Lecture 3 Network Degree (1/16/2018)

1. Average degree $\langle k \rangle$

(1) Undirected networks

k_i — # of edges connected to node i

$$k_i = \sum_{j=1}^{n} A_{ij}$$

Total number of edges $m = \frac{1}{2} \sum_{i=1}^{n} k_i$

$$\langle k \rangle = \frac{1}{n} \sum_{i=1}^{n} k_i = \frac{2m}{n}$$

Given n nodes, what is the maximum possible # of edges?

$$\binom{n}{2} = \frac{1}{2} n(n-1)$$

Link density (connectance) ρ

$$\rho = \frac{m}{\frac{1}{2}n(n-1)} = \frac{\langle k \rangle}{n-1}, \quad 0 \le \rho \le 1$$

$n \to \infty$: Two situations

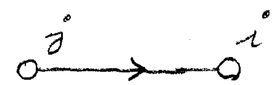
① $\rho \to$ constant $\Rightarrow \langle k \rangle \to \infty$ } dense networks. Example: Food webs

② $\rho \to 0$, $\langle k \rangle \to$ finite } sparse network

Examples: Internet, WWW, Friendship

(2) Directed networks

A — asymmetric



$A_{ij} = 1, \; A_{ji} = 0$

Out- / In-degree : # of outgoing / ingoing edges

$$k_i^{in} = \sum_{j=1}^{n} A_{ij}, \quad k_j^{out} = \sum_{i=1}^{n} A_{ij}$$

total # of edges $m = \sum_{i=1}^{n} k_i^{in} = \sum_{j=1}^{n} k_j^{out} = \sum_{i,j} A_{ij}$

$$\langle k^{in} \rangle = \frac{1}{n} \sum_{i=1}^{n} k_i^{in} = \frac{1}{n} \sum_{j=1}^{n} k_j^{out} = \langle k^{out} \rangle$$

$$= \langle k \rangle = m/n$$

2. Degree distributions

(1) Random networks

p — probability a given node is connected to any of the $(n-1)$ other nodes

$$\binom{n-1}{k} p^k (1-p)^{(n-1)-k} \equiv p_k$$

\# of ways of selecting k nodes among $n-1$ other nodes — probability for the given node to connect to a particular set of k nodes, $1 \le k \le n-1$

p_k — probability of connecting to exactly k other nodes — degree distribution
↑ binomial

Mean degree : $\langle k \rangle = p(n-1) \equiv c$

$n \to \infty$ $\quad p_k \to e^{-c} \frac{c^k}{k!}$ Poisson distribution

(2) Scale-free networks

$$p_k = C_0 k^{-\gamma}$$ γ — power-law exponent

for $k > k_{min}$

$$\sum_{k=k_{min}}^{\infty} p_k = 1 \Rightarrow C_0 = \left(\sum_{k=k_{min}}^{\infty} k^{-\gamma}\right)^{-1}$$

$$\zeta(\gamma, k_{min}) \equiv \sum_{k=k_{min}}^{\infty} k^{-\gamma}$$

— generalized or incomplete Riemann zeta function

Complete : $k_{min} = 1$

$$\sum_{k=k_{min}}^{\infty} k^{-\gamma} \approx \int_{k_{min}}^{\infty} k^{-\gamma} dk = \frac{1}{\gamma-1} k_{min}^{\gamma-1} \equiv C_0^{-1}$$

$$\Rightarrow \quad p_k \cong \frac{\alpha-1}{k_{min}} \left(\frac{k}{k_{min}}\right)^{-\gamma}$$

Statistical moments

$$\langle k^l \rangle = \sum_{k=0}^{\infty} k^l p_k = \underbrace{\sum_{k=0}^{k_{min}-1} k^l p_k}_{\text{finite}} + \underbrace{C_0 \sum_{k=k_{min}}^{\infty} k^{l-\gamma}}$$

$$\cong (\text{a finite number}) + \underbrace{C_0 \int_{k_{min}}^{\infty} k^{l-\gamma} dk}_{\frac{C_0}{l-\gamma+1} \left[k^{l-\gamma+1}\right]_{k_{min}}^{\infty}}$$

ppt
- Robust yet fragile
- Cascading failures

$l-\gamma+1 < 0 \quad \to \quad \langle k^l \rangle$ exists

$l-\gamma+1 \ge 0 \quad \to \quad \langle k^l \rangle \to \infty$

Given γ, all moments diverge for $l \ge \gamma - 1$

$1 < \gamma < 2 \quad \Rightarrow \quad l \ge 1 \quad \to$ all moments ↑ not even the ave.

