Lecture 7 Network Laplacian

1. Diffusion

ψ_i — amount of certain physical quantity at node i

Rate of flow from j to i $= C(\psi_j - \psi_i)$

$$\frac{d\psi_i}{dt} = C\sum_{j=1}^{n} A_{ij}(\psi_j - \psi_i) \qquad C > 0 \text{ — diffusion constant}$$

$$= C\sum_j A_{ij}\psi_j - C\psi_i \overbrace{\sum_j A_{ij}}^{k_i}$$

$$\delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i\neq j \end{cases} \qquad = C\sum_j (A_{ij} - \delta_{ij}k_i)\psi_j$$

$$\psi \equiv \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix} \Rightarrow \frac{d\psi}{dt} = C(A - D)\cdot\psi \text{ — matrix form}$$

$$= -C(D-A)\cdot\psi \equiv -C L\cdot\psi$$

$$D \equiv \begin{pmatrix} k_1 & & 0 \\ & \ddots & \\ 0 & & k_n \end{pmatrix} \qquad L \equiv D - A \text{ — Laplacian matrix}$$

Diffusion equation: $\frac{d\psi}{dt} + C L\cdot\psi = 0$

Why "Diffusion"?

$$L = \begin{cases} k_i, & i=j \\ -1, & i\neq j \text{ but } A_{ij}=1 \\ 0, & \text{otherwise} \end{cases} \qquad \text{c.f. } \frac{\partial\phi}{\partial t} = D\frac{\partial^2\phi}{\partial x^2} \text{ — usual diffusion eq. in 1D}$$

space discretization

$$\phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix} \qquad \frac{\partial^2}{\partial x^2} \rightarrow \frac{1}{\delta}(\phi_{i+1} + \phi_{i-1} - 2\phi_i)$$

$$\Rightarrow \frac{\partial\phi}{\partial t} = D L \phi \quad \text{with } L \equiv \frac{1}{\delta^2}\begin{pmatrix} -2 & 1 & & 0 \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -2 \end{pmatrix}$$

2. Solution of diffusion equation

$$L\cdot v_i = \lambda_i v_i, \qquad v_j^T\cdot v_i = \delta_{ij}$$

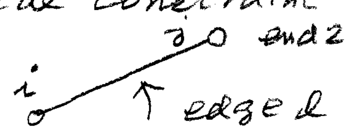
$$\psi(t) = \sum a_i(t) v_i$$

$$\Rightarrow \sum_i \left(\frac{da_i}{dt} + C\lambda_i a_i\right) v_i = 0$$

$$v_j^T\cdot \quad \Rightarrow \frac{da_i}{dt} + C\lambda_i a_i = 0 \Rightarrow \boxed{a_i(t) = a_i(0)e^{-C\lambda_i t}}$$

Physical constraint: $\lambda_i \geq 0$

How?



$$B_{lj} = \begin{cases} +1, & \text{if end 1 of edge } l \rightarrow \text{node } j \\ -1, & \text{" end 2 " "} \\ 0, & \text{otherwise} \end{cases}$$

(l: edge index, j: node index)

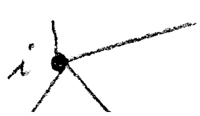
$B_{m\times n}$ — edge incidence matrix

Each row (a fixed edge): one "+1", one "−1"

1 edge between a pair of nodes

$$\sum_l B_{li}B_{lj} = -1 \quad i\neq j$$

$$\sum_l B_{li}B_{li} = \sum_l B_{li}^2 = k_i$$



$$\Rightarrow \quad L_{ij} = \sum_{e} B_{ei} B_{ej} \; ; \quad L = B^T \cdot B$$

$$\lambda_i = \underset{\sim}{v}_i^T \cdot (B^T \cdot B)\, \underset{\sim}{v}_i = (B\, \underset{\sim}{v}_i)^T (B \cdot \underset{\sim}{v}_i) \geq 0 \quad \square$$

Usually, L has at least one zero eigenvalue with $\underset{\sim}{v} = (1, 1, \cdots 1)^T$

