Lecture 8 Random Walks on Networks
— Application of $\mathbb{L}$ matrix

1. Basics

Starting from an arbitrary node. At each time step, the walker travels from one node to another along a link.

$p_i(t)$ — Probability that the walk is at node i at time t

$$p_i(t) = \sum_{j=1}^{n} \frac{A_{ij}}{k_j} p_j(t-1)$$

At time $t-1$, walk must be at one of ith neighbors

Let $\underset{\sim}{p}(t) = (p_1(t), \cdots, p_n(t))^T = A \cdot D^{-1} \cdot \underset{\sim}{p}(t-1)$

$D = \begin{pmatrix} k_1 & & \\ & \ddots & \\ & & k_n \end{pmatrix}$ Let $D^{1/2} \equiv \begin{pmatrix} \sqrt{k_1} & & \\ & \ddots & \\ & & \sqrt{k_n} \end{pmatrix}$, $D^{-1/2} = \begin{pmatrix} \frac{1}{\sqrt{k_1}} & & \\ & \ddots & \\ & & \frac{1}{\sqrt{k_n}} \end{pmatrix}$

$$\Rightarrow [D^{-1/2} \underset{\sim}{p}(t)] = \underbrace{D^{-1/2} A \cdot D^{-1/2}}_{\text{reduced adjacency matrix (RAM)}} [D^{-1/2} \underset{\sim}{p}(t-1)]$$

Random walk: $\text{RAM} \cdot [D^{-1/2} \underset{\sim}{p}(t)]$ at each time step

Why $\mathbb{L}$? $t \to \infty$: $p_i(\infty) = \sum_{j=1}^{n} \frac{A_{ij}}{k_j} p_j(\infty)$

Asymptotic solution

$$\Rightarrow \quad \underset{\sim}{p} = A \cdot D^{-1} \underset{\sim}{p}$$

$$(\mathbb{I} - A \cdot D^{-1}) \underset{\sim}{p} = (D - A) \cdot D^{-1} \underset{\sim}{p}$$
$$= \mathbb{L} \cdot D^{-1} \cdot \underset{\sim}{p} = 0$$

$\Rightarrow$ $D^{-1} \cdot \underset{\sim}{p}$ — eigenvector of $\mathbb{L}$ with zero eigenvalue

$$D^{-1} \cdot \underset{\sim}{p} = a \underset{\sim}{1}$$

$$\underset{\sim}{p} = a D^{-1} \cdot \underset{\sim}{1} \quad \Rightarrow p_i = a k_i = \frac{k_i}{\Sigma k_i} = \frac{k_i}{2m}$$

Asymptotically, $p_i \sim k_i$ — agrees with intuition

2. First passage time (FPT)

τ — mean FPT from u to v

v — absorbing node

$\begin{cases} p_v(t) & \text{— Probability that FPT to } v \le t \\ p_v(t-1) & \text{— " " " " " } \le t-1 \end{cases}$

$p_v(t) - p_v(t-1)$ — Probability that FPT to v $\underline{\underline{=}}$ t

(union of two events. $(\le t-1) \cup (= t)$)

$$\Rightarrow \quad \tau = \sum_{t=0}^{\infty} t\, [p_v(t) - p_v(t-1)]$$

v — absorber $\rightarrow$ need to modify A

Demand $A_{iv} = 0$ for all i

but $A_{vi} \neq 0$ is possible for some nodes

A — now asymmetric

A' — A with vth row and column removed

D' — D " " " " " "

For any $i \neq v$, $\quad p_i(t) = \sum \frac{A_{ij}}{k_j} p_j(t-1) = \sum_{j(\neq v)} \frac{A_{ij}}{k_j} p_j(t-1)$

$\underset{\sim}{p}'$ — $\underset{\sim}{p}$ with vth element removed

$$\underset{\sim}{p}'(t) = A' \cdot (D')^{-1} \cdot \underset{\sim}{p}'(t-1)$$
$$= [A \cdot (D')^{-1}]^t \cdot \underset{\sim}{p}'(0)$$

Since $\sum_i p_i(t) = 1 \Rightarrow p_v(t) = 1 - \sum_{i \neq v} p_i(t)$
$$= 1 - \underset{\sim}{1}^T \cdot \underset{\sim}{p}'(t)$$

$$\Rightarrow \tau = \sum_{t=0}^{\infty} t \cdot \underset{\sim}{1}^T \cdot [\underset{\sim}{p}'(t-1) - \underset{\sim}{p}'(t)]$$
$$= \sum_t \underset{\sim}{1}^T \cdot t \{ [A' \cdot (D'^{-1})]^{t-1} - [A' \cdot (D'^{-1})]^t \} \cdot \underset{\sim}{p}(0)$$

