Effect of resonant-frequency mismatch on attractors

Xingang Wang
Department of Physics, National University of Singapore, Singapore 117542
and Beijing-Hong Kong-Singapore Joint Centre for Nonlinear & Complex Systems (Singapore), National University of Singapore, Kent Ridge, Singapore 119260

Ying-Cheng Lai
Department of Electrical Engineering, Arizona State University, Tempe, Arizona 85287

Choy Heng Lai
Department of Physics, National University of Singapore, Singapore 117542
and Beijing-Hong Kong-Singapore Joint Centre for Nonlinear & Complex Systems (Singapore), National University of Singapore, Kent Ridge, Singapore 119260

(Received 1 February 2006; accepted 8 May 2006; published online 23 June 2006)

Resonant perturbations are effective for harnessing nonlinear oscillators for various applications such as controlling chaos and inducing chaos. Of physical interest is the effect of small frequency mismatch on the attractors of the underlying dynamical systems. By utilizing a prototype of nonlinear oscillators, the periodically forced Duffing oscillator and its variant, we find a phenomenon: resonant-frequency mismatch can result in attractors that are nonchaotic but are apparently strange in the sense that they possess a negative Lyapunov exponent but its information dimension measured using finite numerics assumes a fractional value. We call such attractors *pseudo-strange*. The transition to pesudo-strange attractors as a system parameter changes can be understood analytically by regarding the system as nonstationary and using the Melnikov function. Our results imply that pseudo-strange attractors are common in nonstationary dynamical systems. © *2006 American Institute of Physics*. [DOI: 10.1063/1.2208566]

We learned from freshman physics that for a linear oscillator, resonant forcing with frequency matching the internal frequency of the oscillator can generate oscillations of arbitrarily large amplitude. For a nonlinear oscillator, researchers have discovered that resonant perturbations can cause characteristic changes in the system's asymptotic behavior. For instance, for control of chaos, resonant forcing of small amplitude can convert a chaotic attractor into periodic, and vice versa. But what is the effect of frequency mismatch on the attractors of the nonlinear oscillator under resonant perturbation? By addressing this question, we find a class of attractors that are not chaotic and in principle are not fractal either, but they exhibit a fractal geometry on finite scales. That is, for such an attractor, although its information dimension defined in the mathematical limit of infinitesimal scales is an integer, in finite scales the dimension assumes a fractional value. Physically, this is particularly relevant because extremely small scales are not accessible due to noise. In order to understand the dynamical origin of pseudo-strange attractors, we propose to interpret the effect of frequency mismatch as that due to a time-dependent parameter, so the resonantly forced system is effectively a nonstationary dynamical system where the parameter sweeps adiabatically through both periodic and chaotic regimes. Analysis based on the Melnikov function can be used to provide insights into the transition to pseudo-strange attractors. As nonstationary dynamical systems with adiabatic parameter variations are relevant to physical and biological situations, we expect pseudo-strange attractors to be common.

I. INTRODUCTION

Resonant perturbations have proven to be an effective method for controlling the dynamics of nonlinear oscillators[1–7] in various applications. Early works focused on the control of chaos; i.e., stabilizing some periodic motions from chaos. It has been shown that, for a periodically forced nonlinear oscillator, resonant perturbations of appropriate strength can cause the originally chaotic attractor to be replaced by a periodic attractor.[1,2,5] More recently, it has been demonstrated theoretically and experimentally that resonant perturbations with time-dependent frequencies and phases can be used to continuously excite a stable periodic attractor into a hierarchy of resonant states and eventually to chaos for both Hamiltonian and dissipative systems.[7]

The fundamental requirement for resonant perturbation is frequency match. In physical applications perfect match may not be achieved. Motivated by this, we investigate the effect of resonant-frequency mismatch on the dynamics of the driven nonlinear oscillator. By using the periodically forced Duffing oscillator as a prototype model, we find a surprising phenomenon: there can be parameter regimes where the frequency mismatch results in a type of attractors distinct from those usually seen in the sense that they are not chaotic but are observably strange. In particular, for such an attractor none of the Lyapunov exponents is positive, but the information dimension measured using finite numerics assumes a fractional value. (While the limiting value of the dimension may be an integer, it is physically nonobservable.) We call such attractors *pseudo-strange* attractors.[8] As a system parameter, such as the periodic forcing amplitude, changes, there can be transitions from a periodic attractor to a pseudo-strange attractor, and then to a chaotic attractor. Analytic insights for these phenomena can be obtained by treating the frequency-mismatch term as an adiabatic phase variable and then using the Melnikov function. As we will show, using this approach the system can be regarded effectively as a nonstationary dynamical system with a time-dependent parameter,[9,10] which may find significant applications in biological situations such as neuronal networks with balanced excitatory and inhibitory activity[11] and epilepsy.[12] Our results suggest that pseudo-strange attractors can be expected to arise commonly in nonstationary dynamical systems.

