A Recent Review Article

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Data based identification and prediction of nonlinear and complex dynamical systems

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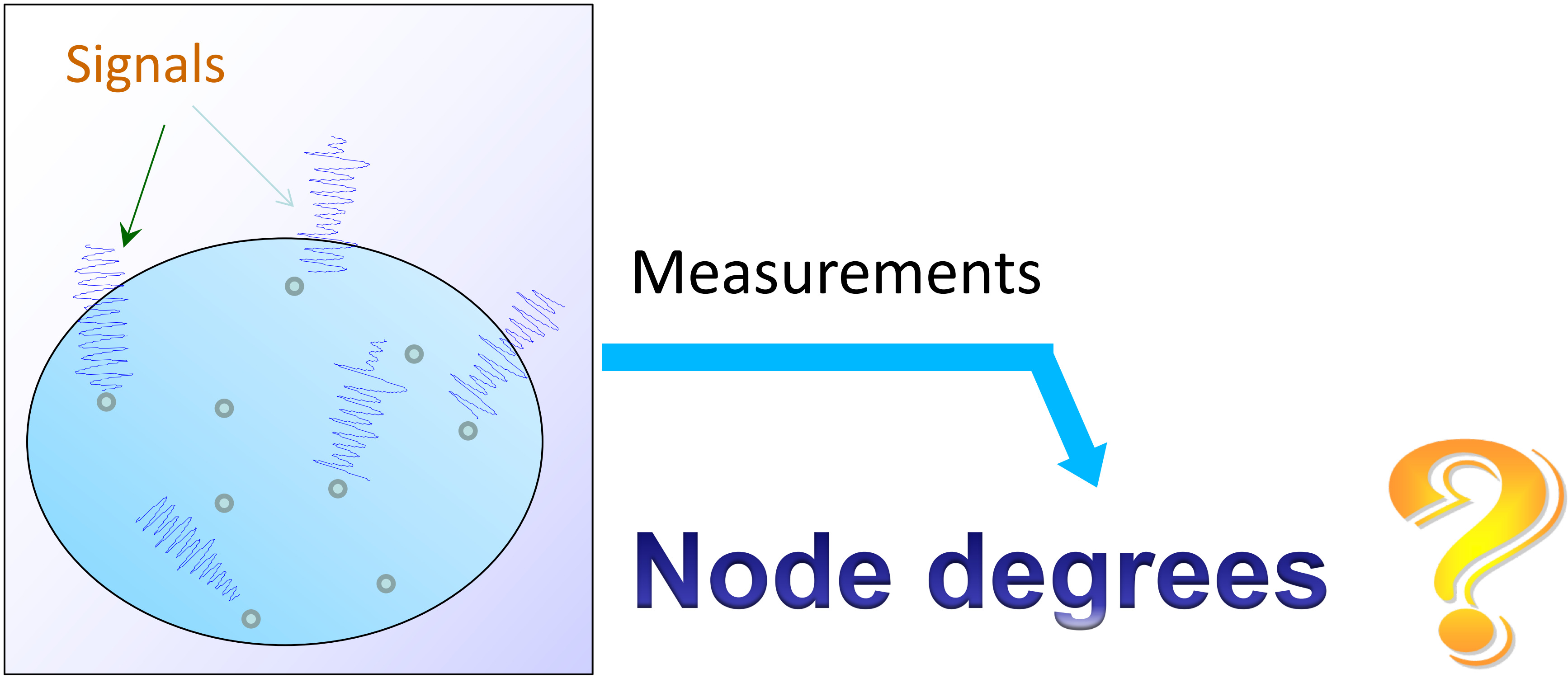
Detecting Nodal Degree – Early Works



An unknown network system

Developed two methods based on

1. **Principal component analysis;**
2. **Universal scaling law of fluctuations about mean field**

Potential Applications

- Adverse social organization detection

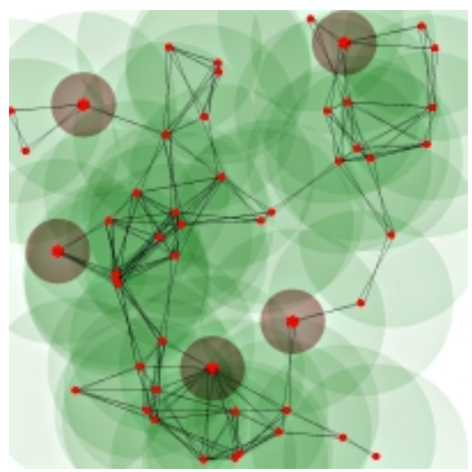
Concurrence of human activities

The most "important" node (person)?
- Detect the hubs of adverse social networks by monitoring a proper public area.

