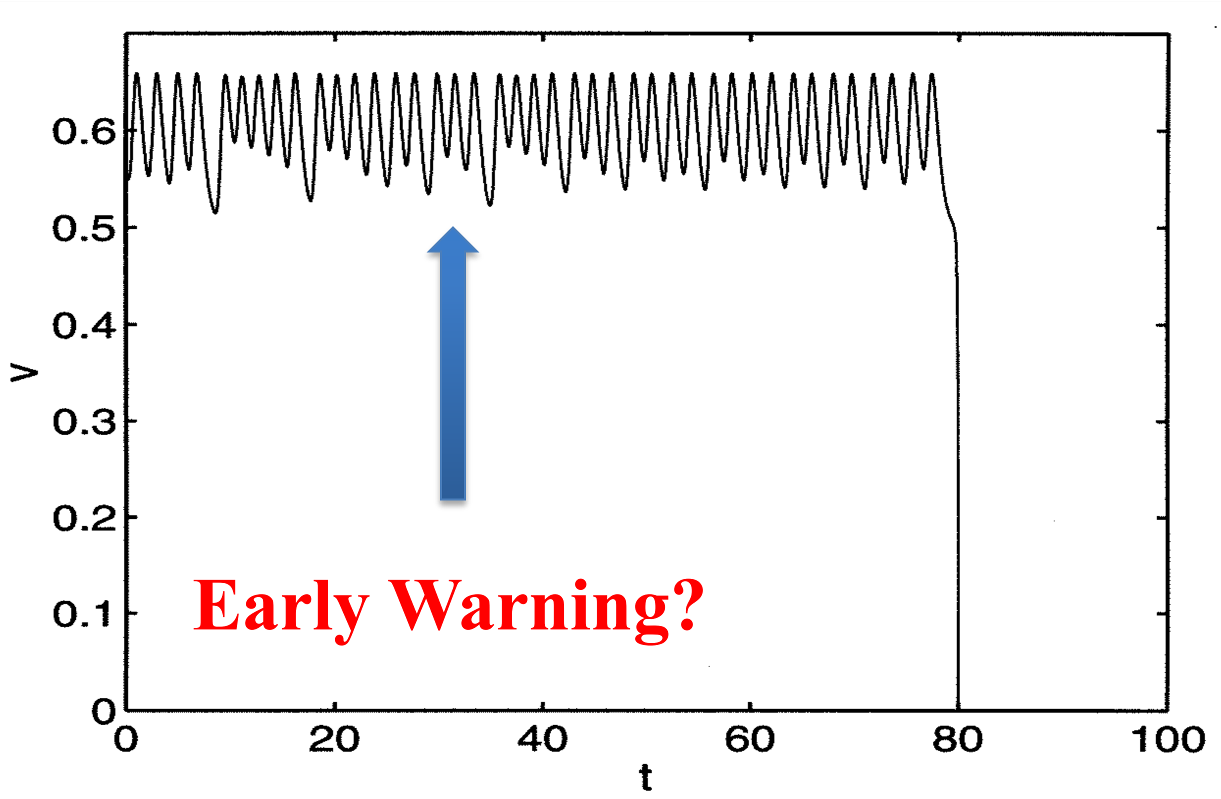
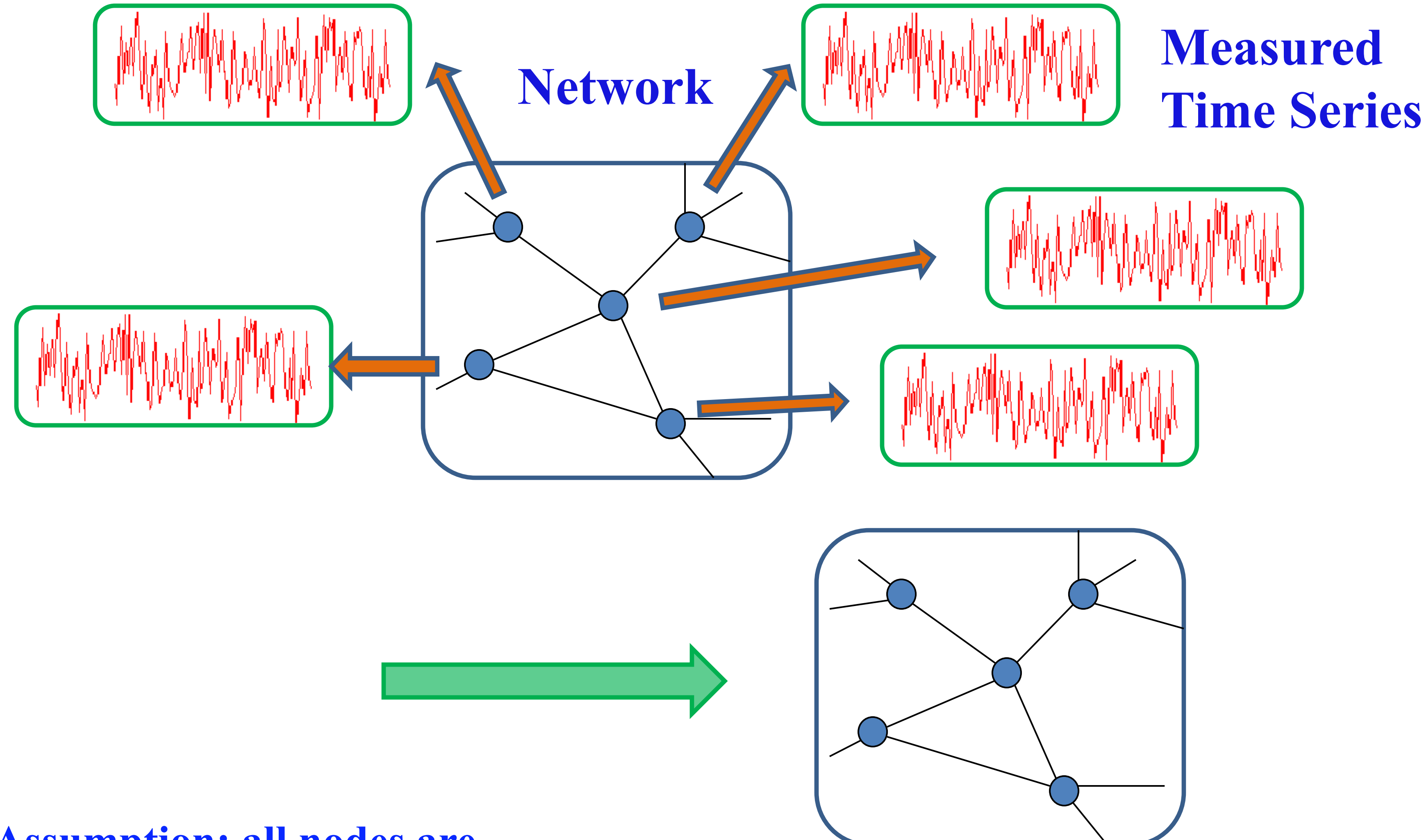
Problem 1: Can Catastrophic Events in Dynamical Systems be Predicted in Advance?



