Frequency dependence of phase-synchronization time in nonlinear dynamical systems

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It has been found recently that the averaged phase-synchronization time between the input and the output signals of a nonlinear dynamical system can exhibit an extremely high sensitivity to variations in the noise level. In real-world signal-processing applications, sensitivity to frequency variations may be of considerable interest. Here we investigate the dependence of the averaged phase-synchronization time on frequency of the input signal. Our finding is that, for typical nonlinear oscillator systems, there can be a frequency regime where the time exhibits significant sensitivity to frequency variations. We obtain an analytic formula to quantify the frequency dependence, provide numerical support, and present experimental evidence from a simple nonlinear circuit system.
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In real-world applications, complete synchronization among signals, in the sense that they approach each other asymptotically, is unlikely. However, phase synchronization can be expected to occur commonly as it characterizes the tendency for signals to follow each other and henceforth is a weaker type of synchronization. Due to factors such as parameter mismatch, nonstationarity, and noise, phase synchronization can typically last for a finite amount of time. The average phase-synchronization time thus stands out as a fundamental quantity that finds usage in a broad spectrum of problems in nonlinear science, such as stochastic resonance, transient chaos, and biomedical signal processing. Here we investigate theoretically, numerically, and experimentally the dependence of the average phase-synchronization time on frequency, regarding the underlying dynamical system as a signal-processing device. We find that the time can be highly sensitive to frequency changes, rendering it useful for tasks such as precise frequency tuning. The result may also have implications to biological systems in terms of their abilities to respond to external excitations for learning and adaptation.

I. INTRODUCTION

Stochastic phase synchronization, since its discovery in nonlinear systems,[1,2] has found wide applications in many areas of science and engineering such as biomedical signal processing[3–5] and lasers.[6] Given a signal $x(t)$, insofar as it is oscillatory, one can define a corresponding phase variable, say, $\phi_x(t)$, intuitively as follows. Set $\phi_x(0)=0$ at $t=0$ and monitor the evolution of $x(t)$. Whenever $x(t)$ completes one cycle of oscillation, $\phi_x(t)$ is increased by 2π. This way, $\phi_x(t)$ can be defined as a nondecreasing function of time t, determined by the oscillations of $x(t)$.[7] Since $x(t)$ can in general be aperiodic (e.g., chaotic or random), one can write $\phi_x(t)=\omega_x t+\theta_x(t)$, where ω_x is the average frequency of $x(t)$ and $\theta_x(t)$ models random fluctuation of the phase, where $|\theta_x(t)|<2\pi$. For a different signal $y(t)$, a phase variable $\phi_y(t)$ can be defined in a similar way: $\phi_y(t)=\omega_y t+\theta_y(t)$. There is a phase synchronization between $x(t)$ and $y(t)$ if the phase difference is bounded within 2π:[2] $\Delta\phi(t)=|\phi_x(t)-\phi_y(t)|<2\pi$. Apparently, phase synchronization requires $\omega_x=\omega_y$. Compared with complete synchronization, where $x(t)$ and $y(t)$ are required to approach each other asymptotically, phase synchronization is a weak type of synchronization and it is thus expected to occur more commonly in real-world systems. In the presence of noise, even phase synchronization cannot be maintained indefinitely, as noise can cause the phase difference to change by more than 2π in relatively short time (so-called 2π-phase slip). Thus, a meaningful quantity to characterize the degree of phase synchronization is the average time during which the condition $\Delta\phi(t)=|\phi_x(t)-\phi_y(t)|<2\pi$ is satisfied. The average phase-synchronization time has found interesting applications in problems in nonlinear science such as superpersistent chaotic transients,[8] characterization of stochastic resonance,[9] and assessment of synchrony in multichannel epileptic brain signals.[5] These suggest that this time is a fundamental quantity in nonlinear and stochastic dynamical systems with a broad spectrum of applications.

In this paper, we investigate the frequency dependence of the average phase-synchronization time, denoted by $\tau(\omega)$. Our motivation comes from the following problem. Imagine a nonlinear system in a noisy environment with input signal $x(t)$ and output signal $y(t)$. Without loss of generality, we shall assume that the input signal is periodic, so it has a unique frequency. (A more general input signal can be Fourier-transformed into a number of periodic components.) The output signal can, however, be significantly more complicated because of nonlinearity and noise. Thus, there can only be phase synchronization between the input and the output signals. Due to noise, it is necessary to focus on the quantity τ. We ask, What is the dependence of τ on frequency ω? The phenomenon that we wish to report is the existence of general conditions under which a strikingly sensitive dependence can occur, mathematically represented by a cusplike behavior in $\tau(\omega)$. This type of dependence can find applications in, for example, frequency-tuning devices. It may also have implications to problems such as how a biological oscillator can effectively respond to external excitations by precise frequency tuning. In what follows, we shall derive a formula for $\tau(\omega)$ for a paradigmatic nonlinear system that is amenable to analysis, and provide both numerical and experimental support for the cusplike behavior in $\tau(\omega)$.