- Distributed adverse electronic system detection

Adverse organization embedded in Internet

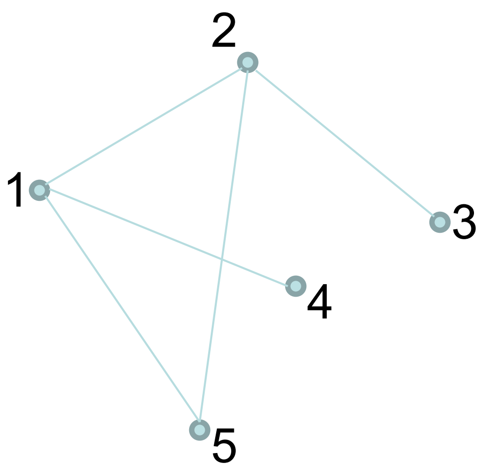
Sensor network



Identify hub nodes to launch attacks to effectively disable the network.

Adjacency Matrix - Network Connectivity

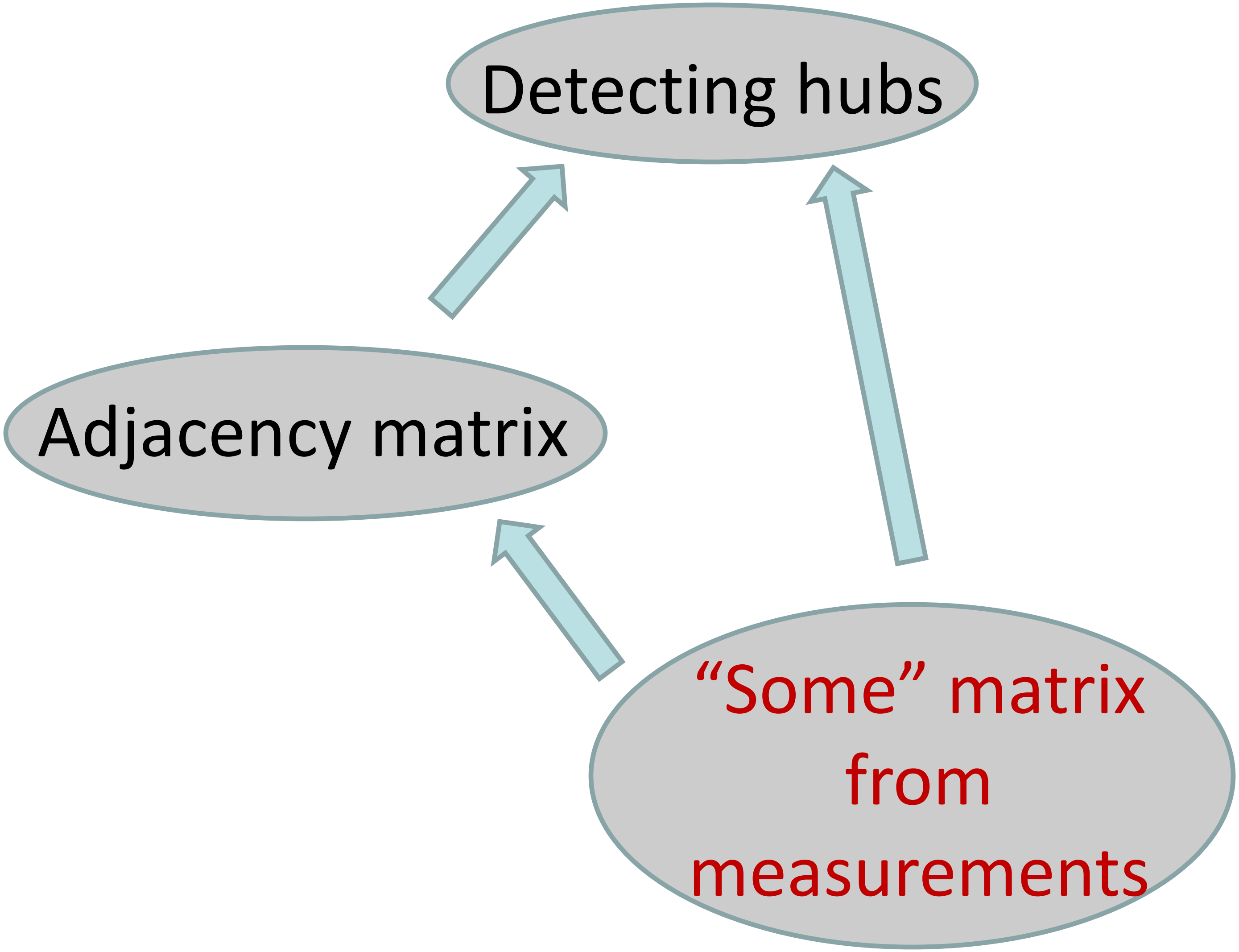
- Adjacency matrix **A**: $A_{ij} = 1$ if nodes i and j are connected, otherwise $A_{ij} = 0$