$\lambda_2 > 0$ $\Rightarrow$ $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \Rightarrow$ L is non-invertible

$\uparrow$ algebraic connectivity — $\lambda_2 > 0$ if the network is connected

3. Network Synchronizability

$\sum_{j=1}^{n} L_{ij} = 0$

$$\frac{d\underset{\sim}{x}_i}{dt} = \underset{\sim}{F}(\underset{\sim}{x}_i) + \varepsilon \sum_{j=1}^{n} L_{ij}\, \underset{\sim}{H}(\underset{\sim}{x}_j) \quad \cdots (1)$$

Isolated oscillator: $\frac{d\underset{\sim}{x}}{dt} = \underset{\sim}{F}(\underset{\sim}{x})$ — identical oscillator

Solution $\underset{\sim}{s}(t)$: $\frac{d\underset{\sim}{s}}{dt} = \underset{\sim}{F}(\underset{\sim}{s})$

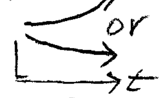
$\underbrace{\underset{\sim}{x}_i(t) = \underset{\sim}{s}(t)}_{\text{substituting this into Eq.(1)}}$ (for all i) — synchronized solution

$$\Rightarrow \quad \frac{d\underset{\sim}{s}}{dt} = \underset{\sim}{F}(\underset{\sim}{s}) + \varepsilon \underbrace{\sum_{j=1}^{n} L_{ij}}_{0}\, \underset{\sim}{H}(\underset{\sim}{s}) = \underset{\sim}{F}(\underset{\sim}{s})$$

Key issue: is $\underset{\sim}{x}_i(t) = \underset{\sim}{s}(t)$ stable?

Variational equation: say $\underset{\sim}{x}_i(t) = \underset{\sim}{s}(t) + \delta \underset{\sim}{x}_i(t)$

How does $\delta \underset{\sim}{x}_i(t)$ evolve?



$$\frac{d(\delta \underset{\sim}{x}_i(t))}{dt} = \underbrace{D\underset{\sim}{F}(\underset{\sim}{s}(t))}_{\text{matrix}} \cdot \delta \underset{\sim}{x}_i(t) + \varepsilon \sum_{j=1}^{n} L_{ij}\, \underbrace{D\underset{\sim}{H}(\underset{\sim}{s}(t))}_{\text{matrix}} \cdot \delta \underset{\sim}{x}_j(t)$$

$$\delta \underset{\sim}{X} \equiv \begin{pmatrix} \delta \underset{\sim}{x}_1 \\ \delta \underset{\sim}{x}_2 \\ \vdots \\ \delta \underset{\sim}{x}_n \end{pmatrix}$$

$$\frac{d\,\delta \underset{\sim}{X}}{dt} = \underbrace{[D\underset{\sim}{F}(\underset{\sim}{s}(t))]}_{\text{block-diagonal}} \cdot \delta \underset{\sim}{X} + \varepsilon \underbrace{[D\underset{\sim}{H}(\underset{\sim}{s}(t))]}_{\text{Block-diagonal}} \cdot L \cdot \delta \underset{\sim}{X}(t)$$

$$L \cdot \underset{\sim}{v}_i = \lambda_i \underset{\sim}{v}_i \qquad L = U^{-1} \cdot \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \cdot U$$

components decoupled — $\underset{\sim}{y}(t) \equiv U \cdot \delta \underset{\sim}{X}(t)$

$$\frac{d\underset{\sim}{y}_i}{dt} = D\underset{\sim}{F}(\underset{\sim}{s}(t)) \cdot \underset{\sim}{y}_i + \varepsilon \lambda_i D\underset{\sim}{H}(\underset{\sim}{s}(t)) \cdot \underset{\sim}{y}_i$$

$\lambda_1 = 0$: $\frac{d\underset{\sim}{y}_1}{dt} = D\underset{\sim}{F}(\underset{\sim}{s}(t)) \cdot \underset{\sim}{y}_1$ — within synchronization manifold

All others : $\frac{d\underset{\sim}{y}}{dt} = [D\underset{\sim}{F}(\underset{\sim}{s}) + \sigma D\underset{\sim}{H}(\underset{\sim}{s})] \cdot \underset{\sim}{y}$ — Linear Diff. Eq.

