Effect of common noise on phase synchronization in coupled chaotic oscillators

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We report a general phenomenon concerning the effect of noise on phase synchronization in coupled chaotic oscillators: the average phase-synchronization time exhibits a nonmonotonic behavior with the noise amplitude. In particular, we find that the time exhibits a local minimum for relatively small noise amplitude but a local maximum for stronger noise. We provide numerical results, experimental evidence from coupled chaotic circuits, and a heuristic argument to establish the generality of this phenomenon. © *2007 American Institute of Physics*.
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A remarkable phenomenon in nonlinear physics is that noise can induce or enhance synchronization in coupled chaotic oscillators. Recent years have seen efforts to explore the effect of noise on chaotic phase synchronization, a "weak" form of synchronization where the oscillators tend to follow (not to approach) each other in phase but their amplitudes remain uncorrelated. This form of synchronization can occur more widely than complete synchronization that requires both the phase and the amplitude of the oscillators to approach each other. As a result, phase synchronization has been increasingly pursued for potential applications in disciplines ranging from physics and chemistry to biology and medical sciences. Here we focus on a measure to characterize phase synchronization, the average phase-synchronization time, and study how noise can affect this time. To understand the behavior of the time is important because, under noise, phase synchronization can occur only for a finite amount of time. We find that, in the presence of weak coupling, common or identical noise can lead to a nonmonotonic behavior in the average phase-synchronization time. In particular, a little noise will cause the time to decrease as a function of the noise amplitude but a moderate amount of noise can maximize the time. We show the nonmonotonic behavior by using numerical computations and laboratory experiments using electronic circuits, and we provide a heuristic argument that the behavior is a result of the interplay between deterministic coupling and noise, and the maximization of the time by noise can be understood as caused by stochastic resonance. We expect the nonmonotonic behavior in the average phase synchronization time to be a general phenomenon in nonlinear dynamical systems in the presence of noise.

I. INTRODUCTION

The problem of noise-induced synchronization has been a subject of interest in statistical and nonlinear physics.[1,2] Consider a system of coupled nonlinear oscillators. In the weakly coupling regime where synchronization does not occur, the presence of noise, identical to each oscillator, can induce synchronization.[1] This seemingly counterintuitive phenomenon can be explained by regarding common noise as being able to generate effectively mutual coherence. In particular, if the root-mean-square value of the noise is large enough, a degree of coherence may be established among the oscillators. An alternative point of view is to regard the noise as a common driving force, or a "master," to each oscillator. If the driving is strong enough, the evolution of each oscillator tends to follow that of the master to some extent. There can then be a generalized synchronization between each individual oscillator and the stochastic driving. Since noise is identical to all oscillators, the motion of each oscillator tends to follow that of their common master more or less in the same manner, leading to effectively coherence among themselves. More recently, numerical and experimental evidence has been presented for noise-induced phase synchronization.[2,3] That is, common noise can induce phase coherence among the oscillators while leaving their amplitudes uncorrelated. More specifically, consider two oscillators and denote their phase variables by $\phi_1(t)$ and $\phi_2(t)$, respectively. In the absence of phase synchronization, the phase difference $\Phi(t)=\phi_1(t)-\phi_2(t)$, normalized to a 2π interval (say, $[-\pi,\pi]$), tends to distribute uniformly in this interval. The presence of common noise can make the distribution more focused about zero.[2] A qualitative explanation for noise-induced phase synchronization has been proposed in Ref. 3.

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In this paper, we focus on the average phase-synchronization time (denoted by τ). Under noise, phase synchronization can be maintained for only a finite amount of time, so the quantity τ is physically relevant and useful. We first present a nonmonotonic behavior of τ through a numerical study of the coupled chaotic Rössler oscillators under common noise (of amplitude D). In the presence of coupling, for small noise, τ decreases with D and reaches a minimum at $D=D_m$.[4] However, as D is increased further, τ can increase and become maximum for some optimal value of D (say, D_{opt}). For $D>D_{\text{opt}}$, the time decreases with D. About D_{opt}, the behavior of τ is indicative of that of a stochastic resonance, but the overall nonmonotonic behavior is quite striking. Heuristically, this appears to be the consequence of the complex interplay between nonlinearity and stochasticity. In order to demonstrate the generality of this behavior, we have carried out experiments using two different electronic circuits: coupled Rössler and coupled Chua circuits. Both experiments give strong evidence for the nonmonotonic behavior in τ. We shall then present a heuristic argument to gain insight. At the present, an analytic theory for the nonmonotonic behavior appears difficult, partly due to the presence of both coupling and common noise.