$$j \longrightarrow \qquad i \downarrow \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

- Degree $k_i = \sum_{j=1}^{N} A_{ij}$.
- Hubs: nodes with many links

Idea


Detecting hubs
Adjacency matrix
"Some" matrix from measurements

Theory -- Adjacency Matrix

- $k_i \sim e_{1,i}$
  - From singular value decomposition:

$$\mathbf{A} = \sum_{i=1}^{N} \lambda_i \mathbf{e}_i \mathbf{e}_i^T$$

    - λ_i , $\mathbf{e}_i$ are the i'th eigenvalue and eigenvector
    - $(\mathbf{e}_i \mathbf{e}_i^T)_{jk} = e_{i,j} \times e_{i,k}$
  - Sort eigenvalues: $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_N|$

Principal Component of Adjacency Matrix

- If $|\lambda_1| >> |\lambda_2|$,

$$\mathbf{A} = \sum_i \lambda_i \mathbf{e}_i \mathbf{e}_i^T \quad \Rightarrow \quad \mathbf{A} \approx \lambda_1 \mathbf{e}_1 \mathbf{e}_1^T$$

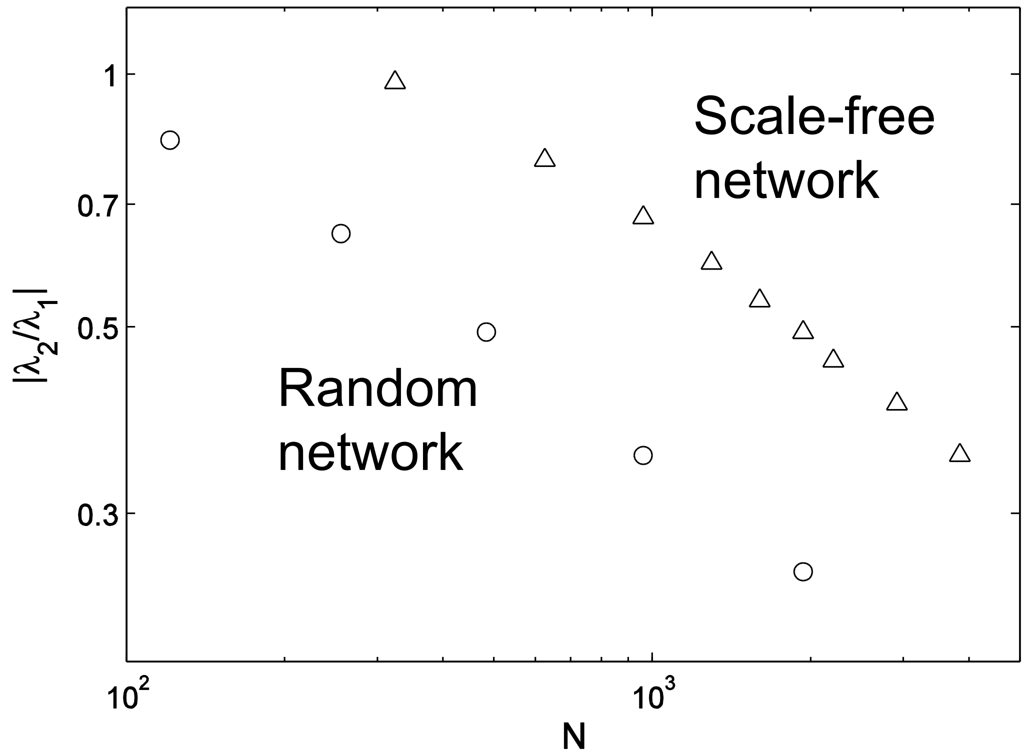
- Summing up rows:

$$k_i = \sum_j A_{ij} \approx \sum_j \lambda_1 e_{1,i} e_{1,j} = C\lambda_1 e_{1,i}$$

$$k_i \sim e_{1,i}$$

Ratio of Eigenvalues

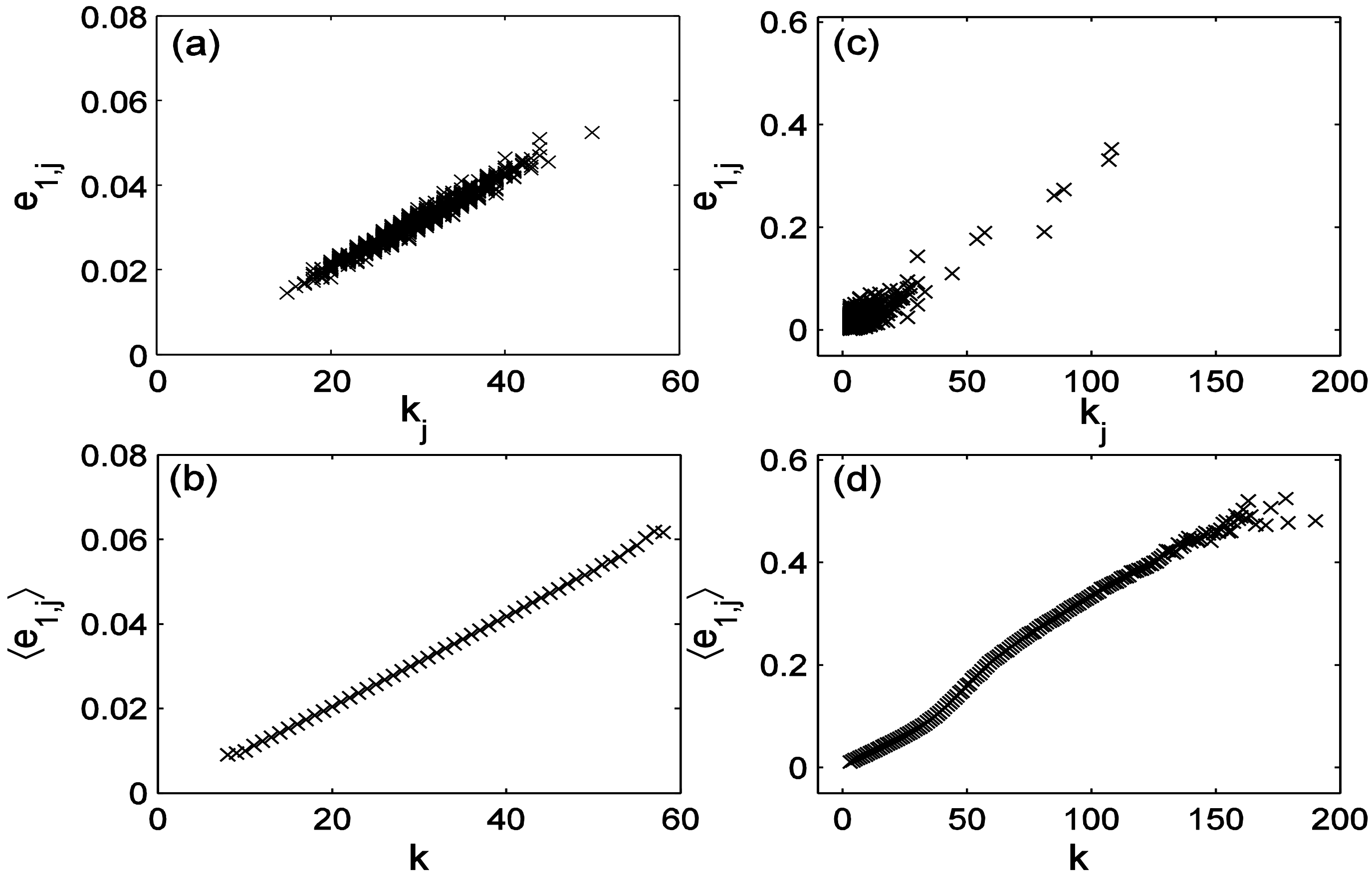
- Validity for $|\lambda_1| >> |\lambda_2|$
  - For random networks, $|\lambda_2/\lambda_1| \sim N^{-1/2}$



$$\langle k \rangle = 0.008N$$

[1] I. J. Farkas, I. Derenyi, A.-L. Barabasi, and T. Vicsek, Phys. Rev. E 64, 026704 (2001).