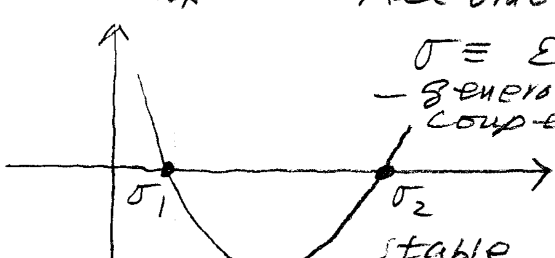
determined by nodal & coupling dyn., not by network structure $\Rightarrow$ Λmax

$\sigma \equiv \varepsilon \lambda$ — generalized coupling parameter

$\underset{\sim}{y}(t) = \underset{\sim}{y}(0)\, e^{\Lambda t}$

Λmax — Master Stability function

$\Lambda max < 0 \Rightarrow$ stable syn.



stable syn. $\Rightarrow \dfrac{\lambda_N}{\lambda_2} < \dfrac{\sigma_2}{\sigma_1}$ $\qquad \boxed{\dfrac{\lambda_N}{\lambda_2}}$ — measure of synchronizability!

Synchronization in Complex Networks

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Heterogeneity in Oscillator Networks: Are Smaller Worlds Easier to Synchronize?

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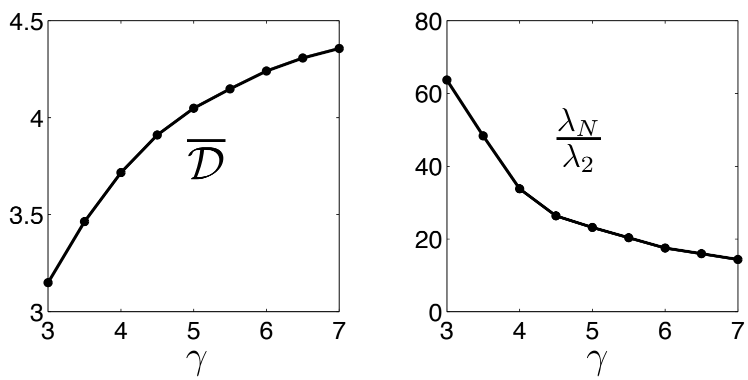
Small-world and scale-free networks are known to be more easily synchronized than regular lattices, which is usually attributed to the smaller network distance between oscillators. Surprisingly, we find that networks with a homogeneous distribution of connectivity are more synchronizable than heterogeneous ones, even though the average network distance is larger. We present numerical computations and analytical estimates on synchronizability of the network in terms of its heterogeneity parameters. Our results suggest that some degree of homogeneity is expected in naturally evolved structures, such as neural networks, where synchronizability is desirable.

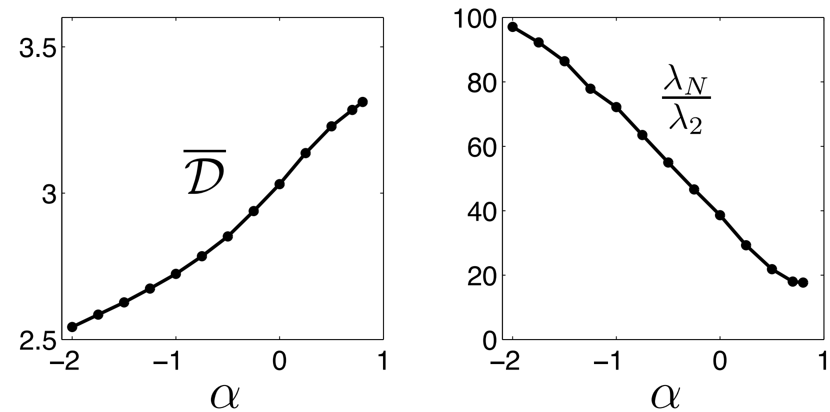
Semi-random scale-free network model

Smaller $\gamma \longleftrightarrow$ More heterogeneous degree distribution



Growing scale-free network model

Smaller $\alpha \longleftrightarrow$ More heterogeneous degree distribution



Heterogeneity $\Longrightarrow \begin{cases} \textit{Smaller } \overline{\mathcal{D}}\text{, but} \\ \textit{More difficult to synchronize} \end{cases}$