II. THEORY

We consider a class of nonlinear dynamical systems that describe the motions of heavily damped particles in a one-dimensional potential, under the influence of noise. The Langevin equation[10] is

$$\dot{x}(t) = -\frac{dV(x)}{dx} + F(t) + \xi(t), \tag{1}$$

where $V(x)=x^4/4-x^2/2$ is a bistable potential, $F(t)$ is a periodic input signal of period T_0, and $\xi(t)$ is a Gaussian random process satisfying $\langle\xi(t)\rangle=0$ and $\langle\xi(t)\xi(t')\rangle=2D\delta(t-t')$, which models noise. This system has been a paradigm in the study of stochastic resonance.[11–13] To facilitate analysis but without loss of generality, we choose $F(t)$ to be a rectangular periodic signal: $F(t)=(-1)^{n(t)}A$, where $n(t)=\lfloor 2t/T_0\rfloor$ and $\lfloor\cdot\rfloor$ denotes the floor function; i.e., $F(t)=A$ $(F(t)=-A)$ if $t\in[nT_0/2,(n+1)T_0/2]$ for even (odd) n. The frequency of the input signal is $\omega=2\pi/T_0$. Equation (1) can be rewritten in the following form:

$$\dot{x}(t) = -\frac{dU(x,t)}{dx} + \xi(t), \tag{2}$$

where $U(x,t)=V(x)-xF(t)$ is a time-dependent, effective potential function. Previous work[12] has indicated that, if the amplitude A of the input signal is less than the threshold value $A_{\text{th}}=\sqrt{4/27}$, the effective potential function $U(x,t)$ can have two minima, located at $x_1(t)<0$ and $x_2(t)>0$, respectively, and one maximum at $x_m(t)$, for all t. Symmetry of $U(x,t)$ stipulates

$$x_m(t) = (-1)^{n(t)}x_m(0), \quad x_1(t)+x_2(t)+x_m(t)=0,$$

$$x_j(t) = (-1)^j\frac{\Delta x(0)}{2} - (-1)^{n(t)}\frac{x_m(0)}{2},$$

where $\Delta x(0)=x_2(0)-x_1(0)$.

To calculate the average phase-synchronization time between the input (the periodic driving force) and the output [the signal $x(t)$], the following two quantities are necessary: the average frequency of the output signal Ω_{out} and the effective diffusion coefficient D_{eff},[10] which are defined as

$$\Omega_{\text{out}} = \langle\omega_{\text{out}}\rangle = \frac{1}{T_0}\int_0^{T_0} dt\,\omega_{\text{out}}(t) \tag{3}$$

and

$$D_{\text{eff}} = \frac{1}{2}\frac{d}{dt}[\langle(\Delta\phi)^2\rangle - \langle\Delta\phi\rangle^2], \tag{4}$$

where $\Delta\phi(t)$ is the phase difference between the input and the output signals: $\Delta\phi(t)\equiv\phi_{\text{out}}(t)-\phi_{\text{in}}(t)$. The quantities Ω_{out} and D_{eff} have recently been derived by Casado-Pascual *et al.*,[10] as follows:

$$\Omega_{\text{out}} = \frac{\pi\gamma}{2}\left\{1-[\Delta P_{\text{eq}}(0)]^2\left[1-\frac{4\tanh\left(\frac{\gamma T_0}{4}\right)}{\gamma T_0}\right]\right\} \tag{5}$$

and

$$\begin{aligned} D_{\text{eff}} = {} & \pi\Omega_{\text{out}} - \frac{2\pi^2}{T_0}[\Delta P_{\text{eq}}(0)]^4\left[\tanh\left(\frac{\gamma T_0}{4}\right)\right]^3 \\ & - \frac{\pi^2}{2T_0}[\Delta P_{\text{eq}}(0)]^2\{1-[\Delta P_{\text{eq}}(0)]^2\}\left(12\tanh\left(\frac{\gamma T_0}{4}\right)\right. \\ & \left. - \gamma T_0\left\{1+2\left[\text{sech}\left(\frac{\gamma T_0}{4}\right)\right]^2\right\}\right), \end{aligned} \tag{6}$$