In Sec. II, we present numerical evidence for pseudo-strange attractors in the resonantly forced Duffing oscillator. In Sec. III, we explain the origin of the attractors in the context of nonstationary dynamical systems and elucidate the transition to a pseudo-strange attractor and further to a chaotic attractor. An additional example is provided in Sec. IV and a brief summary is presented in Sec. V.

II. PSEUDO-STRANGE ATTRACTORS IN RESONANTLY FORCED DUFFING SYSTEMS

We consider a periodically forced Duffing oscillator under resonant perturbations with frequency mismatch, given by

$$\frac{d^2x}{dt^2} = x - x^3 - \nu\frac{dx}{dt} + \rho\cos(\Omega t) + \beta\cos[(\omega+\Delta\omega)t], \quad (1)$$

where ν is the dissipation parameter, ρ and Ω are the periodic forcing amplitude and frequency, respectively, $\beta \ll \rho$ is the amplitude of the resonant perturbation, ω is the resonant frequency which is rotationally related to Ω, and $\Delta\omega \ll (\Omega,\omega)$ is the resonant-frequency mismatch. For the periodically forced Duffing oscillator, if the forcing amplitude is small, the attractor is usually periodic. A chaotic attractor can arise for large forcing amplitude.[13] Utilizing the phase-space variables x, $y \equiv dx/dt$, and $z=\Omega t$, Eq. (1) becomes

$$\frac{dx}{dt} = y,$$

$$\frac{dy}{dt} = x - x^3 - \nu y + \rho\cos z + \beta\cos\left[\frac{\omega+\Delta\omega}{\Omega}z\right], \quad (2)$$

$$\frac{dz}{dt} = \Omega.$$

To gain insights, we choose $\nu=0.4$, $\Omega=\omega=1.0$, $\rho=0.28$, $\beta=0.05$, and $\Delta\omega=5\times10^{-4}$. Figure 1(a) shows a typical time series $x(t)$ from Eq. (1), where we see that $x(t)$ exhibits periodic and chaotic behaviors that occur intermittently in time. Figure 1(b) shows the underlying attractor on the stroboscopic section $z_n=2\pi n$, where $n=0,1,\ldots$. The largest nontrivial Lyapunov exponent of the attractor is calculated to be $\lambda\approx-0.054<0$, indicating that the attractor is nonchaotic. However, the attractor apparently possesses a fractal-like geometry. To gain quantitative insight, we calculate the information dimension as shown in Fig. 2; i.e., the plot of the information sum $I(\epsilon)=\Sigma\mu_i\ln\mu_i$ versus $\ln\epsilon$, where ϵ is the size of the grid of boxes used to cover the attractor, and μ_i is the natural measure in the ith box. The slope of the fit is $d_1=1.33\pm0.01$, so that the information dimension of the attractor in the full phase space is $D_1=1+d_1=2.33\pm0.01$, a fractional value within the feasible numerical resolution.

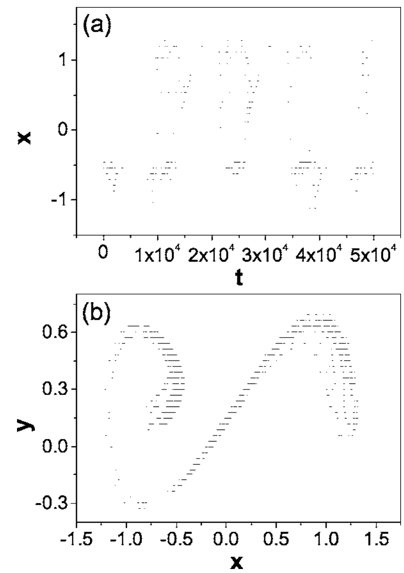

FIG. 1. For the resonantly forced Duffing oscillator Eq. (1): (a) time series $x(t)$ and (b) attractor on the stroboscopic section in the phase space [Eq. (2)].

