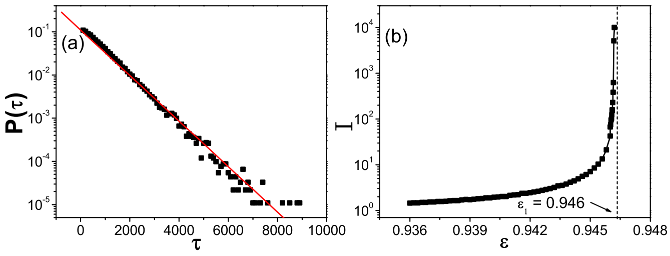

Fig. 2: (Color online) For the random network in fig. 1, (a) the exponential distribution of the laminar phase in the size of the giant synchronous cluster, and (b) the intermittency index *vs.* the coupling parameter. We see that I tends to diverge as the coupling parameter approaches the synchronition/desynchronization transition point.

with the unstable motion of the synchronous manifold in the transverse subspace of $\Lambda(\varepsilon\lambda_2)$, the intermittency of the snapshot attractor in random systems has essentially the same dynamical origin to the intermittency of pattern evolution in desynchronized complex networks. The only difference is that, when the network is synchronized, the synchronous manifold is chaotic; while for a random dynamical system, below the transition point the collective trajectory is periodic. To provide quantitative validation of the applicability of the dynamical-system theory to network synchronization, we quote the main results from the theory [15] and provide tests from direct simulations of the network dynamics. The first result is that the laminar-phase distribution of the size of the intermittent snapshot attractor exhibits a strong exponential tail. This has been confirmed, as shown in fig. 2(a), where the network parameters are the same as in fig. 1, time series $G(t)$ of length 10^8 are used to calculate the distribution, and the "off" state is defined as $G(t) > 950$. (If this threshold is changed, the laminar phase still has an exponential distribution, but with a different exponent.) The second result is that the following intermittency index $I \equiv \langle S^2(t)\rangle/\langle S(t)\rangle^2$ should diverge when the transition point to the chaotic attractor is approached. Physically, the index is roughly the inverse of the fraction of in which that a snapshot attractor has a large size. In our network synchronization problem, I is thus the inverse of the fraction of time in which the size of the giant synchronous cluster becomes small. If the network is effectively temporarily desynchronized, n_c is large and G is small, then S can be roughly treated as constant, and we have $I \approx 1$. As the coupling parameter approaches the transition point, n_c becomes smaller and G becomes larger, the averages $\langle S^2\rangle$ and $\langle S\rangle$ are dominated by the *rare events* of large S or small G, resulting in the divergence of I at the transition point. As the coupling parameter in the desynchronization regime approaches the transition point to synchronization, the intermittency index should then diverge. This behavior is shown in fig. 2(b). The results in figs. 2(a) and (b) thus provide strong support for the applicability of the theory of snapshot attractors to synchronization in random networks.

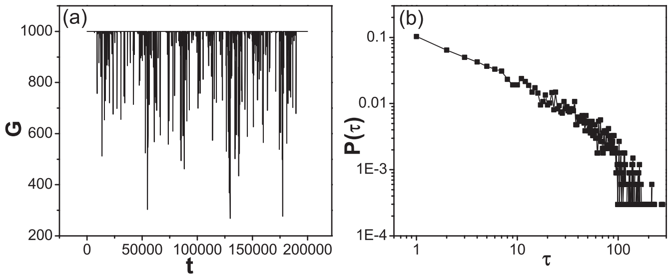

Fig. 3: (a) Example of on-off intermittency in the size of the giant synchronous cluster for a scale-free network of 1000 nodes and average degree 8. The smallest nontrivial and the largest eigenvalues of the coupling matrix are $\lambda_2 \approx 0.6$ and $\lambda_N \approx 1.58$. The long-wave bifurcation point for desynchronization is $\varepsilon_1 \approx 0.835$. The coupling parameter used to generate the intermittency is $\varepsilon = 0.83 \lesssim \varepsilon_1$. (b) Laminar-phase distribution $P(\tau)$. There is a strong exponential tail for $\tau > 50$.

The results presented so far are for random networks. What about scale-free networks? For a scale-free network, there is a small set of nodes with far exceeding average number of links (here we consider scale-free networks with degree distribution exponent $\gamma > 2$ so that an average degree can be defined), the vast majority of nodes possess a relatively small number of links. We thus suspect that the mean-field justification for the applicability of the snapshot attractor theory may not be unreasonable. Indeed, extensive numerical computations reveal a similar behavior of intermittency in the size of the giant synchronous cluster. A representative example is shown in fig. 3, where fig. 3(a) displays an example of intermittency in G, and fig. 3(b) presents the distribution of the laminar phase. Again, there is a strong exponential tail in the distribution. We also find that the intermittency index tends to diverge near the synchronization/desynchronization transition point. These results suggest that the snapshot attractor theory is quite suitable for quantitatively describing the pattern dynamics associated with desynchronization in scale-free networks as well.

In summary, we have uncovered an interesting phenomenon associated with desynchronization in complex oscillator networks: near the transition point there can be a giant cluster of synchronous nodes and the size of the cluster can exhibit on-off intermittency. We have argued and provided strong evidence that the fractal theory of snapshot attractors, originally developed in random dynamical systems, is suitable for explaining the intermittency phenomenon *both qualitatively and quantitatively*. To our knowledge, prior to our work there has been no effort addressing desynchronization patterns in complex networks of small-world and scale-free features. This may be due to the intuition that organized patterns are unlikely due to the complex interactions in the network. Our results indicate that this intuition is incorrect. Indeed, both random and scale-free networks can exhibit quite ordered patterns in the form of a giant synchronous cluster in the desynchronization regime, although the complex-network topology causes the evolution of the cluster to exhibit the highly irregular, intermittent behavior. Our findings suggest that in real-world systems situations can be expected where synchronization is highly dynamic: the number of nodes participated in synchronization can have a strong dependence on time in the sense of on-off intermittency. Such a dynamic synchronization may be advantageous from the standpoint of learning and adaptation in the evolution process.

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