1. Basic network characterization

<table>
<thead>
<tr>
<th>Directed?</th>
<th>N</th>
<th>Y</th>
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<tbody>
<tr>
<td></td>
<td>$A_{ij} = A_{ji} = 1$</td>
<td>$W_{ij} = W_{ji} \neq 0$</td>
</tr>
<tr>
<td>i \rightarrow j</td>
<td>i \rightarrow j</td>
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</tbody>
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\[ A_{ii} = 0, A_{ij} = 0 \Rightarrow \text{no link between } i \text{ & } j \]

$A_{nn \times n}$ — adjacency matrix for NIN case

Examples:

1. Co-Citation networks

\[ C_{ij} = \sum_{k=1}^{n} A_{ik} A_{jk} = \sum_{k=1}^{n} A_{ik} A_{jk}^T \]

\[ C = A \cdot A^T \]

- Cocitation matrix (network)
- Symmetric

\[ C_{ii} = \sum_{k=1}^{n} A_{ik}^2 = \sum_{k=1}^{n} A_{ik} \]

- Weighted / undirected
- Total # of edges pointing to i

2. Acyclic directed networks

- Cyclic

\[ \text{arbitrary path from any node must end at a node w/o outgoing link} \]
Reordering 
\[ A_{ij} = 0 \quad \text{for} \quad j < i \]

A — upper triangular
- All eigenvalues = 0

SO, acyclic network \( \rightarrow \) eigenvalues of \( A = 0 \)

Opposite? \( \leftrightarrow \) nilpotent \( A \)

For a cyclic network, must have at least one non-zero eigenvalue - true or false?

A result from Sec. 6.10
\[ \lambda_r = \sum_{i=1}^{n} \lambda_i \quad \text{of} \quad A \]
\[ \uparrow \text{total # of cycles of length } r \]
Say the network is cyclic \( \Rightarrow \) \( \lambda_r > 0 \)
\( \Rightarrow \) at least one term \( \lambda_i \) \( \lambda_r = 0 \)
\( \Rightarrow \) some \( \lambda_i \neq 0 \) \( \rightarrow \) TRUE

3) Euler's graph
- Königsberg bridge problem (seven-bridge problem)

River

Is there a route that crosses all seven bridges exactly once?

- Eulerian path
- Must both enter and leave a node except the start & end ones
- At most two nodes with odd degree
- All four nodes have odd degree \( \Rightarrow \) no Eulerian path exists!
Real Bipartite Networks: Pollinator-Plant
Networks of Mutualistic Interactions


Network B: Data from Hestehaven, Denmark – 42 pollinators, 8 plants, and 79 mutualistic connections [A. C. Montero, “The ecology of three pollinator network,” Master thesis, Aarhus University, Denmark (2005)]

Data from 59 such networks are currently available: [http://www.web-of-life.es](http://www.web-of-life.es)
Nonlinear Dynamical Network

\[
\frac{dP_i}{dt} = P_i \left( \alpha_i^{(P)} - \sum_{j=1}^{S_p} \beta_{ij}^{(P)} P_j + \frac{\sum_{j=1}^{S_p} \gamma_{ij}^{(P)} A_j}{1 + h \sum_{j=1}^{S_A} \gamma_{ij}^{(A)} A_j} \right) + \mu_P,
\]

Holling type-II dynamics

\[
\frac{dA_i}{dt} = A_i \left( \alpha_i^{(A)} - \kappa_i - \sum_{j=1}^{S_A} \beta_{ij}^{(A)} A_j + \frac{\sum_{j=1}^{S_p} \gamma_{ij}^{(A)} P_j}{1 + h \sum_{j=1}^{S_p} \gamma_{ij}^{(A)} P_j} \right) + \mu_A,
\]

\[\gamma_{ij} = \frac{\varepsilon_{ij} \gamma_0}{(k_i)}, \quad 0 \leq t \leq 1 \quad (t = 0: \text{structure has no effect}; \quad t = 1: \text{structure is important})\]

\[\varepsilon_{ij} = 1 \text{ if plant/pollinator } i \text{ and pollinator/plant } j \text{ are connected}; \quad 0 \text{ otherwise};\]

\[P_i, A_i - \text{Abundance of } i\text{th plant and } i\text{th pollinator};\]

\[S_p, S_A - \text{numbers of plants and pollinators};\]

\[\alpha_i^{(P)}, \alpha_i^{(A)} - \text{intrinsic growth rates of } i\text{th plant and } i\text{th pollinator};\]

\[\beta_{ii}, \beta_{ij} - \text{intraspecific and interspecific competition strength } (\beta_{ii} \gg \beta_{ij});\]

\[\mu_P, \mu_A - \text{immigration of plants and pollinators};\]

\[\gamma_0 - \text{strength of mutualistic interaction};\]

\[\kappa_i - \text{pollinator decay rate - bifurcation parameter}\]


Forcing due to Human Activities

Cause of perturbation: global warming caused climate change, excessive use of pesticides leading to death of pollinators, loss of habitats due to pollution, etc.

Bipartite mutualistic network