where

$$\Delta P_{\text{eq}}(0) = P_{\text{eq}}(2,0) - P_{\text{eq}}(1,0),$$

$$\gamma = \gamma_1 + \gamma_2,$$

$$P_{\text{eq}}(j,0) = [\delta_{j,1}\gamma_2 + \delta_{j,2}\gamma_1]/\gamma,$$

and

$$\gamma_j(t) = \frac{\omega_j(t)\omega_m(t)}{2\pi}\exp\left\{-\frac{U[x_m(t),t]-U[x_j(t),t]}{D}\right\},$$

$$\omega_j(t) = \sqrt{d^2U[x_j(t),t]/dx^2} = \sqrt{3[x_j(t)]^2-1},$$

$$\omega_m(t) = \sqrt{d^2U[x_m(t),t]/dx^2} = \sqrt{1-3[x_m(t)]^2}.$$

The average phase-synchronization time τ can be calculated from Eq. (4):[14]

$$\langle(\Delta\phi)^2\rangle \approx \langle\dot{\Delta\phi}\rangle^2\langle T\rangle^2 + 2D_{\text{eff}}\langle T\rangle, \tag{7}$$

where $\dot{\Delta\phi}=\Delta\omega=\omega_{\text{out}}-\omega$. Since τ is the average time for a 2π change in $\Delta\phi(t)$, we have $\langle\Delta\phi^2(n\tau)\rangle=(2n\pi)^2$, which leads to $\sqrt{\langle\Delta\phi^2(t)\rangle}|_{t=\tau}=2\pi$. The formula for τ can be obtained from this result and Eq. (7),[14,15] as follows:

$$\tau \sim \frac{D_{\text{eff}}}{\langle\dot{\Delta\phi}\rangle^2}\left[\sqrt{1+\left(\frac{2\pi\langle\dot{\Delta\phi}\rangle}{D_{\text{eff}}}\right)^2}-1\right]. \tag{8}$$

We have calculated the dependences of D_{eff}, $\langle\Delta\omega\rangle^2$, and τ on the input frequency. Figures 1(a) and 1(b) show, for $A=0.18$ and $D=0.031$, D_{eff} versus ω and $\langle\Delta\omega\rangle^2$ versus ω,

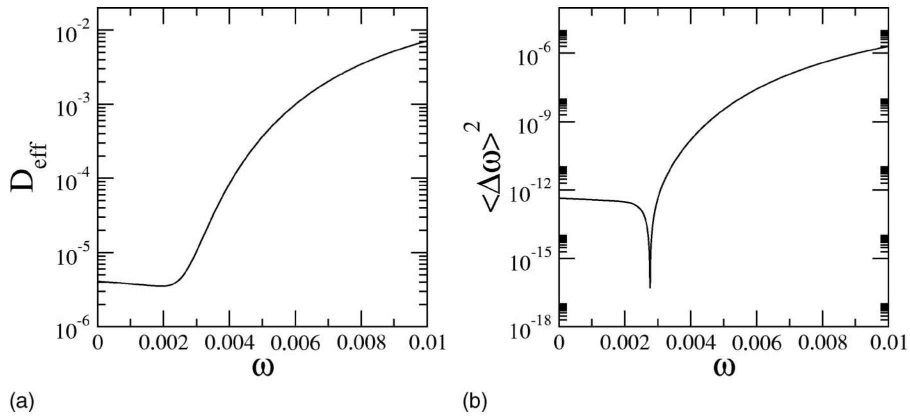

FIG. 1. For the paradigmatic model Eq. (2), (a) the effective diffusion coefficient D_{eff} and (b) $\langle\Delta\omega\rangle^2$ as a function of the frequency ω. Model parameters are $A=0.18$ and $D=0.031$.

respectively. We see that as ω is increased, D_{eff} decreases, reaches the minimum value, and then increases. Interestingly, the frequency difference $\langle\Delta\omega\rangle$, which reaches its minimum value at $\omega\simeq 0.003$, shows a much higher sensitive dependence than D_{eff} on ω. Thus we expect the average phase-synchronization time τ to have a large maximum value near $\omega\simeq 0.003$, as shown in Fig. 2. We see that τ exhibits a cusplike behavior with respect to ω, reaching its maximum value at $\omega\simeq 0.003$. There is thus a high sensitivity of τ to small variations in the frequency of the input signal in a frequency regime about the optimal value, indicating the potential use of τ for precisely tuning the system to the optimal frequency.

