Lecture 6 Small-wored Networks

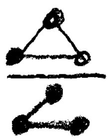
- small diameter
- High Clustering coefficient C

1. C — probability that a friend of a friend is a friend

Random networks: $C \Rightarrow 0$

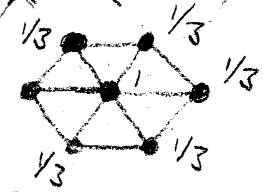
Insights from a triangular lattice

$$C = \frac{(\#\text{ of triangles}) \cdot 3}{\#\text{ of connected triples}}$$



$$= \frac{2n \cdot 3}{\binom{6}{2} n} = 0.4$$

6 — degree of each node



unit cell → has 3 nodes & 6 triangles

Each node has 2 triangles

$C = 6$

Another example: 1D ring

$c/2$ — Longest link length

\# Triangles: two steps within $c/2$ a.w. each node

$$\binom{c/2}{2} = \frac{c}{4}\left(\frac{c}{2} - 1\right)$$

\# of connected triples $\binom{c}{2} = \frac{1}{2}c(c-1)$

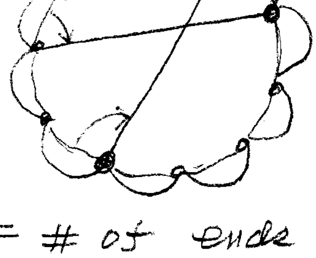
$$C = \frac{\frac{c}{4}\left(\frac{c}{2} - 1\right) \cdot 3}{\frac{1}{2}c(c-1)} = \frac{3}{4}\,\frac{c-2}{c-1} \in [0, \tfrac{3}{4}]$$

These simple, regular networks can generate large C values, but their diameters are also large $\sim n$.

Real social networks: $\begin{cases} d \sim 0 \\ C \gg 0 \end{cases}$ $(C \lesssim 1)$

2. Watts-Strogatz small-wored network model

Ring network + short cuts (small in number)



p — probability of a shortcut between two nodes (randomly chosen)

↑ rewiring probability

s — # of shortcuts attached to any one node

$$k = c + s$$

$$n\langle s \rangle = \#\text{ of ends of shortcuts} = \frac{1}{2}nc \cdot 2 \cdot p = ncp$$

($\frac{1}{2}nc$: # of edges (non-shortcut); 2: two ends on one shortcut; cp: $\langle s \rangle$)

$$p_s = \binom{n-1}{s} p^s (1-p)^{n-1-s}$$

$$\langle s \rangle = cp$$

$$n \to \infty \qquad p_s = e^{-cp} \frac{(cp)^s}{s!}$$

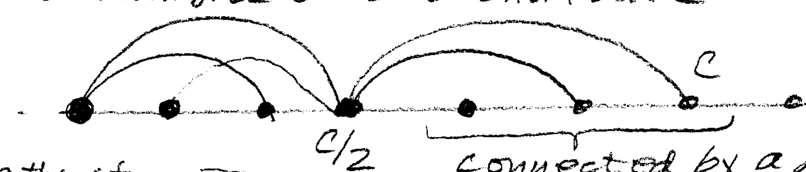
$$s = k - c \quad \Rightarrow \quad p_k = e^{-cp} \frac{(cp)^{k-c}}{(k-c)!}$$

$$(k \geq c)$$

$\mathcal{C} = ?$

- Triangles: ① Regular backbone: $\frac{1}{4} nc\left(\frac{c}{2} - 1\right)$ $n\binom{c/2}{c}$

  ② New triangles due to shortcuts   say $c = 6$



\# of paths of length two $\sim \boxed{n}$ A shortcut $\rightarrow$ triangle

Probability of a node's having such a shortcut $\sim \dfrac{(\frac{1}{2}nc)p}{\binom{n}{2}} = \dfrac{cp}{n-1} \approx \dfrac{cp}{n}$

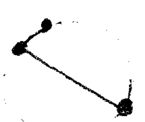
\# of triangles due to shortcuts $= n \cdot \dfrac{cp}{n} = cp$

Total # of triangles $= \frac{1}{4}nc\left(\frac{c}{2} - 1\right) + cp \approx \frac{1}{4}nc\left(\frac{c}{2} - 1\right)$

- Connected triples

  ① From regular backbone: $\frac{1}{2}c(c-1) \cdot n$ $n\binom{c}{2}$

  ②



shortcut + an edge in circle $\left(\frac{1}{2}ncp\right) \cdot c \cdot 2 = nc^2p$