$2 < \gamma < 3 \quad \Rightarrow \quad \langle k \rangle$ finite but $\langle k^2 \rangle \to \infty$

Scale-Free Networks: Robust yet Fragile

Robust: against random nodal failures

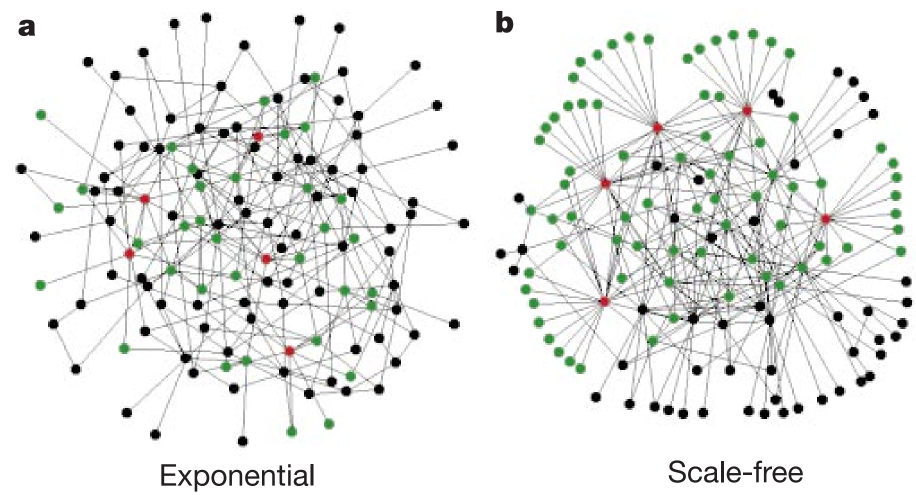
Fragile: when being attacked

Error and attack tolerance of complex networks

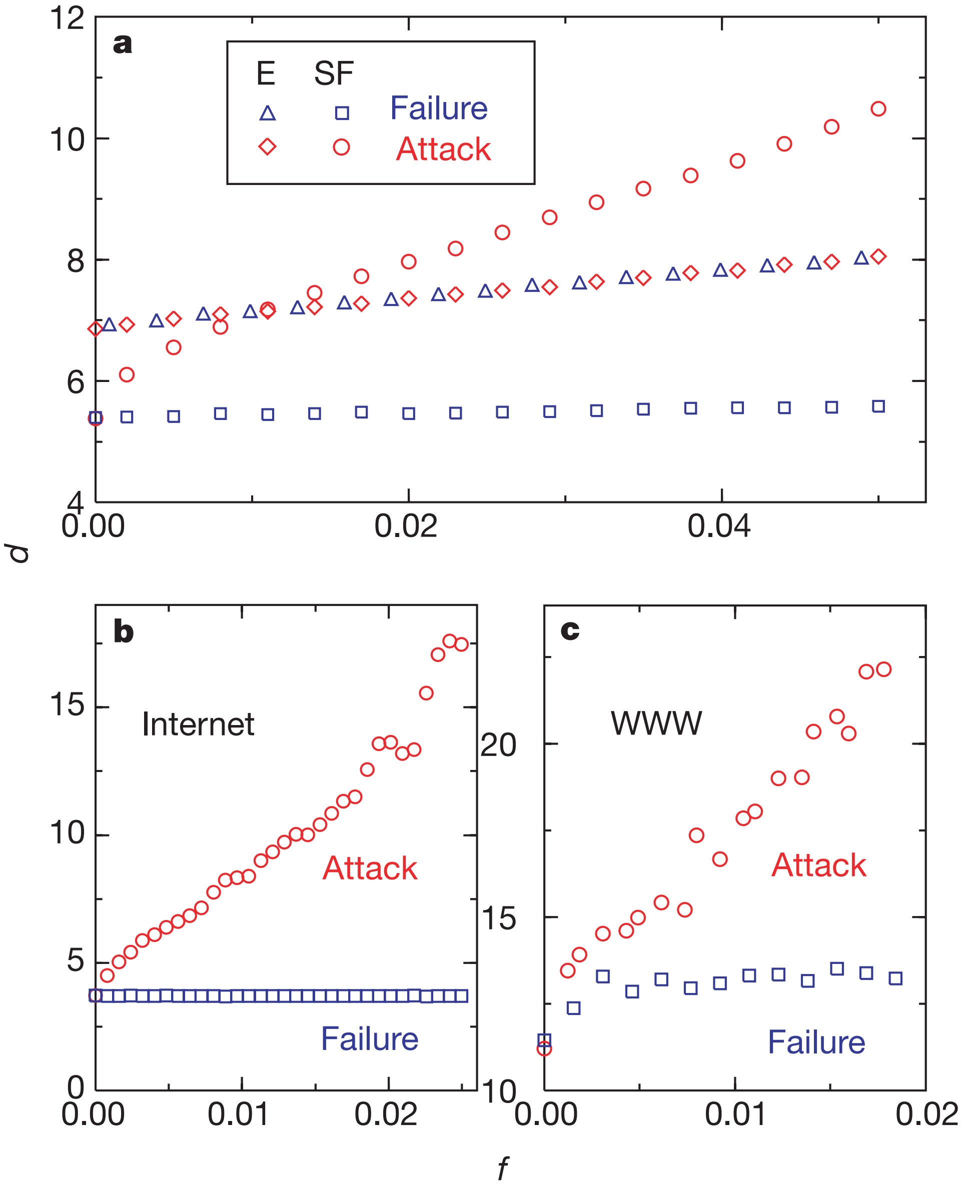
Réka Albert, Hawoong Jeong & Albert-László Barabási