The above fractional value of the information dimension is obtained using finite numerics. The following argument suggests that, if the asymptotic Lyapunov exponent is negative, the "true" information dimension of the attractor should be an integer. Note Fig. 1(a), which suggests that the attractor contains two measures, one chaotic and one periodic. For $\lambda<0$, the measure associated with the periodic orbit is "thicker" in the sense that it weighs over the fractal measure associated with the chaotic set. Consider the dimension spectrum D_q.[14] For $q\geqslant1$, the thicker measure dominates in the limit that the phase-space scale ϵ tends to zero. However, to resolve this numerically requires box-counting and calcula-

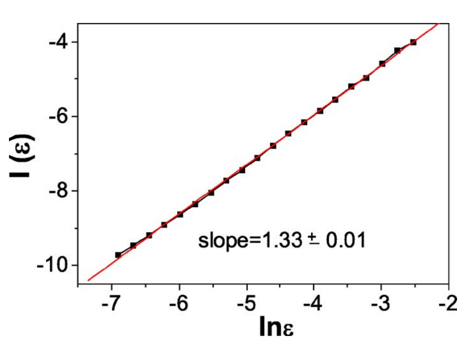

FIG. 2. For the resonantly forced Duffing oscillator Eq. (1), linear scaling between $I(\epsilon)$ and $\ln\epsilon$. The slope of the fit is approximately 1.33 ± 0.01 and the "apparent" information dimension of the attractor is 2.33 ± 0.01.

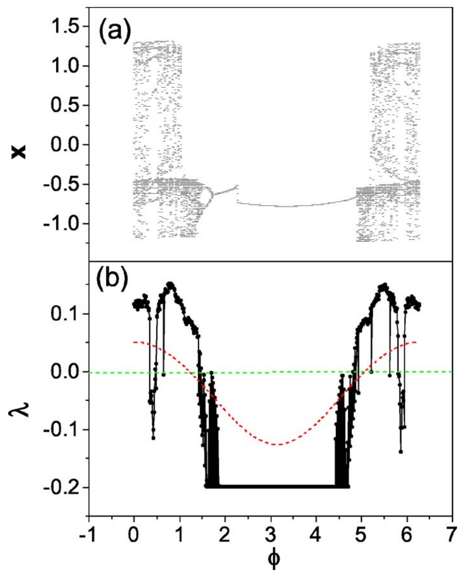

FIG. 3. For the resonantly forced Duffing oscillator Eq. (3): (a) bifurcation diagram with the phase variable ϕ as the bifurcation parameter and (b) the largest Lyapunov exponent λ vs the phase ϕ. We see that different choices of the phase can lead to completely different attractors: periodic or chaotic. The dashed line in (b) is the maximal value of the Melnikov function as a function of ϕ.

tion of the natural measure on prohibitively small scales. Thus, although the asymptotic value of the information dimension is likely to be that associated with the periodic orbit, in numerically or physically accessible scales the dimension is approximately that associated with the chaotic set, which is typically a fractional value (hence the name *pseudo-strange* attractors).

III. UNDERSTANDING BASED ON NONSTATIONARY DYNAMICAL SYSTEMS

To better understand the origin of pseudo-strange attractors, we rewrite Eq. (1) as

$$\frac{d^2x}{dt^2} = x - x^3 - \nu\frac{dx}{dt} + \rho\cos(\Omega t) + \beta\cos[\omega t + \phi(t)], \qquad (3)$$