A potential utility of our result is the following. Given a system of coupled chaotic oscillators, there may be two ways to achieve high-quality phase synchronization in terms of maximizing the average synchronization time: either to eliminate noise as much as possible or to supply common noise to drive the system into resonance. The latter may be relevant, say, to coupled biological systems where noise is inevitable but phase synchronization may be used for information transmission and processing.

In Secs. II and III we present numerical and experimental evidence for the nonmonotonic behavior of the phase synchronization time, respectively. In Sec. IV, we provide a heuristic argument. A brief summary is given in Sec. V.

II. NUMERICAL EVIDENCE

We consider the following system of coupled chaotic Rössler oscillators:

$$\dot{x}_{1,2} = -\omega_{1,2}y_{1,2} - z_{1,2} + K(x_{2,1} - x_{1,2}),$$

$$\dot{y}_{1,2} = \omega_{1,2}x_{1,2} + 0.15y_{1,2} + D\xi(t), \tag{1}$$

$$\dot{z}_{1,2} = 0.4 + (x_{1,2} - 8.5)z_{1,2},$$

where $\omega_1=0.99$, $\omega_2=0.97$, and $\xi(t)$ is a Gaussian random process of zero mean and unit variance: $\langle\xi(t)\rangle=0$ and $\langle\xi(t)\xi(t')\rangle=\delta(t-t')$. Because the Rössler attractor possesses a proper structure of rotation, the phase variables are simply $\phi_{1,2}=\tan^{-1}(y_{1,2}/x_{1,2})$. In the absence of noise, phase synchronization occurs[5] for $K>K_c\cong 0.02$. For $K\gtrsim 0$, the average phase-synchronization time τ is small. As K approaches K_c, the time increases rapidly.[6] For $K\geqslant K_c$, the time becomes infinite. Noise can induce 2π phase slips even for $K\geqslant K_c$, making τ finite.[7] To calculate τ, we integrate the stochastic differential equation Eq. (1) by using a standard second-order routine[8] and obtain time series of length $T=10^5$. In this time interval, there are typically about a dozen of 2π-phase slips in the range of noise amplitude considered. The average synchronization time is obtained by using an ensemble of 2000 independent trajectories. Figure 1 shows, for $k=0.020\,75$, time evolutions of the phase difference for $D=0.015$ (upper trace) and $D=0.07$ (lower trace). We observe that for larger noise, the time intervals of phase synchronization appear longer. Figure 2 shows the dependence of τ on D, which exhibits a nonmonotonic behavior.

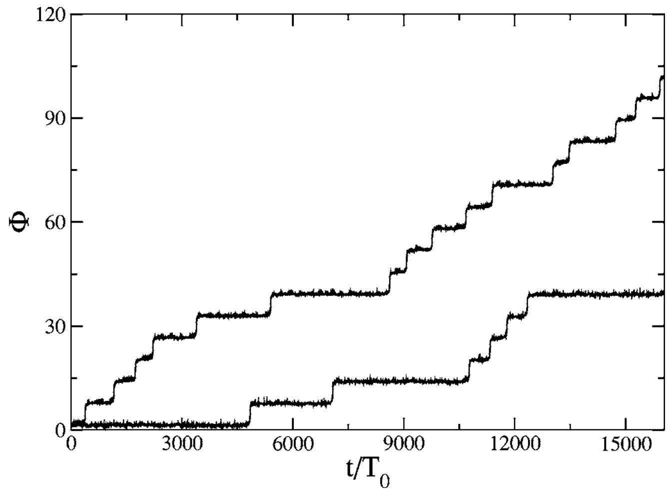

FIG. 1. For the coupled Rössler system for $K=0.020\,75$, time evolution of the phase difference for $D=0.015$ (upper trace) and $D=0.07$ (lower trace), where T_0 is the average period of the chaotic oscillation.