To verify the theoretical prediction Eq. (8), we have carried out extensive numerical simulations of Eq. (2). A set of representative results is shown in Fig. 2 as open circles for $A=0.18$ and $D=0.031$, to enable a direct comparison with theory. Note that the theory does not predict the absolute value of τ and, hence, a proper proportional constant is introduced in Fig. 2. Numerically, to calculate the average phase-synchronization time reliably, we use data with $\tau>2.0$ and, hence, there are no data for $\omega>0.007$ in Fig. 2. Despite these, we observe a reasonable agreement between theory and numerics. The inset of Fig. 2 shows τ versus ω for $D=0.023$, 0.027, 0.031, and 0.035 ($A=0.18$), which is obtained from Eq. (8). As D decreases, τ reaches its maximum value for smaller ω, showing more sensitive dependence on frequency variations.

We remark that recently, Rrager and Schimansky-Geier[16] considered the following Péclet number: $\mathrm{Pe}=2\pi\langle\dot{\Delta}\phi\rangle/\langle\omega/D_{eff}\rangle$, where D_{eff} is the phase-diffusion coefficient, and found that Pe shows a cusplike behavior with respect to frequency variations. This implies that Pe and the average phase-synchronization time are intrinsically related. Indeed, an implicit relation between the two quantities can be established since τ can be expressed in terms of $\Delta\phi$ and D_{eff}, as in Eq. (8).

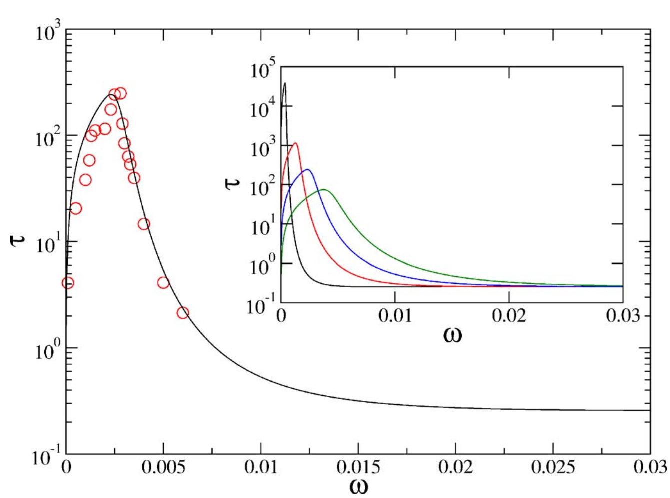

FIG. 2. (Color online) Average phase-synchronization time τ for $A=0.18$ and $D=0.031$. Inset: τ versus ω for $D=0.023$, 0.027, 0.031, and 0.035 ($A=0.18$). The unit of τ is the number of cycles of the input signal. The solid curves are obtained from Eq. (8) and data (circles) are from numerical simulations of Eq. (2).

III. EXPERIMENTS

While the system that we have used to demonstrate a high sensitivity of τ to frequency variations is idealized, the model has proven to be generic for many aspects of typical nonlinear phenomena such as stochastic resonance.[12] Thus, we expect our theoretical prediction of the frequency dependence of τ to be general. To provide further support, we have carried out a set of laboratory experiments utilizing the Schmitt-trigger circuit,[17,18] constructed by microelectronic circuit components, as shown in Fig. 3. The first stage, realized using the operational amplifier U1 and several resistors, is a summing amplifier whose inputs are the sinusoidal signal (periodic input) v_{sin} and noise. The output of the first stage is a voltage equal to the sum of the input and noise. The output of the summing amplifier is fed as input to the Schmitt trigger circuit (operational amplifier U2, and resistors R_1 and R_2). As a result, the output v_{out} of circuit is controlled by both the subthreshold periodic signal v_{sin} and the noise amplitude. Depending on values of v_{sin} and noise, the output is

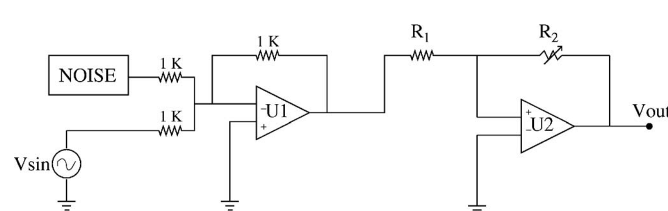

FIG. 3. Microelectronics circuit diagram used in our experimental study of the frequency dependence of the average phase-synchronization time.