(↑ two ends of shortcut (one shortcut → two triples))

③

pairs of shortcuts

Say, a node has s shortcuts

\# of shortcut triples from this node

$$= \binom{s}{2} = \frac{1}{2}s(s-1)$$

s — poisson distribution with mean cp

Expected # of such shortcuts for one node:

$$\left\langle \binom{s}{2} \right\rangle = \frac{1}{2}\left(\langle s^2 \rangle - \langle s \rangle\right) = \frac{1}{2}\left(\left(\text{variance} + \langle s \rangle^2\right) - \langle s \rangle\right)$$

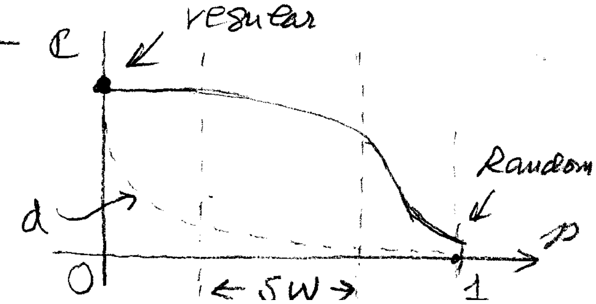
$$= \frac{1}{2}\left(cp + (cp)^2 - cp\right) = \frac{1}{2}(cp)^2$$

Total Expected # $= \frac{1}{2}nc^2p^2$

Total # of triples $= \frac{1}{2}nc(c-1) + nc^2p + \frac{1}{2}nc^2p^2$

$$\mathcal{C} = \frac{\frac{1}{4}nc\left(\frac{1}{2}c - 1\right) \cdot 3}{\frac{1}{2}nc(c-1) + nc^2p + \frac{1}{2}nc^2p^2}$$

$$= \frac{3(c-2)}{4(c-1) + 8cp + 4cp^2}$$



Smallest Small-World Networks

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Smallest small-world network

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Efficiency in passage times is an important issue in designing networks, such as transportation or computer networks. The small-world networks have structures that yield high efficiency, while keeping the network highly clustered. We show that among all networks with the small-world structure, the most efficient ones have a "single center" node, from which all shortcuts are connected to uniformly distributed nodes over the network. The networks with several centers and a connected subnetwork of shortcuts are shown to be "almost" as efficient. Genetic-algorithm simulations further support our results.

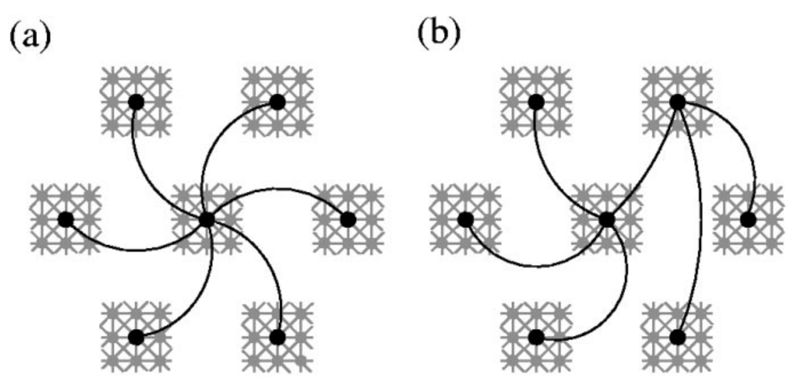

FIG. 1. Examples of shortcut configuration with (a) a single center and (b) two centers.