III. EXPERIMENTAL SUPPORT

A. Coupled Rössler circuits

Our first experimental system consists of two mutually coupled, nearly identical Rössler circuits[9] driven by a common noise source, as shown schematically in Fig. 3. A single Rössler circuit consists of five operational amplifiers (opamps, TL082 or TL084 in our experiments), three capacitors, a diode (1N4007), a number of resistors, and a potentiometer R. The nonlinearity in the circuit (piecewise linear

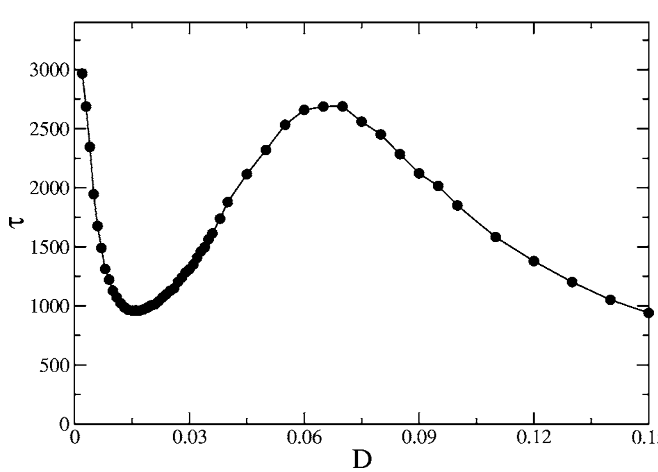

FIG. 2. For the coupled Rössler system, the average phase-synchronization time τ vs the amplitude D of common noise for $K=0.020\,75$. The unit for τ is the average period of the chaotic oscillation.

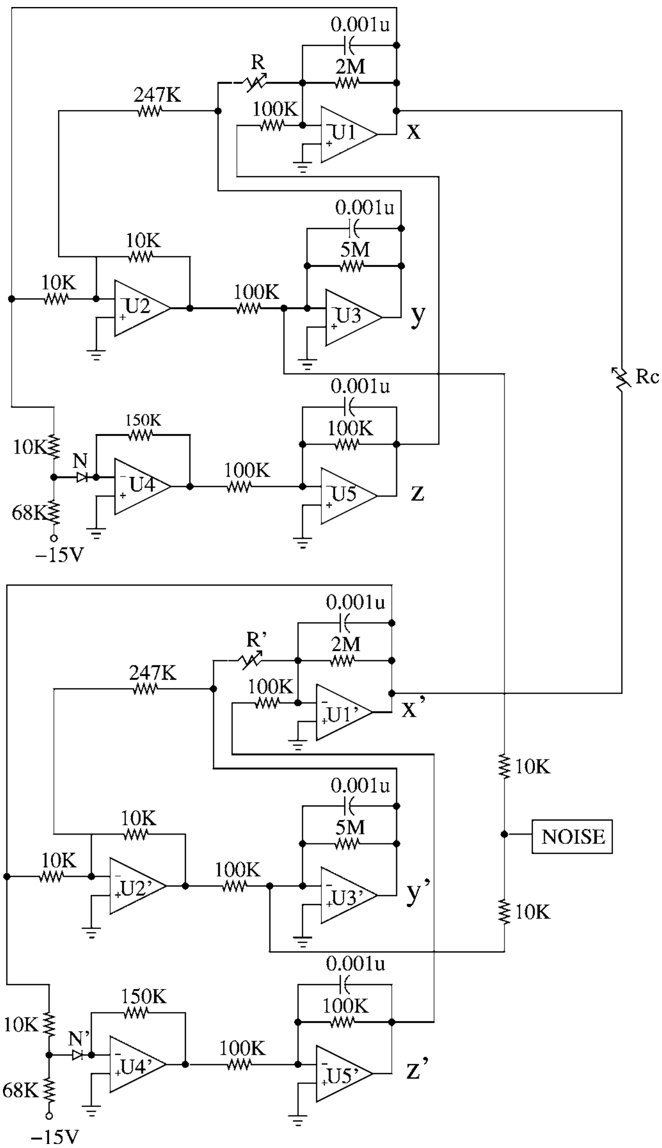

FIG. 3. Experimental circuit diagram of two mutually coupled Rössler oscillators driven by a common noise source.

function) is introduced by the opamp $U4$ along with the three resistors and the diode N. Different periodic and chaotic behaviors can be observed by tuning the potentiometer R. Mutual coupling between the two circuits is realized through the potentiometer R_c, the value of which can be adjusted. Common noise is generated by a commercial signal generator (SRS-DS345) and is applied to the inputs of opamps $U3$ and $U3'$ through the 10 KΩ resistors.