Autapses Promote Synchronization in Neuronal Networks

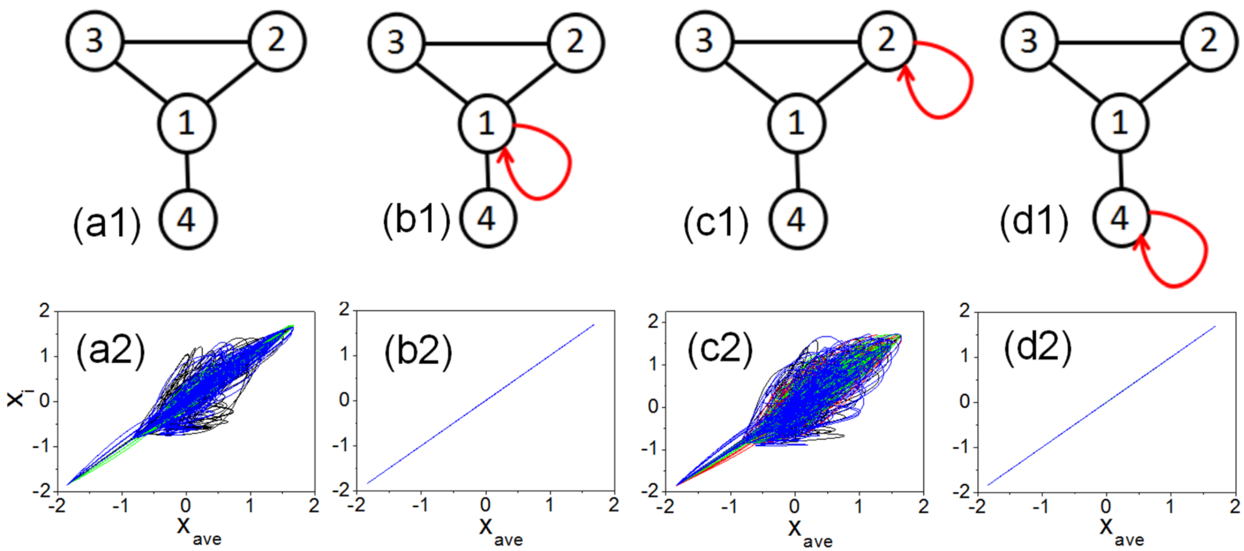


Figure 1. The impact of a single autapse on synchronization in a toy neuronal network. (**a1**) Without any autapse, the network has four nodes and four edges, where each node is a Hindmarsh-Rose neuron. (**b1**–**d1**) Network structure when a single autapse (represented by the red curve with an arrow) is present at node 1, 2, and 4, respectively. (**a2**–**d2**) For the network structures in (**a1**–**d1**), respectively, the dynamical behaviors of the network in terms of synchronization. Shown in each panel is a plot of the x variable from each node versus the averaged value of this variable over all the nodes during the time evolution. When there is global synchronization, all the variables are equal to their average value at any instant of time, tracing out a straight line segment along the diagonal. Any deviation from the diagonal signifies lack of synchronization. The uniform coupling parameter is $\varepsilon = 1$ and the time delay associated with the autapse is $\tau = 4$.

H.-W. Fan, Y.-F. Wang, H.-T. Wang, Y.-C. Lai, and X.-G. Wang, ``Autapses promote synchronization in neuronal networks,'' *Scientific Reports* **8**, article number 580 (2018).

- Autapses – first discovered in 1972
- For 25 years, autapses were thought to be "useless"
- After 1997 – biological roles of autapses gradually recognized
- Still much is to be learned