Note $\sum_{t=0}^{\infty} t(M^{t-1} - M^t) = (I - M) + 2(M - M^2) + 3 \cdots$
$$= I + M + M^2 + \cdots = (I - M)^{-1}$$

$$\Rightarrow \quad \tau = \underset{\sim}{1}^T \cdot \underbrace{(I - A' \cdot D'^{-1})^{-1}}_{[(D' - A') \cdot D'^{-1}]^{-1} = D' \cdot (D' - A')^{-1} = D' \cdot L'^{-1}} \cdot \underset{\sim}{p}'(0)$$

Final expression: $\quad \tau = \underset{\sim}{1}^T \cdot D' \cdot (L')^{-1} \cdot \underset{\sim}{p}'(0)$

L' — · vth reduced Laplacian

· invertible because $\underset{\sim}{1}$ is no longer an eigen-vector

$L'_{(n-1)\times(n-1)}$

$L'^{-1}_{(n-1)\times(n-1)}$

Let $\Lambda^{(v)}$ — $(L')^{-1}$ with vth row & column reintroduced

$$(L')^{-1}_{(n-1)\times(n-1)} \Rightarrow \begin{array}{c} \\ v \\ \\ \end{array} \begin{pmatrix} & 0 & \\ & \vdots & \\ 0\ 0 \cdots & & 0 \\ & 0 & \end{pmatrix}_{n \times n} = \Lambda^{(v)}$$
$$\qquad\qquad\qquad v$$

Initial condition: Walk starting from u $\Rightarrow \begin{cases} [\underset{\sim}{p}'(0)]_i = 0, & i \neq u \\ [\underset{\sim}{p}'(0)]_u = 1 \end{cases}$

$$\Rightarrow \quad \boxed{\tau = \sum_i k_i (\Lambda^{(v)})_{iu}}$$

$$\underset{\sim}{1}^T \cdot [D' \cdot (L')^{-1}] \cdot \underset{\sim}{p}'(0) \Rightarrow (1, 1, \cdots 1) \begin{pmatrix} & x & \\ & x & \\ & x & \\ & \vdots & \\ & x & \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} - u$$
$$\qquad\qquad u$$

Scientific Innovation Processes – Random Walk on Networks

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Network Dynamics of Innovation Processes

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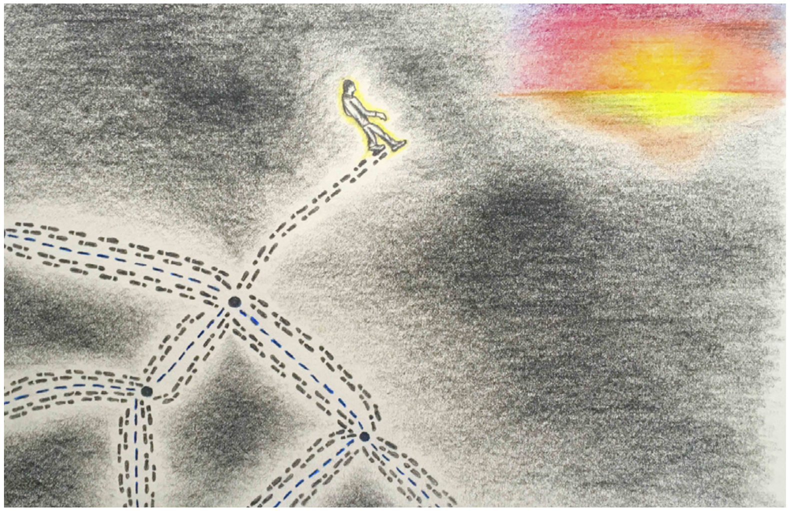
We introduce a model for the emergence of innovations, in which cognitive processes are described as random walks on the network of links among ideas or concepts, and an innovation corresponds to the first visit of a node. The transition matrix of the random walk depends on the network weights, while in turn the weight of an edge is reinforced by the passage of a walker. The presence of the network naturally accounts for the mechanism of the "adjacent possible," and the model reproduces both the rate at which novelties emerge and the correlations among them observed empirically. We show this by using synthetic networks and by studying real data sets on the growth of knowledge in different scientific disciplines. Edge-reinforced random walks on complex topologies offer a new modeling framework for the dynamics of correlated novelties and are another example of coevolution of processes and networks.