Components of Principal Eigenvector



Random networks ($\langle k \rangle = 30$) (a)(b); Scale-free networks ($\langle k \rangle = 6$) (c)(d). N=1000.

Proposal

- From measurements, if we can construct a matrix with properties similar to those of the adjacency matrix, the components of the principal eigenvector can be an indicator of the degrees.
- How?

  Synchronization-probability matrix

$$\Phi_{ij} \sim A_{ij}$$

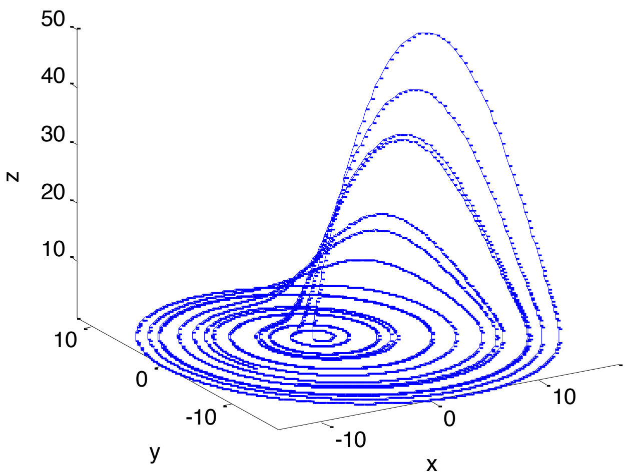
Example: Coupled Chaotic Network

- Coupled Rössler System:

$$\frac{\mathrm{d}\mathbf{x}_i}{\mathrm{d}t} = \mathbf{F}(\mathbf{x}_i) - \varepsilon \sum_{j=1}^{N} G_{ij} \mathbf{H}(\mathbf{x}_j),$$

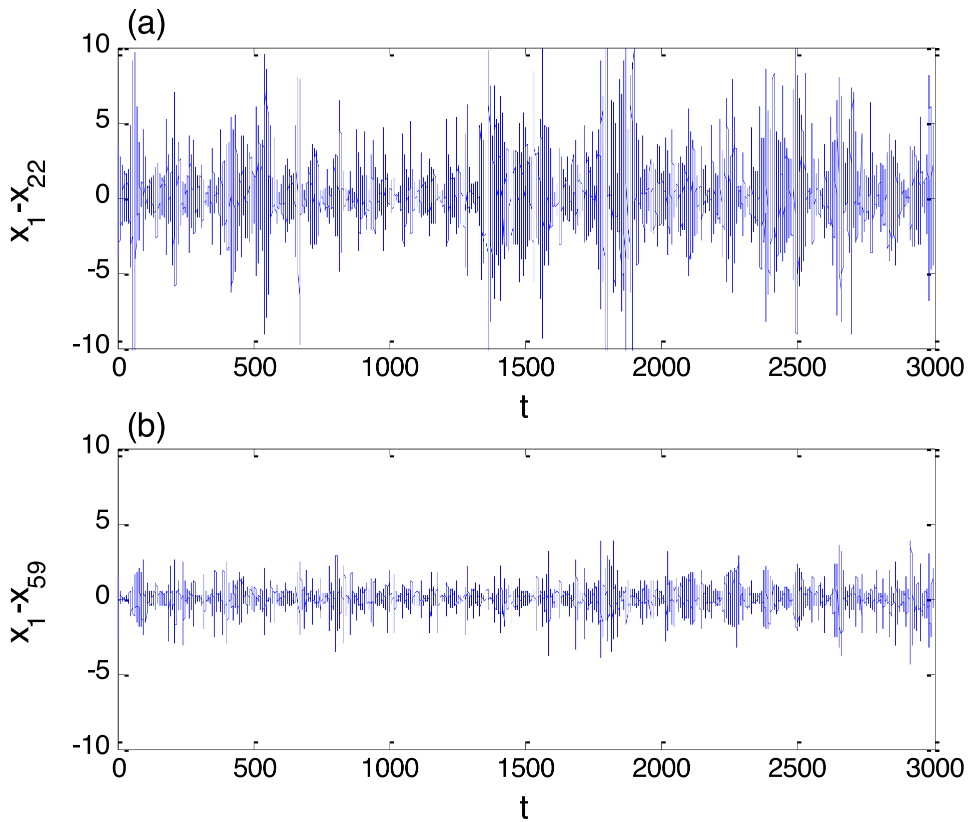
  - $\mathbf{F}(\mathbf{x}) = [-(y+z), x + a_i y, 0.2 + z(x-9)]^T$
  - a_i is random in [0.16, 0.24]
  - ε is coupling constant
  - $G_{ii} = 1$, $G_{ij} = -A_{ij} / k_i$
  - $\mathbf{H}(\mathbf{x}) = [x, 0, 0]^T$