Network Diameter

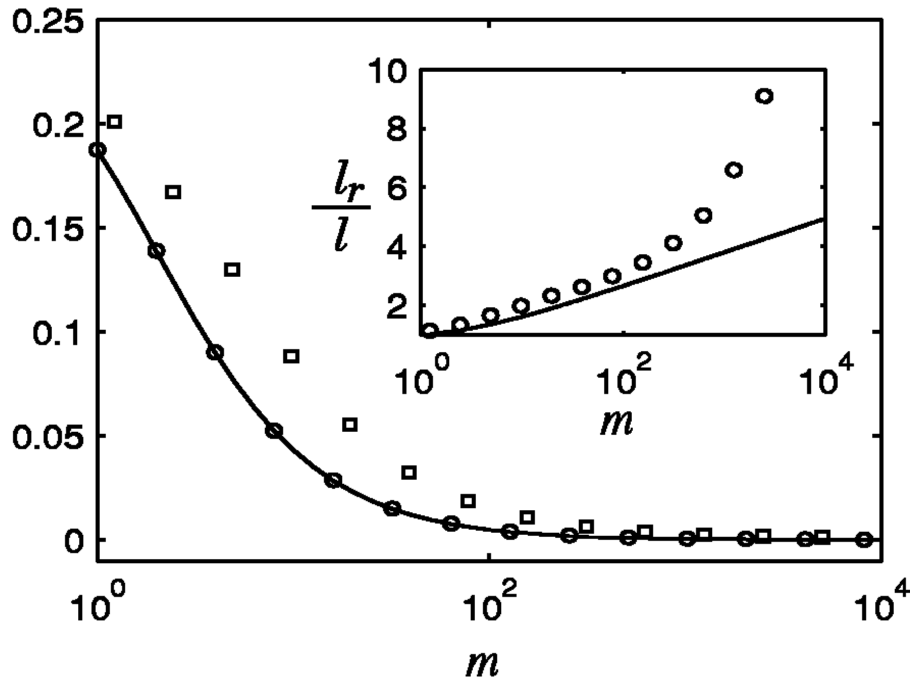

FIG. 4. Normalized path length of the network as a function of the number m of shortcuts for $k=1$. The continuous curve is Eq. (1). The circles and squares are the numerical computation of l for the configuration with a single center and of l_r over 10 random shortcut configurations, respectively. The inset shows the ratio l_r/l computed from numerical simulations (circles) and from theoretical results (1) and (2) for $N=\infty$ (continuous line). $N=10^4$ was used for numerical computations.

m shortcuts from one center:

$$l=\frac{1}{k}\overline{d(P,Q)}=\frac{2m+1}{4k(m+1)^2}$$

m randomly distributed shortcuts:

$$l_r=\frac{1}{2k\sqrt{m^2+2m}}\tanh^{-1}\left(\frac{m}{\sqrt{m^2+2m}}\right)$$

Simulation Results from Genetic Algorithm

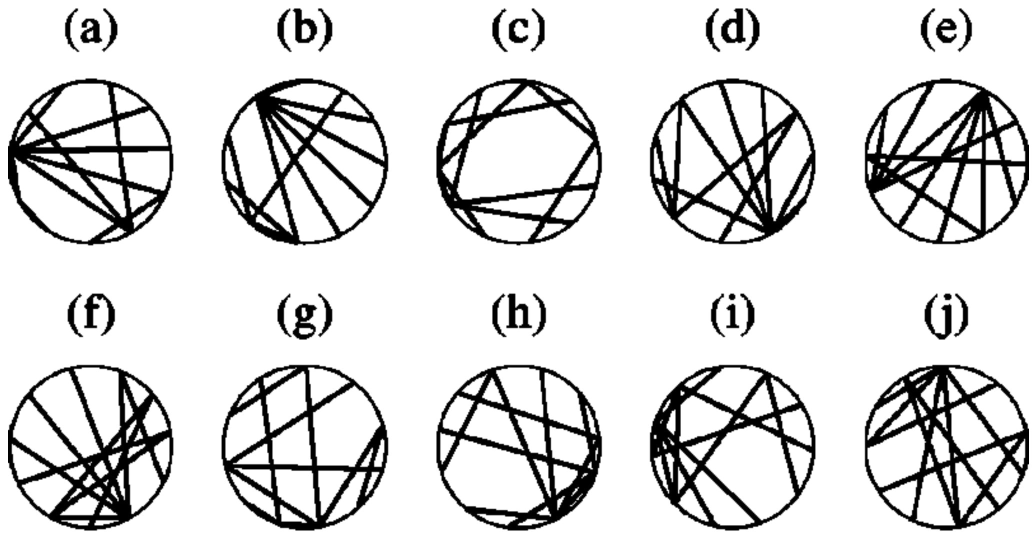

FIG. 6. Ten best solutions from 81 independent runs of GA simulation with the population size of 30, $N=1000$, $m=10$, and $k=2$. The corresponding average path lengths are (a) $L=24.309$, (b) $L=24.379$, (c) $L=24.622$, (d) $L=24.627$, (e) $L=24.640$, (f) $L=24.650$, (g) $L=24.653$, (h) $L=24.660$, (i) $L=24.779$, (j) $L=24.798$. The average path length is 23.795 for the single-center configuration, while it is approximately 43 for random shortcuts.

- Initial population: m pairs of integers
- Fitness: 1/L
- What does GA do? - Maximize fitness for different choices of the m pairs of integers