A related problem: Can future behaviors of time-varying dynamical systems be forecasted?

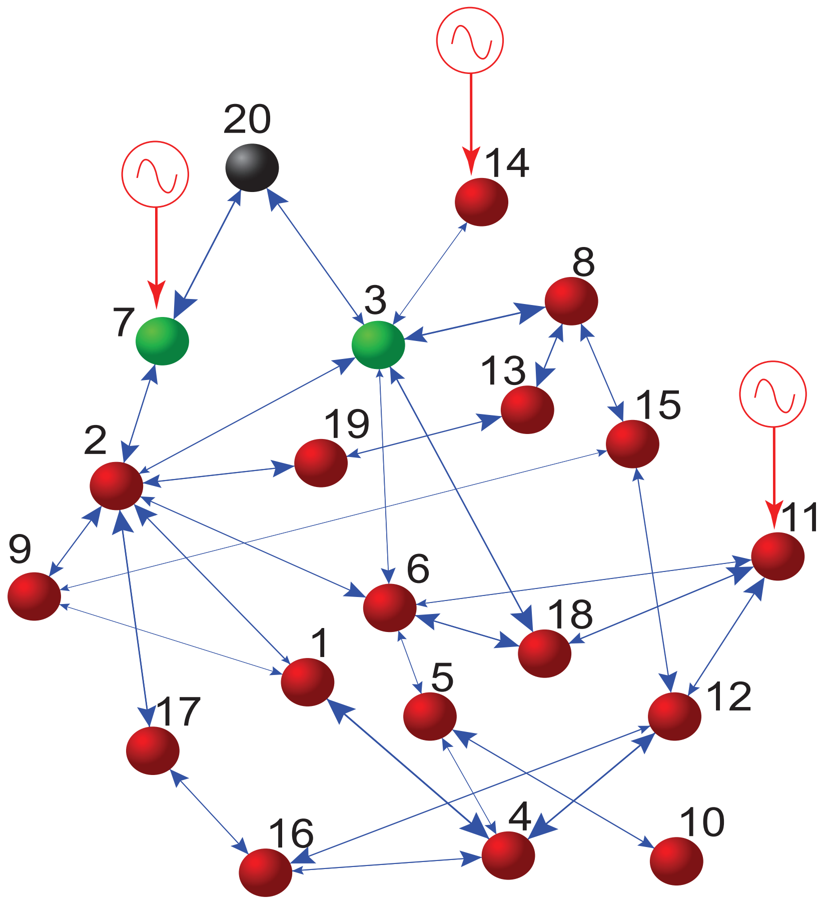
Problem 2: Reverse Engineering of Complex Networks



Assumption: all nodes are externally accessible

Full network topology?

Problem 3: Detecting Hidden Nodes



No information is available from the black node. How can we ascertain its existence and its location in the network? How can we distinguish hidden node from local noise sources?

Basic Idea (1)

Dynamical system: $d\mathbf{x}/dt = \mathbf{F}(\mathbf{x})$, $\mathbf{x} \in R^m$

Goal: to determine $\mathbf{F}(\mathbf{x})$ from measured time series $\mathbf{x}(t)$!

Power-series expansion of jth component of vector field $\mathbf{F}(\mathbf{x})$

$$[\mathbf{F}(\mathbf{x})]_j = \sum_{l_1=0}^{n} \sum_{l_2=0}^{n} \cdots \sum_{l_m=0}^{n} (a_j)_{l_1 l_2 \ldots l_m} x_1^{l_1} x_2^{l_2} \ldots . x_m^{l_m}$$

x_k – kth component of $\mathbf{x}$; Highest-order power: n

$(a_j)_{l_1 l_2 \ldots l_m}$ - coefficients to be estimated from time series

- $(1+n)^m$ coefficients altogether

If $\mathbf{F}(\mathbf{x})$ contains only a few power-series terms, most of the coefficients will be zero.

Basic Idea (2)

Concrete example: m = 3 (phase-space dimension): (x,y,z)

n = 3 (highest order in power-series expansion)

total $(1 + n)^m = (1+3)^3 = 64$ unknown coefficients

$$[\mathbf{F}(\mathbf{x})]_1 = (a_1)_{0,0,0}x^0y^0z^0 + (a_1)_{1,0,0}x^1y^0z^0 + \ldots + (a_1)_{3,3,3}x^3y^3z^3$$

Coefficient vector $\mathbf{a}_1 = \begin{pmatrix} (a_1)_{0,0,0} \\ (a_1)_{1,0,0} \\ \ldots \\ (a_1)_{3,3,3} \end{pmatrix}$ - 64×1

Measurement vector $\mathbf{g}(t) = [x(t)^0y(t)^0z(t)^0, x(t)^1y(t)^0z(t)^0, \ldots, x(t)^3y(t)^3z(t)^3]$

1×64

So $[\mathbf{F}(\mathbf{x}(t))]_1 = \mathbf{g}(t) \bullet \mathbf{a}_1$

Basic Idea (3)

Suppose $\mathbf{x}(t)$ is available at times $t_{0,} t_1, t_2, \ldots, t_{10}$ (11 vector data points)

$$\frac{dx}{dt}(t_1) = [\mathbf{F}(\mathbf{x}(t_1))]_1 = \mathbf{g}(t_1) \bullet \mathbf{a}_1$$

$$\frac{dx}{dt}(t_2) = [\mathbf{F}(\mathbf{x}(t_2))]_1 = \mathbf{g}(t_2) \bullet \mathbf{a}_1$$

...

$$\frac{dx}{dt}(t_{10}) = [\mathbf{F}(\mathbf{x}(t_{10}))]_1 = \mathbf{g}(t_{10}) \bullet \mathbf{a}_1$$

$$\text{Derivative vector } d\mathbf{X} = \begin{pmatrix} (dx/dt)(t_1) \\ (dx/dt)(t_2) \\ \ldots \\ (dx/dt)(t_{10}) \end{pmatrix}_{10\times 1} \text{; Measurement matrix } \mathbf{G} = \begin{pmatrix} \mathbf{g}(t_1) \\ \mathbf{g}(t_2) \\ \vdots \\ \mathbf{g}(t_{10}) \end{pmatrix}_{10\times 64}$$

We finally have $d\mathbf{X} = \mathbf{G} \bullet \mathbf{a}_1$ or $d\mathbf{X}_{10\times 1} = \mathbf{G}_{10\times 64} \bullet (\mathbf{a}_1)_{64\times 1}$

Basic Idea (4)

$$d\mathbf{X} = \mathbf{G} \bullet \mathbf{a}_1 \qquad \text{or} \qquad d\mathbf{X}_{10\times 1} = \mathbf{G}_{10\times 64} \bullet (\mathbf{a}_1)_{64\times 1}$$

Reminder: $\mathbf{a}_1$ is the coefficient vector for the first dynamical variable x.