$a = 0.2$



Synchronization Probability

- Synchronization probability Φ_{ij} : fraction of time during which $|x_i - x_j| < \delta$



- $\Phi_{ij} \sim A_{ij}$

(a) Not connected

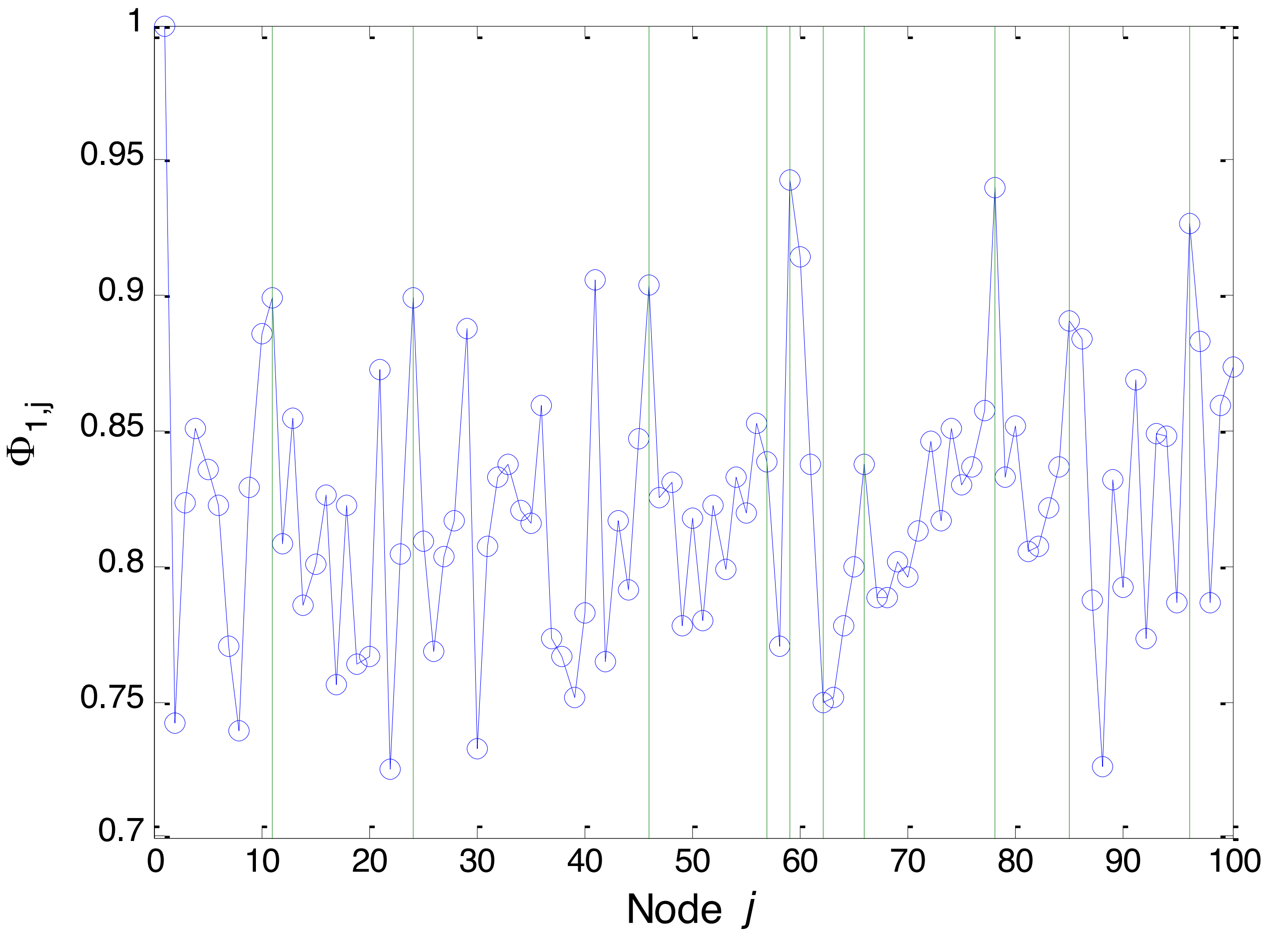