where $\phi(t) \equiv \Delta\omega \cdot t$ is a time-dependent phase variable. Our idea is to treat $\phi(t)$ as a bifurcation *parameter* of the system and to examine the attractors of the system as the parameter varies. Because of the dependence of ϕ on time, the underlying dynamical system is effectively *nonstationary*. A typical ϕ-bifurcation diagram is shown in Fig. 3(a), where the parameter setting is the same as that for Fig. 1. The corresponding Lyapunov bifurcation diagram is shown in Fig. 3(b). We see that, when ϕ is varied in its natural range $[0, 2\pi)$, the asymptotic attractor can be either periodic or chaotic. In particular, there is an interval of ϕ values for which λ is negative, indicating that the attractor is periodic, and for the complementary ϕ interval λ is mostly positive, signifying chaotic attractors.[4] While there are periodic windows in the chaotic windows in the chaotic region, as indicated by the dips of λ to negative values, there are chaotic saddles (nonattracting chaotic invariant sets) coexisting with the periodic attractors in these windows. Because of the slow phase modulation $(\Delta\omega)t$ in time, the system "selects" periodic and chaotic behavior in different time intervals. Pseudo-strange attractors are generated if a trajectory spends relatively more time near some periodic attractors than near chaotic attractors.

We imagine that, as a system parameter (e.g., the forcing amplitude ρ) changes through a critical value, there is a transition from a periodic attractor to a pseudo-strange attractor. How does this transition occur? To gain insight, we calculate the Melnikov function,[13] a time-dependent distance function between the stable and the unstable manifold. A zero of this function indicates homoclinic tangencies and, homoclinic intersections can occur if this function is positive. Since the Smale horseshoe (chaotic) dynamics is a consequence of homoclinic intersections, chaos exists (at least locally) when such intersections occur. The Melnikov criterion is, however, necessary but not sufficient for chaos. That is, chaotic dynamics may arise when the Melnikov function is positive, although the underlying chaotic set may be globally attracting, or nonattracting such that there is only transient chaos. For Eq. (3), a standard procedure[13] yields the following Melnikov distance:

$$M(t_0) = -\frac{4\nu}{3} + A\rho\sin(\Omega t_0) + B\beta\sin(\omega t_0 + \phi), \qquad (4)$$

where

$$A = A(\Omega) = \sqrt{2}\pi\Omega\,\mathrm{sech}\left(\frac{\pi\Omega}{2}\right),$$

and

$$B = B(\omega) = \sqrt{2}\pi\omega\,\mathrm{sech}\left(\frac{\pi\omega}{2}\right).$$

If $\nu > 0$ and

$$\max_{0\le t_0\le 2\pi}[A\rho\sin(\Omega t_0) + B\beta\sin(\omega t_0 + \phi)] > \frac{4\nu}{3},$$

$M(t_0)$ has simple zeros, implying that homoclinic intersections can occur and, hence, chaos is likely. For instance, if $\Omega = \omega$, we have $A = B$ and

$$M(t_0) \le -\frac{4\nu}{3} + A\sqrt{\rho^2 + 2\rho\beta\cos(\phi) + \beta^2} \equiv M(\phi). \qquad (5)$$

In Fig. 3(b), the maximal Melnikov distance $M(\phi)$ as a function of ϕ is plotted (the dashed curve). We see that there is a qualitative agreement between the behaviors of $M(\phi)$ and of the maximum Lyapunov exponent. In particular, the regions where $M(\phi)$ is positive correspond to regions where the largest Lyapunov exponent is positive. The correspondence is only approximate because the Melnikov function takes into account only the first-order correction in the underlying perturbation treatment. Also note that, when there are periodic windows in the chaotic region, the Melnikov function is still positive and, hence, it cannot be used to distinguish periodic windows from chaotic attractors. From Eq. (5), we see that the maximal value of the Melnikov function as a function of the forcing amplitude ρ is

$$M_m(\rho) = M(\phi = 0) = -\frac{4\nu}{3} + A(\Omega)(\rho + \beta). \quad (6)$$

As ρ is increased, pseudo-strange attractors can occur when $M_m(\rho)$ becomes positive. The transition point ρ_s is thus given by

$$\rho_s = \frac{4\nu}{3A(\Omega)} - \beta. \quad (7)$$

Because of the perturbative nature of the Melnikov theory, the value of the transition point given by Eq. (7) is approximate.

As ρ is increased through ρ_s, the attractor from the driven system Eq. (1) suddenly acquires a chaotic component to become a pseudo-fractal set, but the largest nontrivial Lyapunov exponent remains negative. There is then an increase in the "apparent" fractal dimension from 1 for $\rho < \rho_s$ to a fractional value above 1 for $\rho > \rho_s$. Numerically, we obtain $\rho_s \approx 0.243$, which agrees with the prediction given by Eq. (7).