To obtain $[\mathbf{F}(\mathbf{x})]_2$, we expand

$$[\mathbf{F}(\mathbf{x})]_2 = (a_2)_{0,0,0} x^0 y^0 z^0 + (a_2)_{1,0,0} x^1 y^0 z^0 + \ldots + (a_2)_{3,3,3} x^3 y^3 z^3$$

with $\mathbf{a}_2$, the coefficient vector for the second dynamical variable y. We have

$$d\mathbf{Y} = \mathbf{G} \bullet \mathbf{a}_2 \qquad \text{or} \qquad d\mathbf{Y}_{10\times 1} = \mathbf{G}_{10\times 64} \bullet (\mathbf{a}_2)_{64\times 1}$$

where

$$d\mathbf{Y} = \begin{pmatrix} (dy/dt)(t_1) \\ (dy/dt)(t_2) \\ \ldots \\ (dy/dt)(t_{10}) \end{pmatrix}_{10\times 1} .$$

Note: measurement matrix **G** is the same.

Similar expressions can be obtained for all components of the velocity field.

Compressive Sensing (1)

Look at

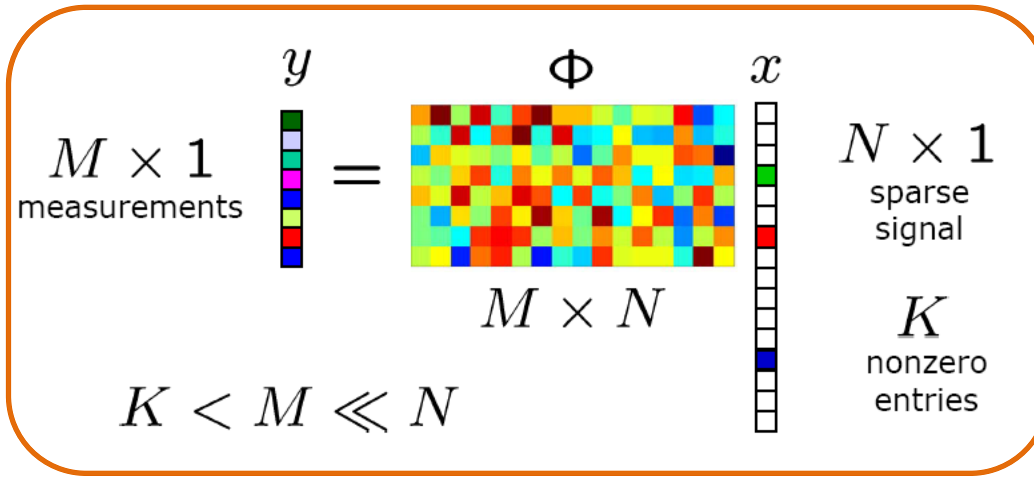
$\mathrm{d}\mathbf{X} = \mathbf{G} \bullet \mathbf{a}_1$ or $\mathrm{d}\mathbf{X}_{10\times 1} = \mathbf{G}_{10\times 64} \bullet (\mathbf{a}_1)_{64\times 1}$

Note that $\mathbf{a}_1$ is sparse - Compressive sensing!

Data/Image compression:

Φ: Random projection (not full rank)

x - sparse vector to be recovered

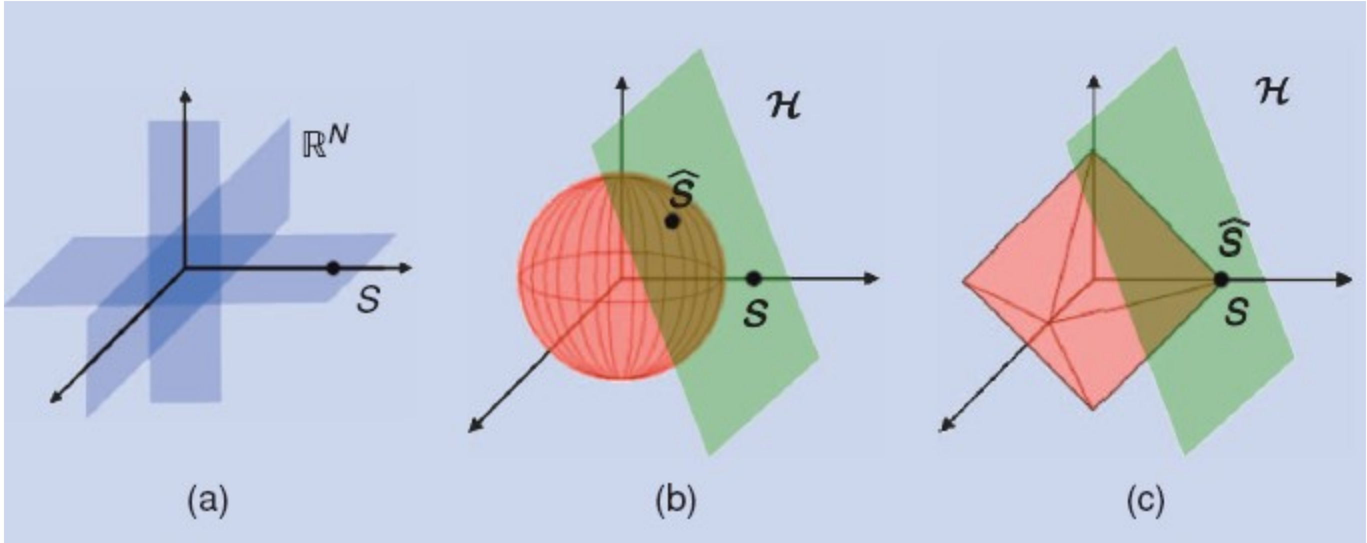


Goal of compressive sensing: Find a vector x with minimum number of entries subject to the constraint $y = \Phi \bullet x$

Compressive Sensing (2)

Find a vector x with minimum number of entries subject to the constraint $y = \Phi \bullet x$: $l_1 - \text{norm}$

Why $l_1 - \text{norm}$? - Simple example in three dimensions



E. Candes, J. Romberg, and T. Tao, *IEEE Trans. Information Theory* **52**, 489 (2006), *Comm. Pure. Appl. Math*. **59**, 1207 (2006);
D. Donoho, *IEEE Trans. Information Theory* **52**, 1289 (2006));
Special review: *IEEE Signal Process. Mag*. **24**, 2008

Predicting Catastrophe (1)