$$\Phi_{1,22} = 0.725$$

(b) Connected

$$\Phi_{1,59} = 0.943$$

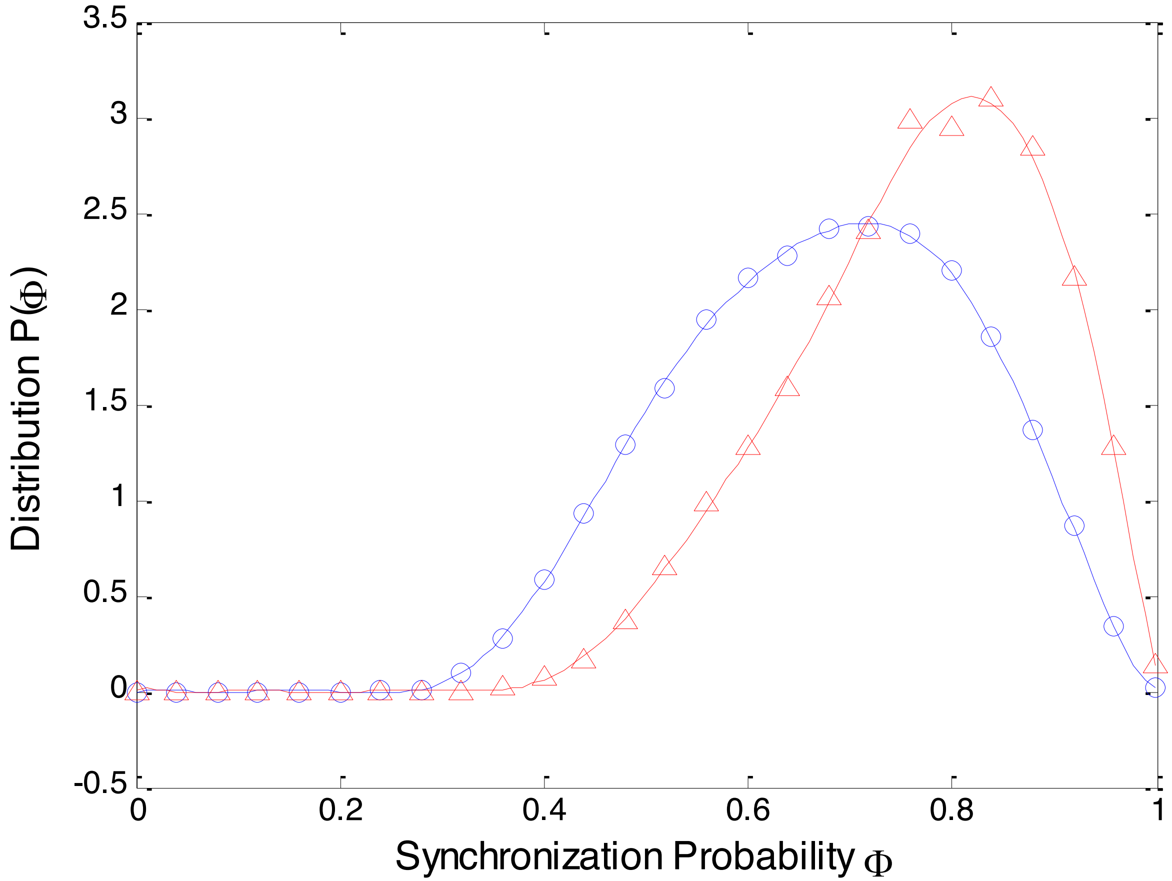
(Random network p=0.1, N=100)

Synchronization probability



(Random network, p=0.1, N=100)