Edge Reinforced Random Walks (ERRWs)

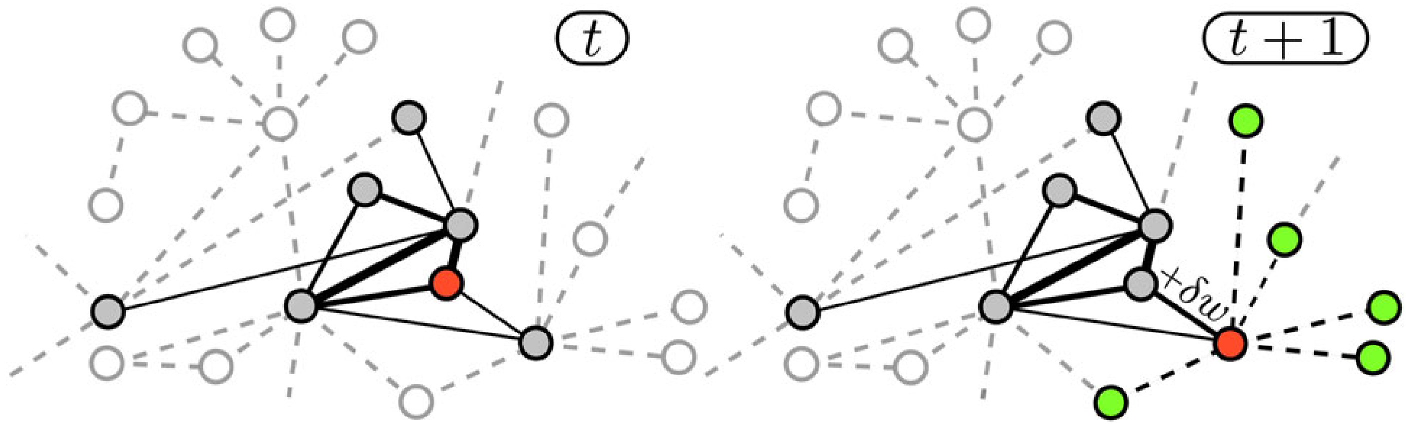


FIG. 1. Edge-reinforced random walks (ERRWs) produce a coevolution of the network with the dynamics of the walkers. At time t, the walker is on the red node and has already visited the gray nodes, while the shaded nodes are still unexplored. The widths of edges are proportional to their weights. At time $t + 1$, the walker has moved to a neighbor (red) with probability as in Eq. (1), and the weight of the used edge has been reinforced by δw. At this point, the walker will preferentially go back, although it can also access the set of "adjacent possible" (green).

Heaps' Law $S(t) \sim t^{\beta}$

- S(t) – number of scientific concepts (innovations) at time t
- Originally introduced by Heaps to describe the number of distinct words encountered at time t in a text document

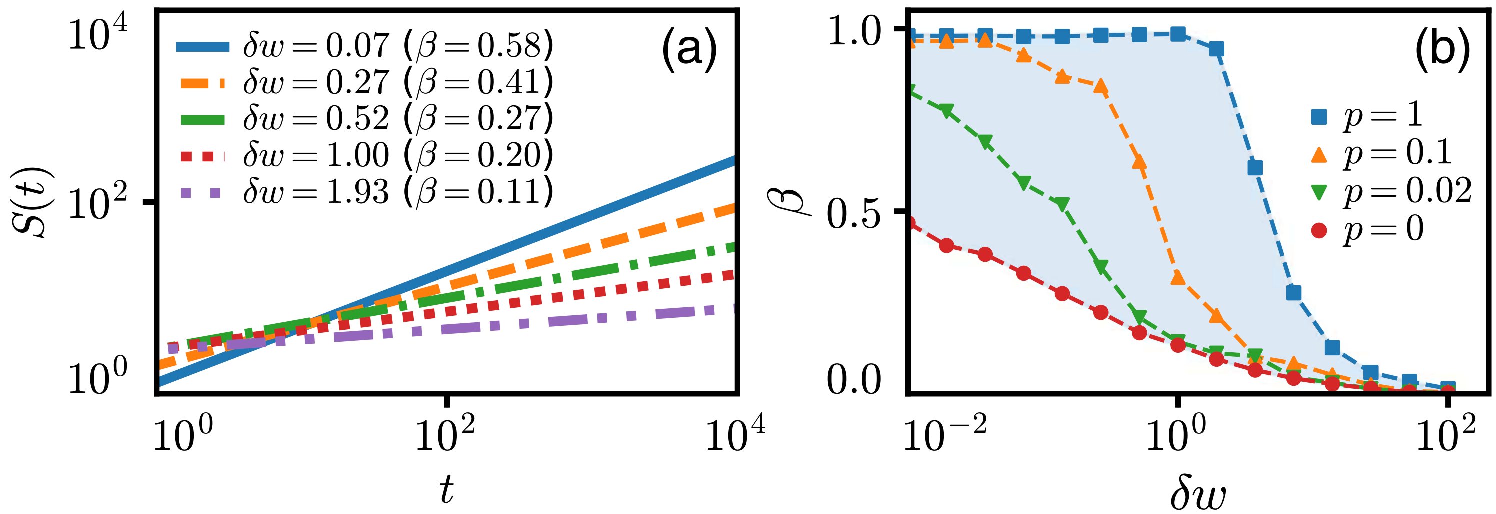


FIG. 2. ERRW on SW networks with $N = 10^5$ and $m = 1$. (a) Heaps's law and associated exponents β obtained for different values of reinforcement δw on a network with $p = 0.02$. (b) Exponent β as a function of the reinforcement δw for networks with different rewiring probabilities p.