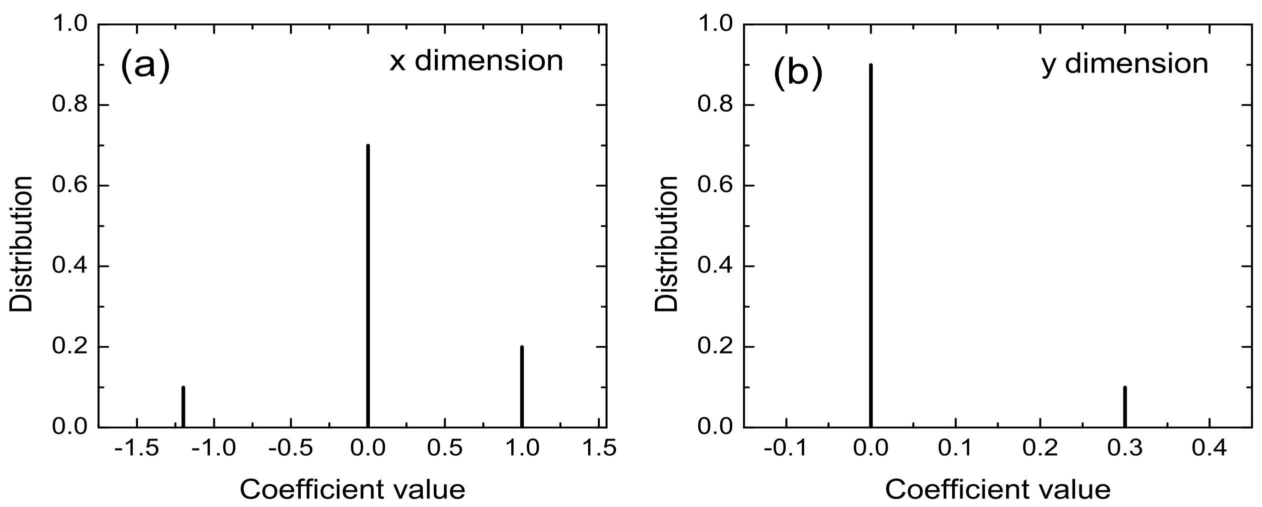
Henon map: $(x_{n+1}, y_{n+1}) = (1 - ax_n^2 + y_n, bx_n)$

Say the system operates at parameter values: a = 1.2 and b = 0.3.

There is a chaotic attractor.

Can we assess if a catastrophic bifurcation (e.g., crisis) is imminent based on a limited set of measurements?

Step 1: Predicting system equations



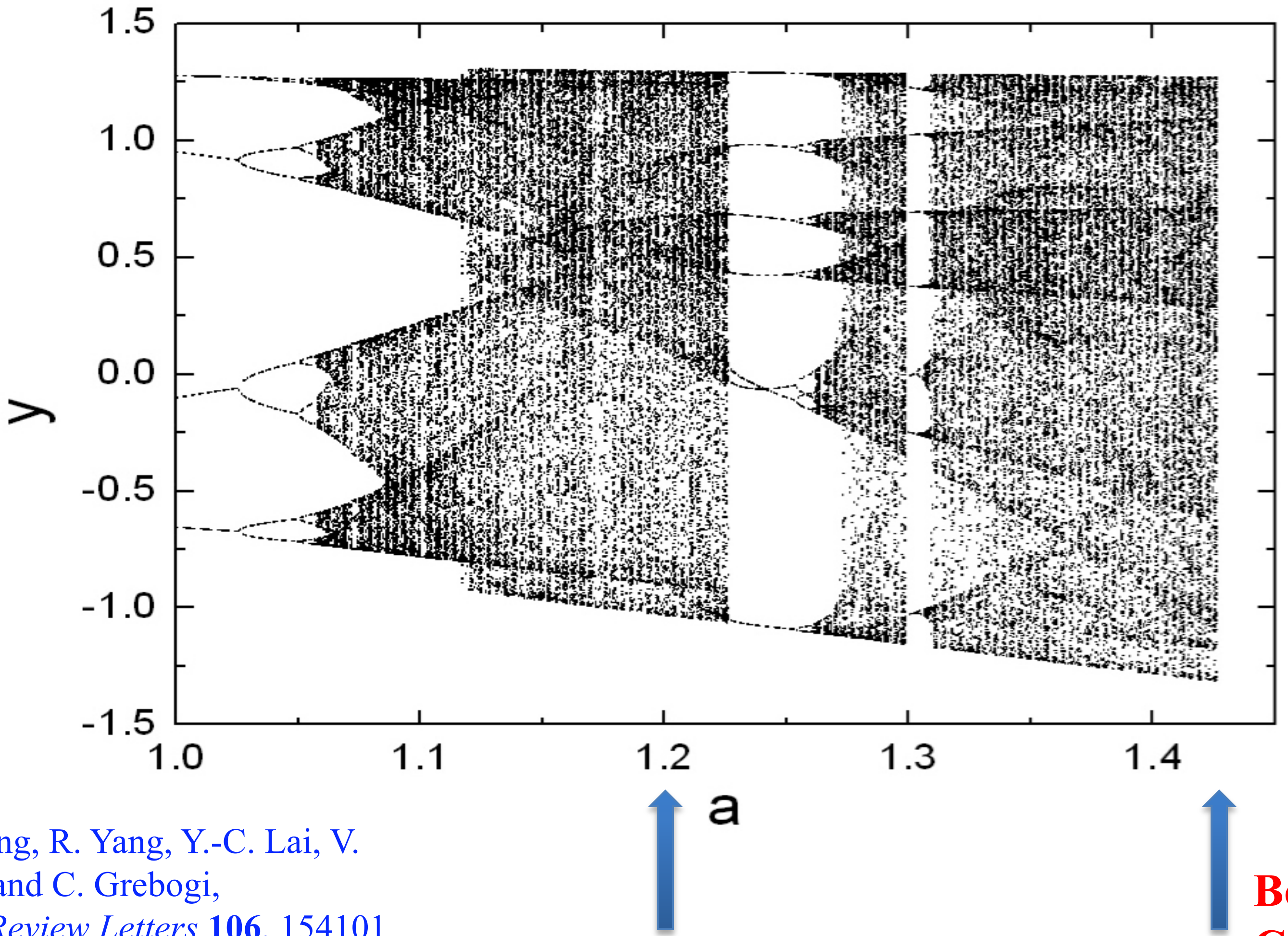
Distribution of predicted values of ten power-series coefficients:

constant, y, y^2, y^3, x, xy, xy^2, x^2, x^2y, x^3

# of data points used: 8

Predicting Catastrophe (2)

Step 2: Performing numerical bifurcation analysis



Current operation point

Boundary Crisis

W.-X. Wang, R. Yang, Y.-C. Lai, V. Kovanis, and C. Grebogi, *Physical Review Letters* **106**, 154101 (2011).

Reconstructing Full Topology of Oscillator Networks (1)

A class of commonly studied oscillator -network models:

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{F}_i(\mathbf{x}_i) + \sum_{j=1, j\neq i}^{N} \mathbf{C}_{ij} \bullet (\mathbf{x}_j - \mathbf{x}_i) \qquad (i = 1, \ldots, N)$$

- dynamical equation of node i

N - size of network, $\mathbf{x}_i \in R^m$, $\mathbf{C}_{ij}$ is the *local* coupling matrix

$$\mathbf{C}_{ij} = \begin{pmatrix} C_{ij}^{1,1} & C_{ij}^{1,2} & \cdots & C_{ij}^{1,m} \\ C_{ij}^{2,1} & C_{ij}^{2,2} & \cdots & C_{ij}^{2,m} \\ \vdots & \vdots & \vdots & \vdots \\ C_{ij}^{m,1} & C_{ij}^{m,2} & \cdots & C_{ij}^{m,m} \end{pmatrix}$$

- determines full topology

If there is at least one nonzero element in $\mathbf{C}_{ij}$, nodes i and j are coupled.