Initially, coupling is removed and noise is absent. In this case, two circuits are independent oscillators. The potentiometers R and R' are adjusted (≈ 10 KΩ) so that the circuits exhibit a similar chaotic attractor, which can be observed by using the voltage signals $x(t)$ and $y(t)$ [or $x'(t)$ and $y'(t)$]. The circuits are then connected via the potentiometer R_c. Voltage signals $y(t)$ and $y'(t)$ are digitally recorded using a data acquisition system (National Instrument, PXI-6115 DAQ Board) at the sampling frequency of 40 KHz. While the attractors in the $x(t)$-$y(t)$ planes are apparently chaotic, the voltage signals are topologically equivalent to a sinusoidal signal, and thus correspond to a proper rotation. The Hilbert transform is used to construct the corresponding analytic signal, which gives the phase variable.[5] Using the signals $y(t)$ and $y'(t)$ from both Rössler circuits, the phase difference $\Phi(t)$ can be obtained for any fixed noise level, which typically contains 2π-phase slips. This yields the average phase-synchronization time. Long voltage signals (60 s) typically containing between 700–1200 phase slips are acquired and the experiment is repeated three times to reduce the statistical variation. An example of the nonmonotonic phenomenon is shown in Fig. 4 for $R_c = 9.6$ KΩ, where the average phase-synchronization time is plotted versus the common-noise amplitude. We find that the nonmonotonic phenomenon can be readily generated in this experimental system.

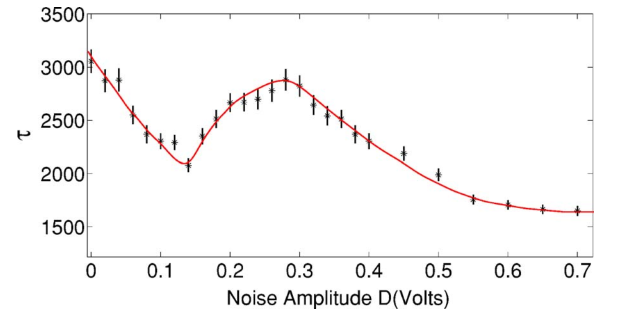

FIG. 4. Nonmonotonic behavior in the average phase-synchronization time observed from the experimental circuit system of two coupled Rössler oscillators for $R_c = 9.6$ KΩ, where τ is in the unit of average period of the voltage oscillations.

B. Coupled Chua circuits

To show that the nonmonotonic behavior of the average phase-synchronization time is a general phenomenon independent of the type of chaotic oscillators, we have conducted another experiment with two mutually coupled, nearly identical Chua circuits[10] under common noise, as shown schematically in Fig. 5. A single Chua circuit consists of an inductor, two capacitors, a potentiometer R, six resistors, and a nonlinear diode constructed using two operational amplifiers (TL084 and TL082). The mutual coupling between the two circuits is realized using the potentiometer R_c, and the common noise is applied to the inductor of each circuit using the operational amplifier U_3 as a buffer.

The equations of the complete circuit are given by $L_1 di_{L_1}/dt = -v_{C_1} + \xi(t)$, $C_1 dv_{C_1}/dt = i_{L_1} - (v_{C_1} - v_{C_2})/R$, $C_2 dv_{C_2}/dt = (v_{C_1} - v_{C_2})/R - f(v_{C_2}) - (v_{C_2} - v'_{C_2})/R_c$, $L_1 di'_{L_1}/dt = -v'_{C_1} + \xi(t)$, $C_1 dv'_{C_1}/dt = i'_{L_1} - (v'_{C_1} - v'_{C_2})/R'$, and $C_2 dv'_{C_2}/dt = (v'_{C_1} - v'_{C_2})/R' - f'(v'_{C_2}) - (v'_{C_2} - v''_{C_2})/R_c$, where $\xi(t)$ denotes the common noise added to the circuit, and $f(\cdot)$ and $f'(\cdot)$ represent the current-voltage relation of the nonlinear diode, as follows: $f(x) = m_0 x + [(m_1 - m_0)(|x + B_p| - |x - B_p|)]/2$ and $f'(x) = m'_0 x + [(m'_1 - m'_0)(|x + B_p| - |x - B_p|)]/2$, where $B_p = 1.0$ V, $m_0 = R[-R_2/(R_1 R_3) - R_5/(R_4 R_6)]$, $m_1 = R[-1/R_4 - R_2/(R_1 R_3)]$ for the first circuit, and similar equations for the second circuit. Initially, the coupling is removed and noise is absent as for the Rössler circuits. The potentiometers R and R' are chosen (≈ 1.73 KΩ) such that each circuit exhibits a double-scroll chaotic attractor, which can be ob-