We now discuss the transition from a pseudo-strange to a chaotic attractor as the forcing amplitude ρ is increased further. The dynamical mechanism for pseudo-strange attractors implies that the transition is necessarily smooth in the sense that, as ρ changes through the transition point ρ_c, the largest nontrivial Lyapunov exponent passes through zero smoothly. For a given value of ρ, let $[t_1(\rho), t_2(\rho)]$ be the time interval for which the Melnikov function is negative, and let $[0, t_1(\rho)]$ and $[t_2(\rho), T_M]$ be the intervals for which the function is positive, where $T_M \equiv 2\pi/\Delta\omega$. Thus, for $t \in [t_1(\rho), t_2(\rho)]$, the finite-time largest nontrivial Lyapunov exponent is negative and Fig. 3(b) indicates that the variation of the negative exponent is smooth (in fact, almost constant). We write $\lambda^-(t) < 0$. For $t \in [0, t_1(\rho))$ or $t \in [t_2(\rho), T_M)$, the exponent is positive. In these two intervals, there are periodic windows but in such a case, we ignore the periodic attractor but instead focus on the coexisting chaotic saddle. In so doing, the Lyapunov exponent can also be regarded as smooth,[15] and we write $\lambda^+(t) > 0$. In principle, $\lambda^+(t)$ and $\lambda^-(t)$ also depends on the bifurcation parameter ρ, but if we focus on a small interval of ρ about the transition, the dependence is relatively weak, comparing with the dependences of $t_1(\rho)$ and $t_2(\rho)$ on ρ. The largest nontrivial Lyapunov exponent of the asymptotic attractor can then be expressed as

$$\lambda(\rho) \approx \frac{1}{T_M}\left[\int_{t_1(\rho)}^{t_2(\rho)} \lambda^-(t)dt + \int_0^{t_1(\rho)} \lambda^+(t)dt + \int_{t_2(\rho)}^{T_M} \lambda^+(t)\right]dt$$
$$= \frac{1}{T_M}\{\langle\lambda^-\rangle[t_2(\rho) - t_1(\rho)] + \langle\lambda^+\rangle[T_M - t_2(\rho) + t_1(\rho)]\}, \quad (8)$$

where $\langle\lambda^-\rangle$ and $\langle\lambda^+\rangle$ are the average values of $\lambda^-(t)$ and $\lambda^+(t)$ in their respective intervals, and the symmetry of the Lyapunov exponent with respect to $\phi = \pi$ in Fig. 3(b) has been used. Let ρ_c be the transition point for which $\lambda(\rho) = 0$. Because of the smooth dependences of $t_1(\rho)$ and $t_2(\rho)$ on ρ, we can expand these quantities near the transition point to the first order. This yields

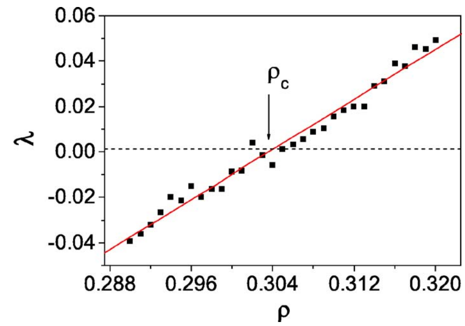

FIG. 4. For the periodically forced Duffing oscillator Eq. (1): largest Lyapunov exponent vs the bifurcation parameter ρ as it passes through ρ_c for which the exponent becomes positive. The transition from a pseudo-strange to a chaotic attractor is apparently smooth, as characterized by Eq. (9).

$$\lambda(\rho) \sim (\rho - \rho_c) \quad (9)$$

for ρ in the vicinity of the transition point ρ_c. The transition is thus smooth. Numerical evidence for (9) is shown in Fig. 4.

IV. PSEUDO-STRANGE ATTRACTORS IN A SYSTEM WITH FRACTAL BASIN BOUNDARIES

We now consider a variant of the resonantly driven Duffing oscillator where the double-well potential is replaced by a single well:[4]

$$\frac{d^2x}{dt^2} + 0.3\frac{dx}{dt} + x^3 = 10\cos(\Omega t) + 0.075\cos[(\omega + \Delta\omega)t]. \quad (10)$$

Choosing $\Omega = 1.0$ and $\omega = 3.0$, and treating $\phi(t) \equiv \Delta\omega \cdot t$ as a bifurcation parameter, we obtain the bifurcation and the Lyapunov bifurcation diagram similar to those in Fig. 3. An example of pseudo-strange attractor is shown in Fig. 5(a), where $\Delta\omega = 0.007$ and the two-dimensional stroboscopic section is defined by the driving frequency Ω. The largest nontrivial Lyapunov exponent is estimated to be $\lambda \approx -0.12$. Figure 5(b) shows the scaling of the information sum. The apparent value of the information dimension is estimated to be $D_1 \approx 2.74 \pm 0.01$. These results point to a pseudo-strange attractor.