Goal: to determine all $\mathbf{F}_i(\mathbf{x}_i)$ and $\mathbf{C}_{ij}$ from time series.

Reconstructing Full Topology of Oscillator Networks (2)

$$\mathbf{X} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \cdots \\ \mathbf{x}_N \end{pmatrix}_{Nm\times 1}$$

– Network equation is $\frac{d\mathbf{X}}{dt} = \mathbf{G}(\mathbf{X})$, where

$$[\mathbf{G}(\mathbf{X})]_i = \mathbf{F}_i(\mathbf{x}_i) + \sum_{j=1, j\neq i}^{N} \mathbf{C}_{ij} \bullet (\mathbf{x}_j - \mathbf{x}_i)$$

- A very high-dimensional (*Nm*-dimensional) dynamical system;
- For complex networks (e.g, random, small-world, scale-free), node-to-node connections are typically sparse;
- In power-series expansion of $[\mathbf{G}(\mathbf{X})]_i$, most coefficients will be zero - guaranteeing sparsity condition for compressive sensing.

W.-X. Wang, R. Yang, Y.-C. Lai, V. Kovanis, M. A. F. Harrison, "Time-series based prediction of complex oscillator networks via compressive sensing", *Europhysics Letters* **94**, 48006 (2011).

Evolutionary-Game Dynamics

Prisoner's dilemma game

	Cooperate	Defect
Cooperate	win-win	lose much-win much
Defect	win much-lose much	lose-lose

Strategies: cooperation $\mathbf{S}(C) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; defection $\mathbf{S}(D) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Payoff matrix: $\mathbf{P}(PD) = \begin{pmatrix} 1 & 0 \\ b & 0 \end{pmatrix}$ b - parameter

Payoff of agent x from playing PDG with agent y: $M_{x \leftarrow y} = \mathbf{S}_x^T \mathbf{P} \mathbf{S}_y$

For example, $M_{C \leftarrow C} = 1$

$M_{D \leftarrow D} = 0$

$M_{C \leftarrow D} = 0$

$M_{D \leftarrow C} = b$

Evolutionary Game on Network
(Social and Economical Systems)

A network of agents playing games with one another:

$$\text{Adjacency matrix} = \begin{pmatrix} \cdots & \cdots & \cdots \\ \cdots & a_{xy} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}: \begin{cases} a_{xy} = 1 & \text{if } x \text{ connects with } y \\ a_{xy} = 0 & \text{if no connection} \end{cases}$$

Payoff of agent x from agent y: $M_{x \leftarrow y} = a_{xy} \mathbf{S}_x^T \mathbf{P} \mathbf{S}_y$

Time series of agents (Detectable)
- (1) payoffs
- (2) strategies

Compressive sensing →

Full social network structure

Prediction as a CS Problem

Payoff of x at time t: $M_x(t) = a_{x1}\mathbf{S}_x^T(t)\mathbf{P}\mathbf{S}_1(t) + a_{x2}\mathbf{S}_x^T(t)\mathbf{P}\mathbf{S}_2(t) + \cdots + a_{xN}\mathbf{S}_x^T(t)\mathbf{P}\mathbf{S}_N(t)$

$$\mathbf{Y} = \begin{pmatrix} M_x(t_1) \\ M_x(t_2) \\ \vdots \\ M_x(t_m) \end{pmatrix} \qquad \mathbf{X} = \begin{pmatrix} a_{x1} \\ a_{x2} \\ \vdots \\ a_{xN} \end{pmatrix}$$

$\mathbf{X}$: connection vector of agent x (to be predicted)

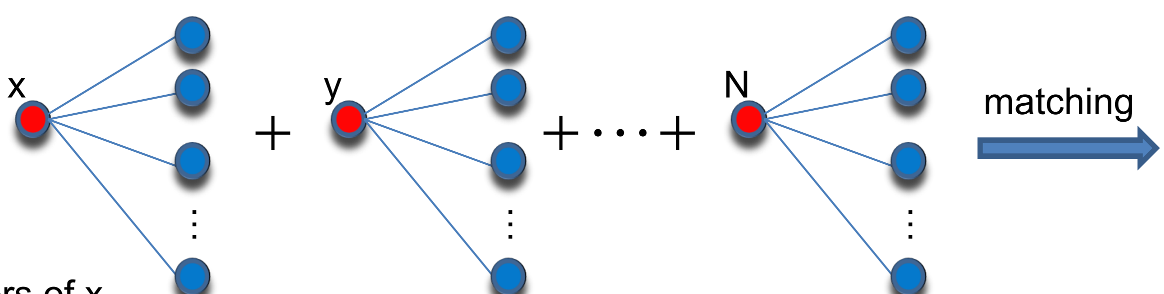
$$\Phi = \begin{pmatrix} \mathbf{S}_x^T(t_1)\mathbf{P}\mathbf{S}_1(t_1) & \mathbf{S}_x^T(t_1)\mathbf{P}\mathbf{S}_2(t_1) & \cdots & \mathbf{S}_x^T(t_1)\mathbf{P}\mathbf{S}_N(t_1) \\ \mathbf{S}_x^T(t_2)\mathbf{P}\mathbf{S}_1(t_2) & \mathbf{S}_x^T(t_2)\mathbf{P}\mathbf{S}_2(t_2) & \cdots & \mathbf{S}_x^T(t_2)\mathbf{P}\mathbf{S}_N(t_2) \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{S}_x^T(t_m)\mathbf{P}\mathbf{S}_1(t_m) & \mathbf{S}_x^T(t_m)\mathbf{P}\mathbf{S}_2(t_m) & \cdots & \mathbf{S}_x^T(t_m)\mathbf{P}\mathbf{S}_N(t_m) \end{pmatrix}$$

- W.-X. Wang, Y.-C. Lai, C. Grebogi, and J.-P. Ye, "Network reconstruction based on evolutionary game data," *Physical Review X* **1**, 021021, 1-7 (2011).