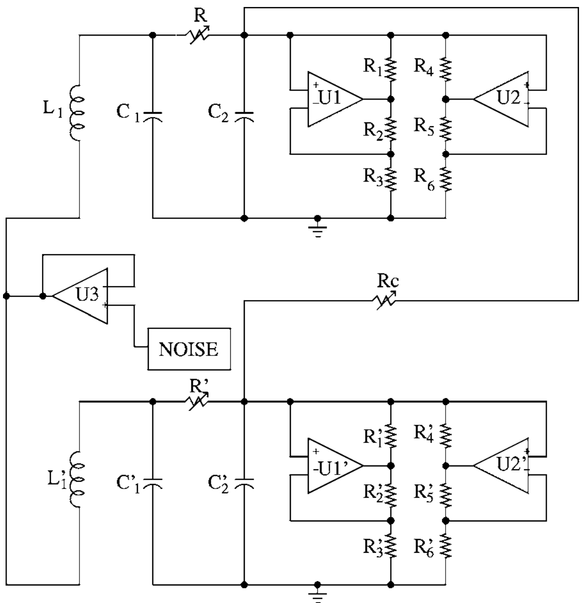

FIG. 5. Circuit diagram of two mutually coupled Chua circuits driven by common noise source. The circuit parameters are $L_1=L_1'=18$ mH, $C_1=C_1'=100$ nF, $C_2=C_2'=10$ nF, $R_1=R_1'=R_2=R_2'=20.4$ KΩ, $R_3=R_3'=3.39$ KΩ, $R_4=R_4'=R_5=R_5'=205$ Ω and $R_6=R_6'=2.0$ KΩ.

served by using the voltage signals $v_{C_1}(t)$ and $v_{C_2}(t)$ from the two capacitors. The circuits are then connected via the potentiometer R_c, to which the coupling strength is inversely proportional. Voltage signals $v_{C_1}(t)$ and $v_{C_2}(t)$ are digitally recorded using a data acquisition system at the sampling frequency of 100 KHz. While the attractor in the $v_{C_1}(t)-v_{C_2}(t)$ plane exhibits the double-scroll characteristic, one of the voltage signals is topologically equivalent to a sinusoidal signal and thus corresponds to a proper rotation. The average phase-synchronization time can be calculated by using long voltage signals (80 s) that typically contain between 600 and 1000 phase slips. An example of the nonmonotonic phenomenon is shown in Fig. 6 for $R_c=1.2$ KΩ. Similar plots can be

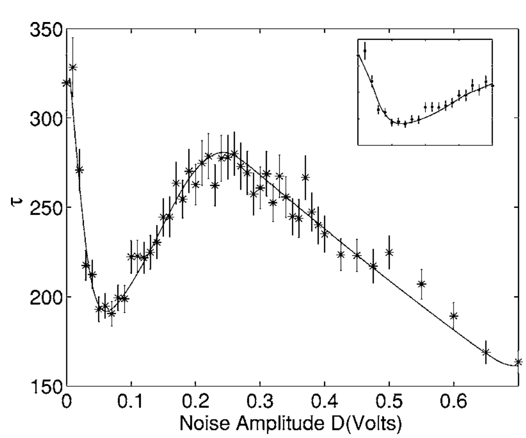

FIG. 6. Nonmonotonic behavior in the average phase-synchronization time observed from our experimental system of Chua circuits for $R_c=1.2$ KΩ, where τ is in the unit of average period of the voltage oscillations. Inset: τ for $0\leqslant D\leqslant 0.2$.

obtained for different sets of parameter values. This result suggests that the nonmonotonic behavior is a robust and general phenomenon.

IV. HEURISTIC ARGUMENT

To obtain a qualitative understanding of phase synchronization under both coupling and common noise, we consider a prototype model of two mutually coupled, phase-coherent chaotic attractors. The phase dynamics of each attractor in such a model can be described by $\dot{\phi}=\omega+g[r(t)]$,[5,11] where ω is the average frequency of the chaotic oscillation, $r(t)$ represents the chaotic amplitude, and the function g describes the influence of $r(t)$ on the phase dynamics. For two coupled oscillators with coupling constant K and without noise, we can write

$$\dot{\phi}_{1,2}=\omega_{1,2}+g_{1,2}[r_{1,2}(t)]+Kh(\phi_{2,1},\phi_{1,2}), \tag{2}$$