Growth of Knowledge in Science – Empirical Data

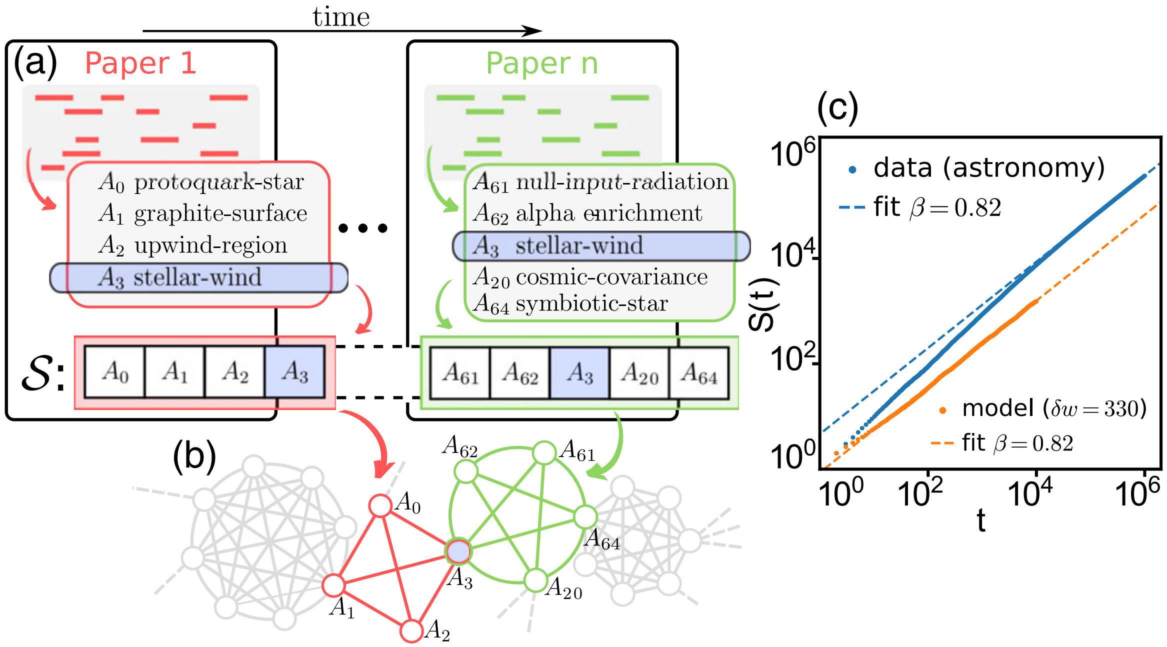


FIG. 3. Growth of knowledge in science. (a) For each scientific field, an empirical sequence of scientific concepts $\mathcal{S}$ is extracted from the abstracts of the temporally ordered sequence of papers. (b) The network of relations among concepts is constructed by linking two concepts if they appear in the same abstract. The network is then used as the underlying structure for the ERRW model. (c) The model is tuned to the empirical data by choosing the value of the reinforcement δw that reproduces the Heaps exponent β associated to $\mathcal{S}$.

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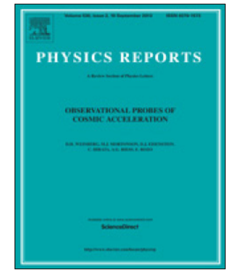

Random walks and diffusion on networks

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A B S T R A C T

Random walks are ubiquitous in the sciences, and they are interesting from both theoretical and practical perspectives. They are one of the most fundamental types of stochastic processes; can be used to model numerous phenomena, including diffusion, interactions, and opinions among humans and animals; and can be used to extract information about important entities or dense groups of entities in a network. Random walks have been studied for many decades on both regular lattices and (especially in the last couple of decades) on networks with a variety of structures. In the present article, we survey the theory and applications of random walks on networks, restricting ourselves to simple cases of single and non-adaptive random walkers. We distinguish three main types of random walks: discrete-time random walks, node-centric continuous-time random walks, and edge-centric continuous-time random walks. We first briefly survey random walks on a line, and then we consider random walks on various types of networks. We extensively discuss applications of random walks, including ranking of nodes (e.g., PageRank), community detection, respondent-driven sampling, and opinion models such as voter models.