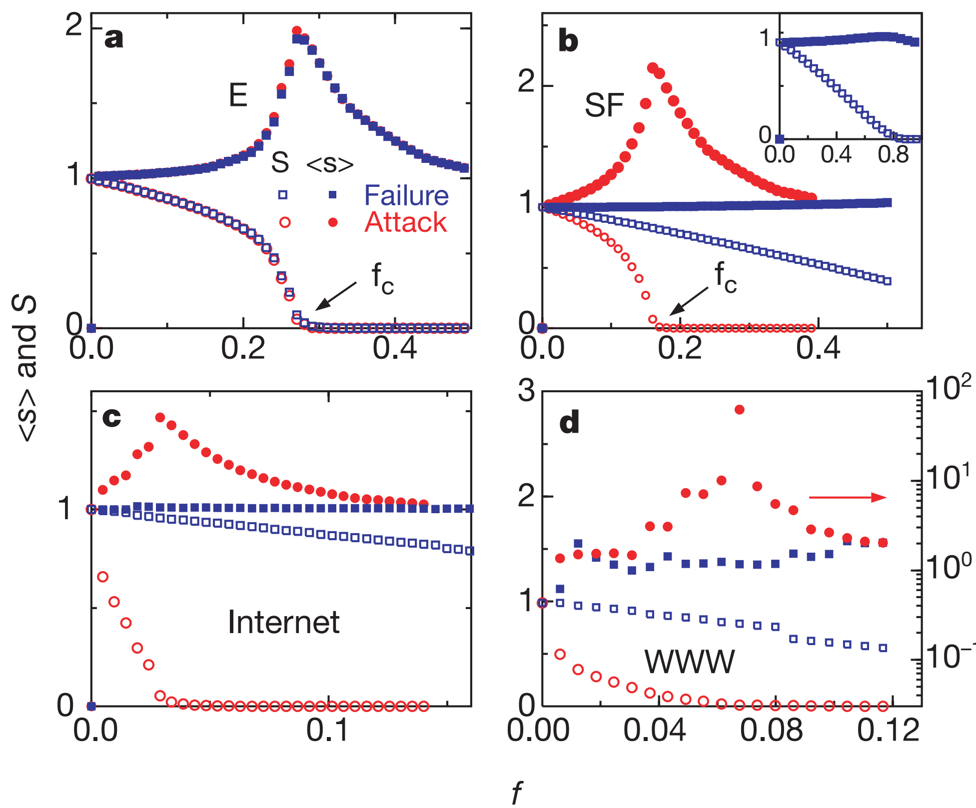
Department of Physics, 225 Nieuwland Science Hall, University of Notre Dame, Notre Dame, Indiana 46556, USA

Many complex systems display a surprising degree of tolerance against errors. For example, relatively simple organisms grow, persist and reproduce despite drastic pharmaceutical or environmental interventions, an error tolerance attributed to the robustness of the underlying metabolic network[1]. Complex communication networks[2] display a surprising degree of robustness: although key components regularly malfunction, local failures rarely lead to the loss of the global information-carrying ability of the network. The stability of these and other complex systems is often attributed to the redundant wiring of the functional web defined by the systems' components. Here we demonstrate that error tolerance is not shared by all redundant systems: it is displayed only by a class of inhomogeneously wired networks, called scale-free networks, which include the World-Wide Web[3–5], the Internet[6], social networks[7] and cells[8]. We find that such networks display an unexpected degree of robustness, the ability of their nodes to communicate being unaffected even by unrealistically high failure rates. However, error tolerance comes at a high price in that these networks are extremely vulnerable to attacks (that is, to the selection and removal of a few nodes that play a vital role in maintaining the network's connectivity). Such error tolerance and attack vulnerability are generic properties of communication networks.



Network Diameter and Fragmentation





f – fraction of removed nodes

S – Relative size of the largest cluster

<s> - average size of the isolated clusters

Summary- Response of Random and Scale-Free Networks to Failures and Attacks