For system equation (10), there are choices of the phase variable that lead to two coexisting, limit-cycle attractors, whose basins of attraction are separated by fractal boundaries.[16] Figure 6(a) shows, for $\phi = 5.0$, the two attractors in the two-dimensional plane $[x(t), y(t) = dx(t)/dt]$ (not on a stroboscopic section), one represented by the solid curve while another by the dashed curve. The fractal basin boundaries separating their basins are shown in Fig. 6(b) on the stroboscopic section defined by the resonant frequency Ω. We note that for other choices of values of ϕ in the 2π interval, chaotic attractors can occur. Thus intervals of ϕ for which the attractors are limit cycles can be viewed as periodic windows, where there is a chaotic saddle that gives rise to the observed fractal basin boundaries. In this sense, there is chaos, attracting or nonattracting, for almost every value of ϕ in the 2π interval. This is unlike the forced Duffing system equation (1), where there is a relatively large interval

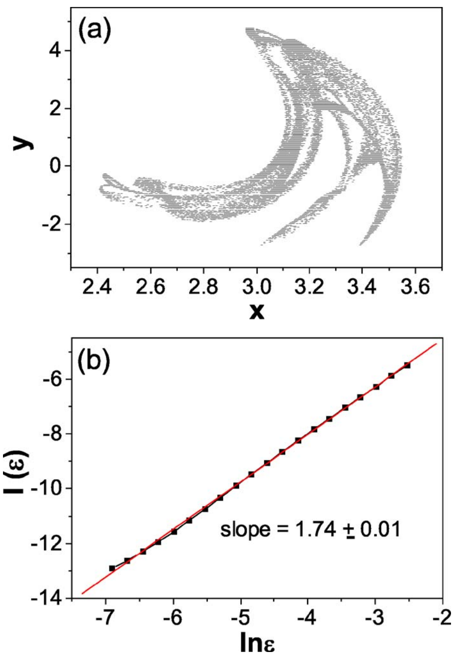

FIG. 5. For the single-well, forced Duffing oscillator equation (10) for $\Delta\omega=0.007$: (a) attractor on the two-dimensional stroboscopic section defined by the driving frequency Ω and (b) scaling of the information sum $I(\epsilon)$ with $\ln \epsilon$. Within the feasible numerical resolution, the information dimension is estimated to be $D_1 \approx 2.74 \pm 0.01$.

of ϕ for which there is no chaos (Fig. 3). Thus, our modified system equation (10) can be viewed as quite different from the forced Duffing system equation (1) in detail. Our point is, despite the difference, pseudo-strange attractors are common.

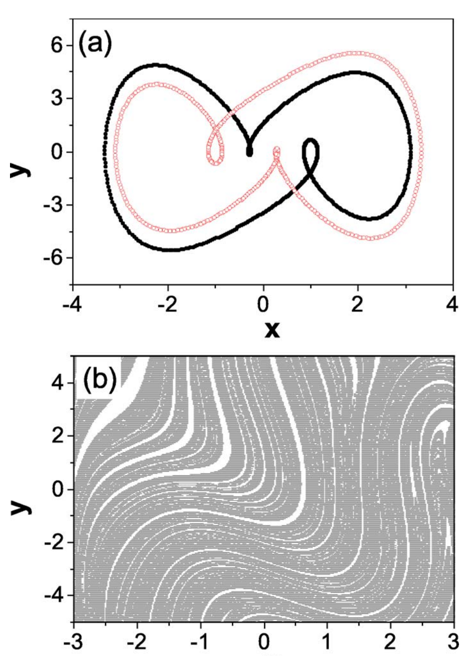

FIG. 6. For $\phi=5.0$ in the modified forced Duffing oscillator equation (10): (a) two coexisting limit-cycle attractors and (b) fractal basin boundaries between their basins of attraction.