$\mathbf{Y} = \Phi \bullet \mathbf{X}$ $\quad \mathbf{Y}, \Phi$: from time series

Compressive sensing

$$\mathbf{X} = \begin{pmatrix} a_{x1} \\ a_{x2} \\ \vdots \\ a_{xN} \end{pmatrix} \Leftrightarrow$$



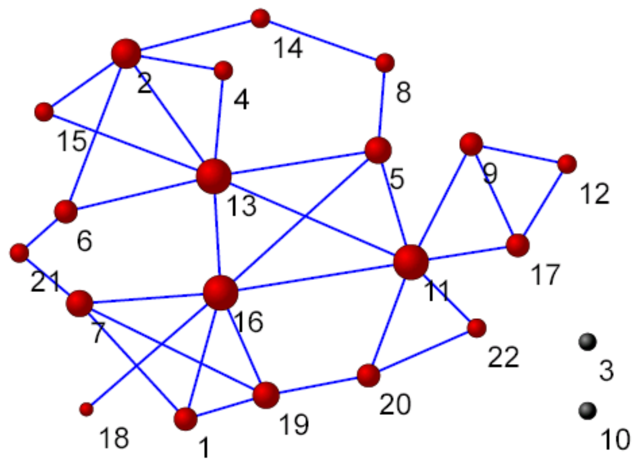
Full network structure

Neighbors of x

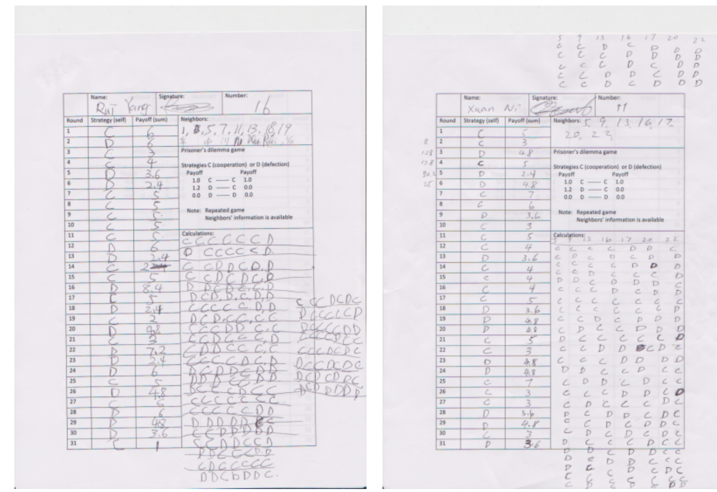
Reverse Engineering of a Real Social Network

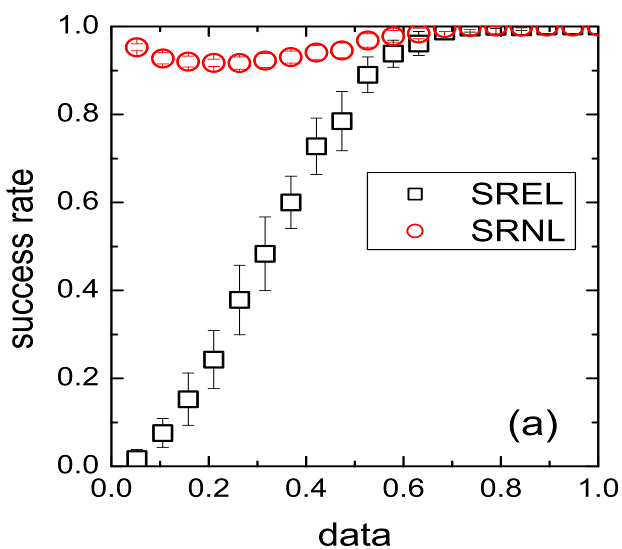
22 students play PDG together and
write down their payoffs and strategies