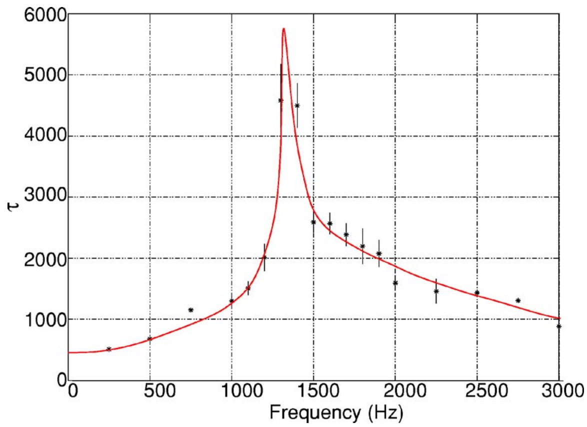

FIG. 4. Representative experimental result of the frequency dependence of the average phase-synchronization time (in units of the number of cycles of the input periodic signal). The noise voltage is fixed at $D=0.8$ V. We observe a cusplike behavior, as predicted by theory. (The solid curve is a guide for the eye.)

in either one of the two stable states. The resistor R_2 is utilized as a potentiometer in order to set the threshold voltages to some required value. The input periodic signal is biased below the threshold voltages. In our experiments, we fix the noise amplitude and vary the input frequency over a reasonable range, and then calculate τ from measured voltage signals.

In one experiment, the noise voltage is fixed as 0.8 V. The frequency of the input sinusoidal signal is varied from 0 to 3 KHz in steps of 250 Hz. The sinusoidal input and the Schmitt-trigger output data are recorded at the sampling frequency of 40 kHz using a standard data-acquisition device (National Instruments) and τ is calculated by using long voltage signals (200 s) that typically contain a large number of 2π phase slips. The experiment is repeated ten times to reduce the statistical variation in the measurement of τ. A representative result is shown in Fig. 4. The optimal frequency for this noise intensity is estimated to be about 1.3 kHz. The average phase-synchronization time exhibits a high sensitivity to frequency variations near an optimal frequency value, as predicted.

IV. DISCUSSION

In conclusion, studying a nonlinear signal-processing system from the standpoint of synchronization, we have addressed the frequency dependence of a fundamental quantity in synchronization: the average phase-synchronization time. Our theoretical, numerical, and experimental explorations have revealed that this time can typically be highly sensitive to frequency variations near an optimal frequency value. The sensitive frequency dependence can be well explained by the Langevin dynamics for heavily damped particle motion in the paradigmatic one-dimensional, double potential well system, which is representative of a large class of stochastic systems. Thus, we expect our finding to be general. For example, for a system that exhibits two distinct time scales (or frequencies), we expect the average phase-synchronization time to depend sensitively on frequency variations in the vicinities of these frequencies. In excitable systems where the double-well approximation generally holds,[16,19,20] we expect a similar phenomenon of sensitive frequency dependence.

The sensitive dependence of the average phase-synchronization time on frequency may find potential use in signal-processing tasks such as high-precision frequency tuning. It can also provide insight into how frequency tuning may be achieved in biological systems that need to constantly find optimal environment for survival, adaptation, and evolution in the presence of noise.[21] While for many biological systems, better synchronization means better performance, there are situations where synchronization can lead to disastrous events such as certain types of epileptic seizures. To prevent strong synchronization from happening can be of interest, which can be realized, for instance, by applying a small external signal with proper frequency mismatch so that the average synchronization time is small.

We remark that nonlinear systems can exhibit a resonant behavior with the change of the frequency of the weak input signal under a fixed noise intensity, the so-called *bona fide* resonance.[22] This phenomenon has been observed in the residence time distributions through numerical simulations of a bistable system and subsequently in experiments.[22] Typically, *bona fide* resonances are not as strong as stochastic resonance in the sense that the system usually shows a much higher sensitivity to noise variations about some optimal value. Thus, it can be quite difficult to use measures such as the signal-to-noise ratio and correlations to detect *bona fide* resonance. Our finding that the average phase-synchronization time can be sensitive to frequency variations can significantly facilitate the detection of *bona fide* resonance in computations or in laboratory experiments.

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