V. CONCLUSION

In summary, we find that small frequency mismatch in a resonantly driven nonlinear oscillator can generate pseudo-strange attractors that, to our knowledge, have not been noticed previously. The attractors are nonchaotic in that they possess no positive Lyapunov exponent, but their geometries examined on finite, accessible scales are apparently fractal. Our finding may be important because, in any physical situation noise is inevitable, so that scales smaller than one determined by the noise level are inaccessible and the fractal geometry needs to be examined on finite scales. In this sense, the value of the "true" fractal dimension, mathematically defined on infinitesimal scales, is irrelevant. The mechanism for the pseudo-strange attractors can be explained in the context of nonstationary dynamical systems where a parameter sweeps adiabatically through periodic and chaotic regimes. Because of this connection, we expect these attractors to be common in nonstationary dynamical systems.

ACKNOWLEDGMENTS

Y.C.L. acknowledges the great hospitality of the National University of Singapore, where part of this work was done during a visit. He is supported by AFOSR under Grant Nos. F49620-03-1-0290 and FA9550-06-1-0024.

[1]R. Lima and M. Pettini, Phys. Rev. A **41**, 726 (1990).
[2]L. Fronzoni, M. Giocondo, and M. Pettini, Phys. Rev. A **43**, 6483 (1991).
[3]Y. Braiman and I. Goldhirsch, Phys. Rev. Lett. **66**, 2545 (1991).
[4]A figure similar to Fig. 3 has been used to demonstrate that chaos in a nonlinear oscillator can be converted to periodic motion by properly selecting the phase of a weak resonant driving force [Z. Qu, G. Hu, G. Yang, and G. Qin, Phys. Rev. Lett. **74**, 1736 (1995)].
[5]F. Cuadros and R. Chacón, Phys. Rev. E **47**, 4628 (1993); R. Chacón, Phys. Rev. E **51**, 761 (1995); Phys. Rev. Lett. **86**, 1737 (2001); Europhys. Lett. **54**, 148 (2001).
[6]S. M. Booker, P. D. Smith, P. V. Brennan, and R. J. Bullock, IEEE Trans. Circuits Syst., I: Fundam. Theory Appl. **49**, 639 (2002).
[7]Y.-C. Lai, A. Kandangath, S. Krishnamoorthy, J. A. Gaudt, and A. P. S. de Moura, Phys. Rev. Lett. **94**, 214101 (2005).
[8]In dynamical systems, there can be strange nonchaotic attractors, attractors that are geometrically strange but with nonpositive Lyapunov exponents (Ref. 17). So far such attractors have been identified in quasi-periodically forced systems (Ref. 14) and in dynamical systems under noise (Ref. 18). There has been no consensus as to whether strange nonchaotic attractor can occur in periodically forced, deterministic dynamical systems.
[9]J.-L. Chen, F.-J. Kao, and I.-M. Jiang, Phys. Lett. A **218**, 268 (1996).
[10]L. H. Juárez, H. Kantz, O. Martinez, E. Ramos, and R. Rechtman, Phys. Rev. E **70**, 056202 (2004).
[11]C. van Vreeswijk and H. Sompolinsky, Science **274**, 1724 (1996).
[12]See, for example, S. J. Schiff, Nat. Med. (N.Y.) **4**, 1117 (1998).
[13]J. Guckenheimer and P. J. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer-Verlag, Berlin, 1983).
[14]E. Ott, *Chaos in Dynamical Systems*, 2nd ed. (Cambridge University Press, Cambridge, 2002).
[15]Y.-C. Lai, U. Feudel, and C. Grebogi, Phys. Rev. E **54**, 6070 (1996).
[16]C. Grebogi, S. W. McDonald, E. Ott, and J. A. Yorke, Phys. Lett. A **99**, 415 (1983); S. W. McDonald, C. Grebogi, E. Ott, and J. A. Yorke, Physica D **17**, 125 (1985).
[17]C. Grebogi, E. Ott, S. Pelikan, and J. A. Yorke, Physica D **13**, 261 (1984).
[18]X. Wang, M. Zhan, C. H. Lai, and Y.-C. Lai, Phys. Rev. Lett. **92**, 074102 (2004).