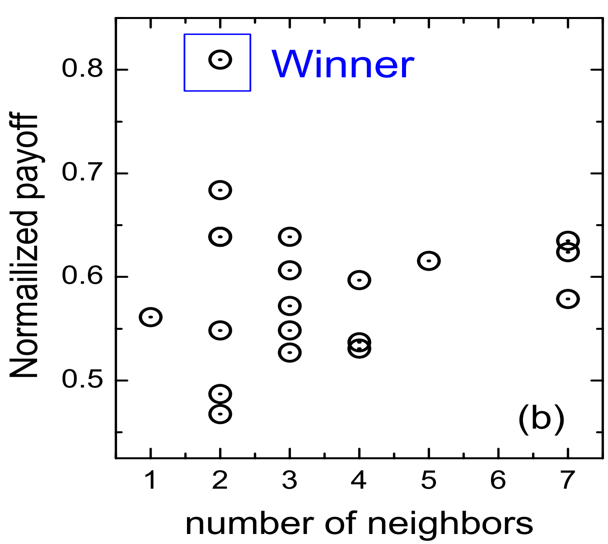
Friendship network



Experimental record of two players



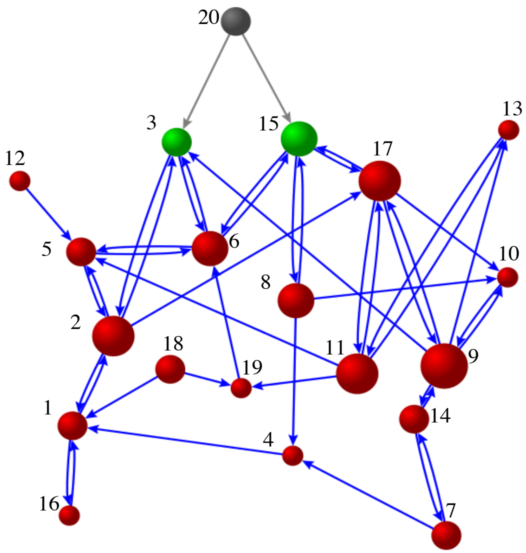


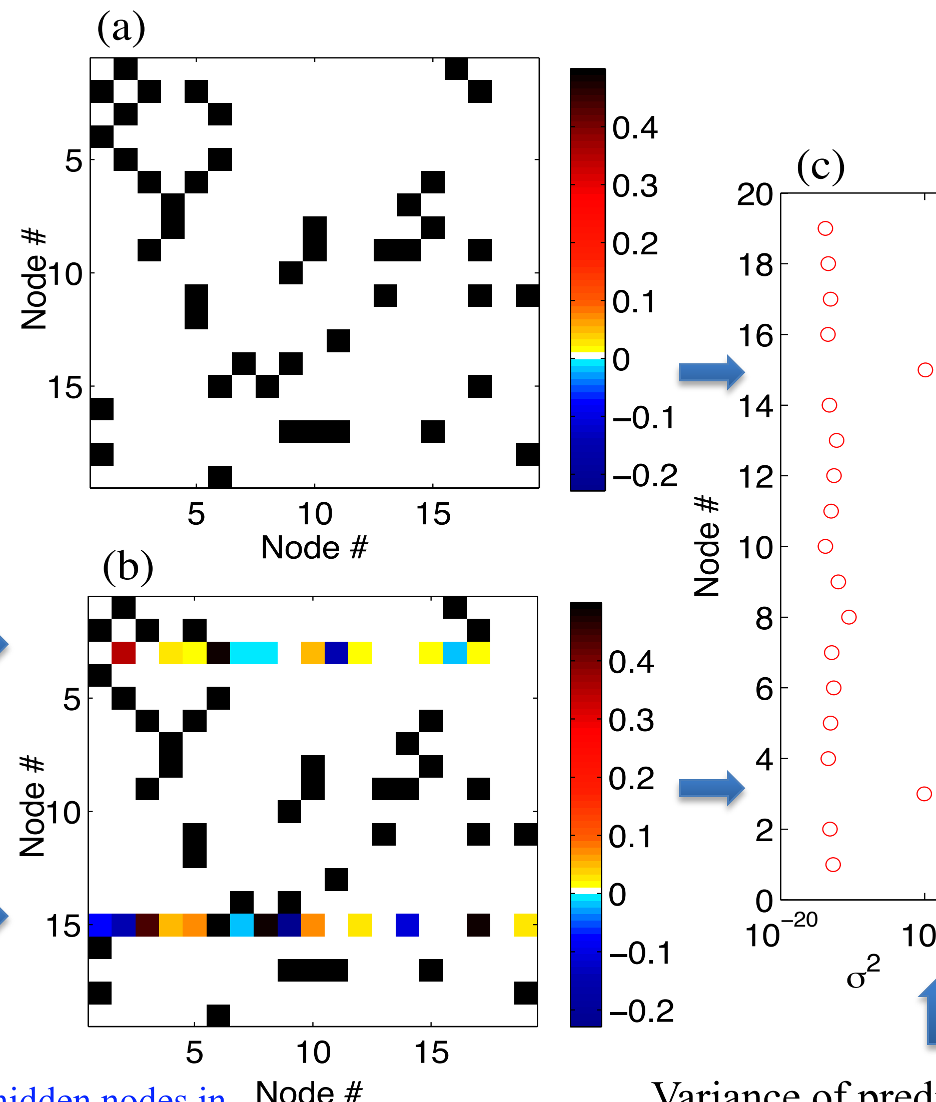


Observation: Large-degree nodes are not necessarily winners

W.-X. Wang, Y.-C. Lai, C. Grebogi, and J.-P. Ye, "Network reconstruction based on evolutionary game data," *Physical Review X* **1**, 021021 (2011).

Detecting Hidden Node





Variance of predicted coefficients

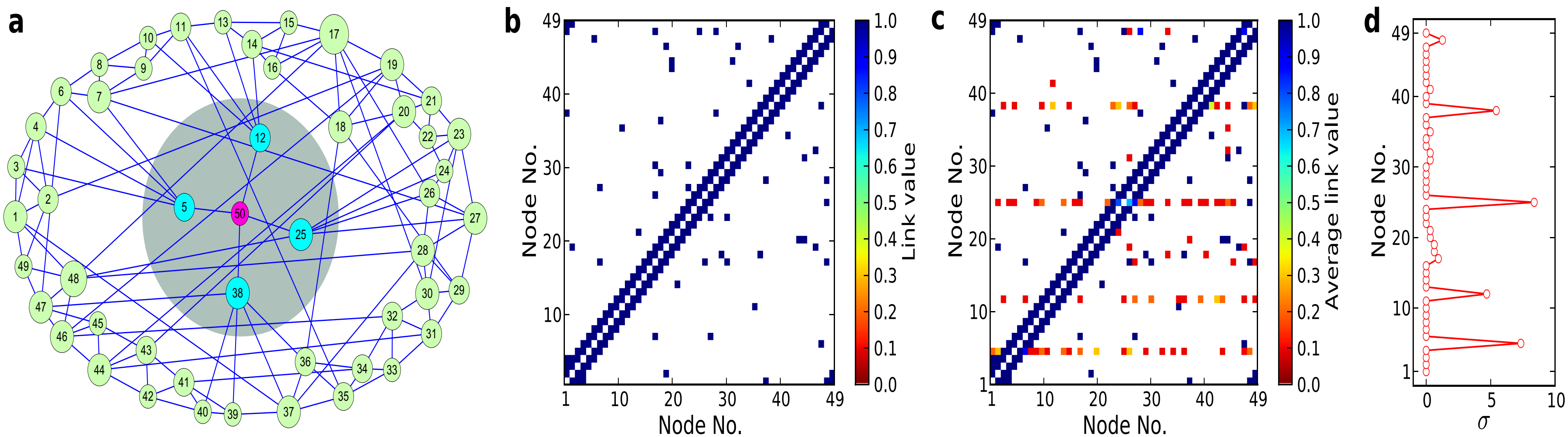
<u>Idea</u>

- Two green nodes: immediate neighbors of hidden node
- Information from green nodes is not complete
- Anomalies in the prediction of connections of green nodes

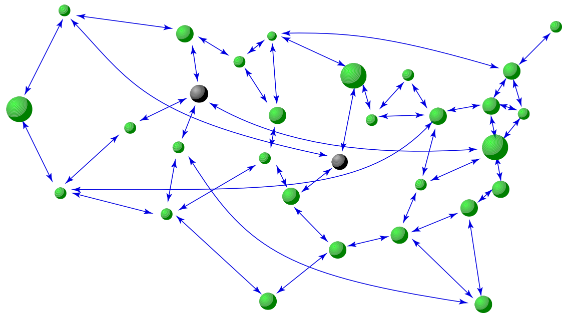
- R.-Q. Su, W.-X. Wang, and Y.-C. Lai, "Detecting hidden nodes in Complex networks from time series," *Physical Review E* 106, 058701R (2012).
- R.-Q. Su, Y.-C. Lai, X. Wang, and Y.-H. Do, "Uncovering hidden nodes in Complex networks in the presence of noise," *Scientific Reports* **4**, Article number 3944 (2014).

Locating Hidden Source

in Complex Spreading Network Using **Binary** Time Series

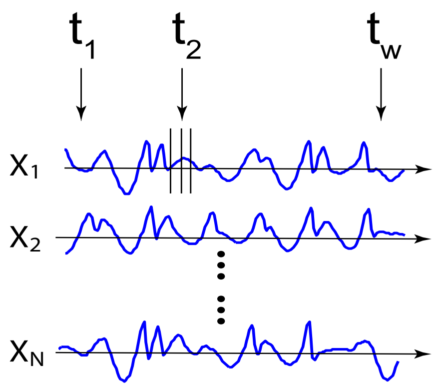


- Z.-S. Shen, W.-X. Wang, Y. Fan, Z.-R. Di, and Y.-C. Lai, "Reconstructing propagation networks with natural diversity and identifying hidden source," *Nature Communications* **5**, Article number 4323 (2014).
- Z.-L. Hu, X. Han, Y.-C. Lai, and W.-X. Wang, "Optimal localization of diffusion sources in complex networks," *Royal Society Open Science* **4**, Article number 170091 (2017).