where h is a 2π-periodic function in each of its arguments and we assume that $\omega_1\neq\omega_2$. A simple choice for the coupling term $h(\phi_{2,1},\phi_{1,2})$ is

$$h(\phi_{2,1},\phi_{1,2})=\sin(\phi_2-\phi_1). \tag{3}$$

Equation (2) is representative of typical systems in the study of chaotic phase synchronization such as the system of coupled phase-coherent Rössler attractors.[2] In the coupled system, the time evolution of the phase difference $\Phi(t)$ between the two oscillators is determined by K. For a phase-coherent chaotic attractor, the dependence of the frequency on the amplitude is typically weak,[5] so $g_2[r_2(t)]\approx g_1[r_1(t)]$. The equation for Φ can thus be written as

$$\dot{\Phi}\approx\Delta\omega-2K\sin\Phi, \tag{4}$$

where $\Delta\omega=\omega_2-\omega_1$. Equation (4) describes the motion of a heavily damped particle in the potential $V(\Phi)$ given by

$$-dV/d\Phi=\Delta\omega-2K\sin\Phi,$$

which possesses an infinite number of local minima separated by 2π in the phase variable Φ.[7] Because of chaos and noise, the values of the potential minima fluctuate with time. A particle starting near a potential well can be trapped for a finite amount of time and hoppings of the particle between adjacent potential wells correspond to 2π-phase slips. The probability for a hopping event to occur is given by the Kramers formula: $P\sim e^{-\Delta E/T(D)}$, where ΔE is the average height of the potential barrier and $T(D)$ denotes the effective "temperature" that depends on the common noise amplitude in the following way: $T(D)\sim D^2$. The average phase-synchronization time is given by

$$\tau(D)\sim e^{\Delta E/T(D)}. \tag{5}$$

To explain the behavior of $\tau(D)$ we have to know the explicit form of the potential-barrier height ΔE, but it depends on details of the system and in general cannot be obtained analytically.

For $D=0$, the interaction is purely deterministic. Although K is finite and small, on average the height of the potential barrier is large despite chaotic fluctuation. In this case, we expect τ to be large. As D is turned on, the height of the potential barrier is reduced due to random noise, making particle hopping easier. We should thus see a decrease in τ as D is increased from zero. This trend continues until for $D \equiv D_m$, where common noise begins to induce coherence. The value of τ reaches a local minimum for $D=D_m$. This minimum value cannot be arbitrarily small because, for any small time interval there is still coupling and, consequently, the height of the potential barrier is not zero. As D is increased from D_m, the coherence induced by common noise becomes stronger, making ΔE larger. For $D \gtrsim D_m$, τ then increases with D. This trend cannot continue indefinitely because for large noise, the increase of the "temperature" $T(D)$ with D becomes dominant. Thus, although the effective potential wells become deeper, random fluctuations are enhanced as well. The interplay between these two factors is similar to what happens typically in a stochastic resonance:[12] as D increases, τ can reach a maximum value for $D=D_{\text{opt}} > D_m$ and then decreases. Summarizing, we expect to observe the following behavior in the average phase-synchronization time as a function of the common-noise amplitude: as D is increased from zero, τ first decreases and reaches a minimum at D_m. The time then increases with D and reaches a maximum at $D_{\text{opt}} > D_m$. For $D > D_{\text{opt}}$, τ decreases continuously with D.

V. DISCUSSION

In summary, we have presented a phenomenon in noisy phase synchronization: a nonmonotonic relation between the average phase-synchronization time and the amplitude of the common noise. We have provided numerical and experimental evidence for the nonmonotonic behavior. Our heuristic argument suggests that the local minimum occurring at the relatively small noise amplitude is due to the dynamical competition between the deterministic coupling and the effective coupling due to the common noise, and the local maximum at a larger noise amplitude is generated by a mechanism similar to stochastic resonance. Previous works have demonstrated that in settings different from ours, quantities such as the effective diffusion coefficient or the signal-to-noise ratio also exhibit a nonmonotonic behavior with respect to the noise intensity.[13–15] It would be interesting to relate our finding to these and to work out a more quantitative theory to explain the nonmonotonic behavior. Practically, the behavior suggests that fine tuning of the noise strength is required if one attempts to achieve robust phase synchronization by applying common noise. Noise-induced phase synchronization is a recently discovered phenomenon that has a number of potential applications.[2] We expect our finding to be useful for applications of phase synchronization.

ACKNOWLEDGMENT

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