Reconstruction of complex geospatial networks

(A) Time Series collection

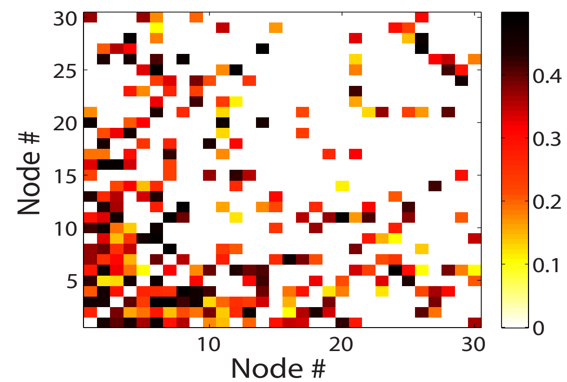


(B) Compressive Sensing

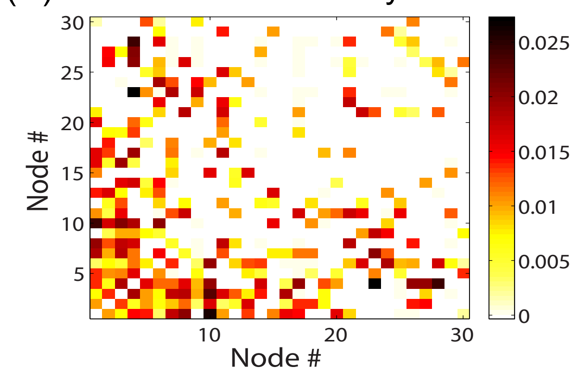
$$\dot{x}_i(t) = \frac{x_i(t + \Delta t) - x_i(t - \Delta t)}{2 \times \Delta t}$$

$$\begin{pmatrix} \dot{x}_i(t_1) \\ \dot{x}_i(t_2) \\ \vdots \\ \dot{x}_i(t_w) \end{pmatrix} = \begin{pmatrix} \mathbf{A}_i(t_1) & \mathbf{B}_i(t_1) & \mathbf{C}_i(t_1) \\ \mathbf{A}_i(t_2) & \mathbf{B}_i(t_2) & \mathbf{C}_i(t_2) \\ \vdots & \vdots & \vdots \\ \mathbf{A}_i(t_w) & \mathbf{B}_i(t_w) & \mathbf{C}_i(t_w) \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{a}}_i \\ \mathbf{b}_i \\ \mathbf{c}_i \end{pmatrix}$$

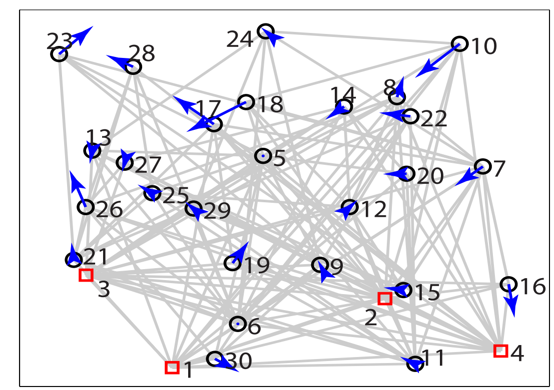
(C) Reconstructed Weight



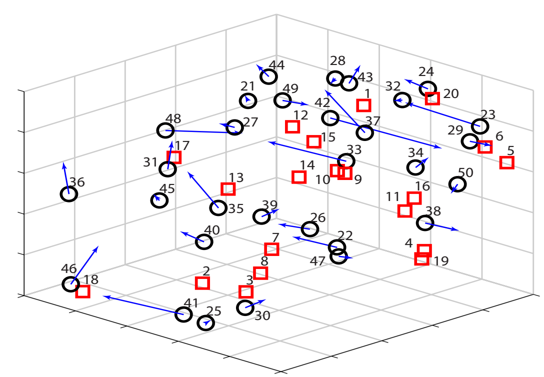
(D) Reconstructed Delays



(E) Final geo-spatial network (2D)



(F) Final geo-spatial network (3D)



R.-Q. Su, W.-X. Wang, X. Wang, and Y.-C. Lai, "Data-based reconstruction of complex geospatial networks, nodal positioning and detection of hidden nodes," *Royal Society Open Science* **3**, 150577 (2016).

Discussion

1. Key requirement of compressive sensing – the vector to be determined must be sparse.

   Dynamical systems - three cases:

- Vector field/map contains a few Fourier-series terms - Yes
- Vector field/map contains a few power-series terms - Yes
- Vector field /map contains many terms – not known

Ikeda Map: $F(x,y) = [A + B(x\cos\phi - y\sin\phi), B(x\sin\phi + y\cos\phi)]$

where $\phi \equiv p - \frac{k}{1 + x^2 + y^2}$ - describes dynamics in an optical cavity

Mathematical question: given an arbitrary function, can one find a <u>suitable base of expansion</u> so that the function can be represented by a limited number of terms?

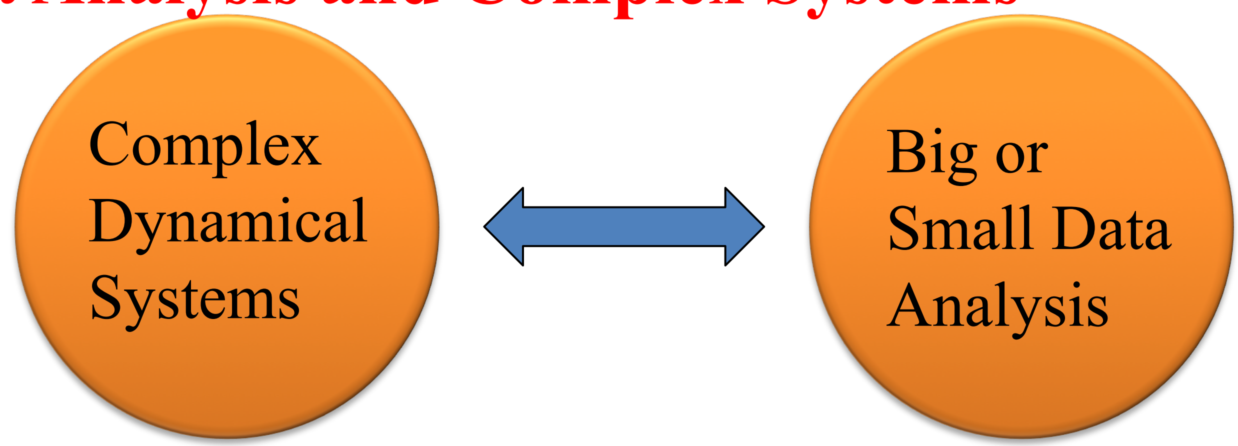
Discussion

2. Networked systems described by evolutionary games – Yes
3. Measurements of ALL dynamical variables are needed.

   Outstanding issue

If this is not the case, say, if only one dynamical variable can be measured, the CS-based method would not work. Delay-coordinate embedding method? - gives only a topological equivalent of the underlying dynamical system (e.g., Takens' embedding theorem guarantees only a one-to-one correspondence between the true system and the reconstructed system).

Data Analysis and Complex Systems



Big Data or Small Data?

Data-based Research is becoming increasingly important for understanding, predicting, and controlling Complex Dynamical Systems