Distribution of sync probability



Circles: for node-pairs without a link;

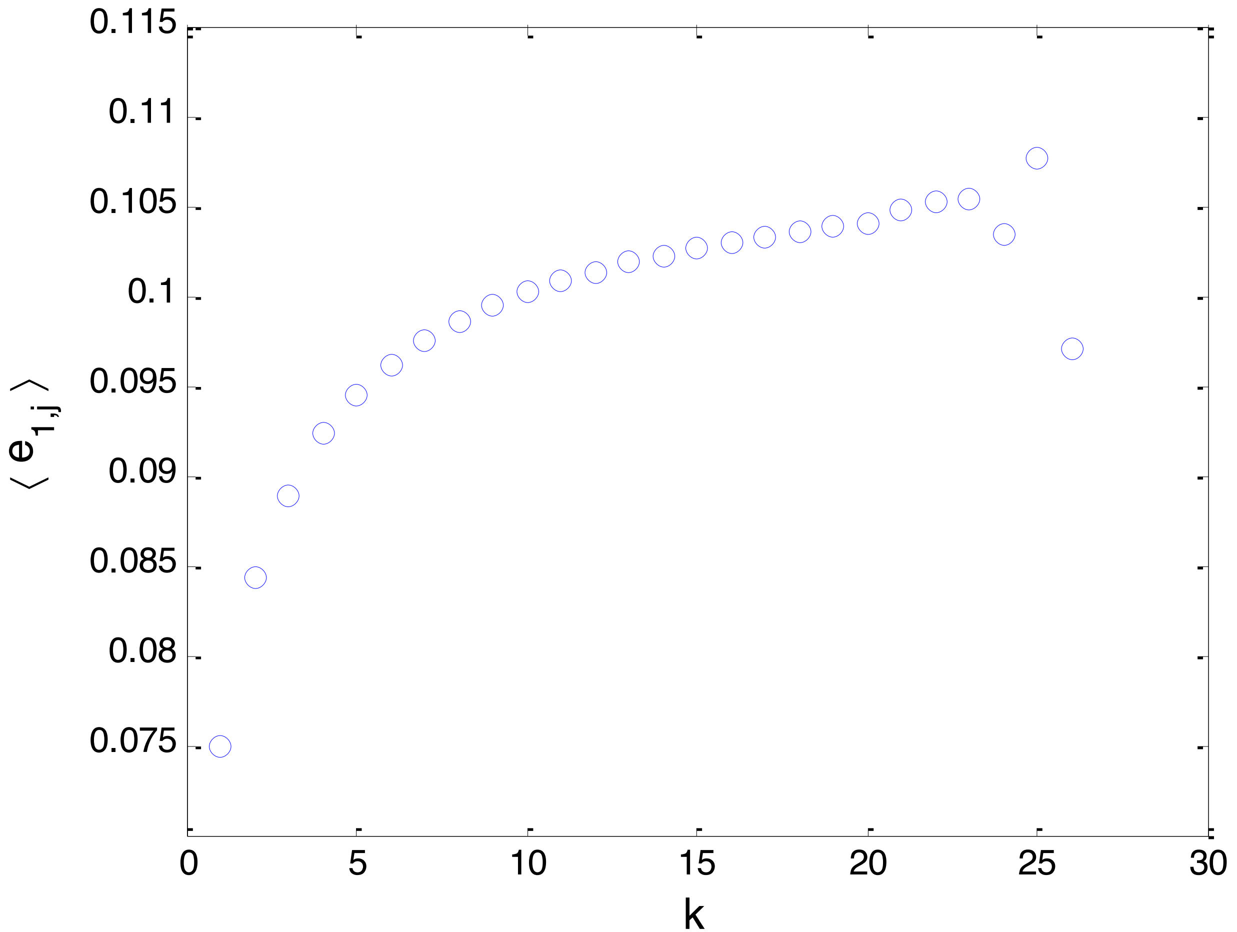
Triangles: for node-pairs with a direct link.

Random networks: p=0.1, N=100, ε = 0.5, δ = 1. T0=3000.

Synchronization-Probability Matrix

- Expect $e_{1,i}$ of matrix $\lfloor \Phi_{jk} \rfloor$ to be positively correlated with k_i (a heuristic argument can be worked out)
  - Hubs can be identified as nodes with large $e_{1,i}$ values

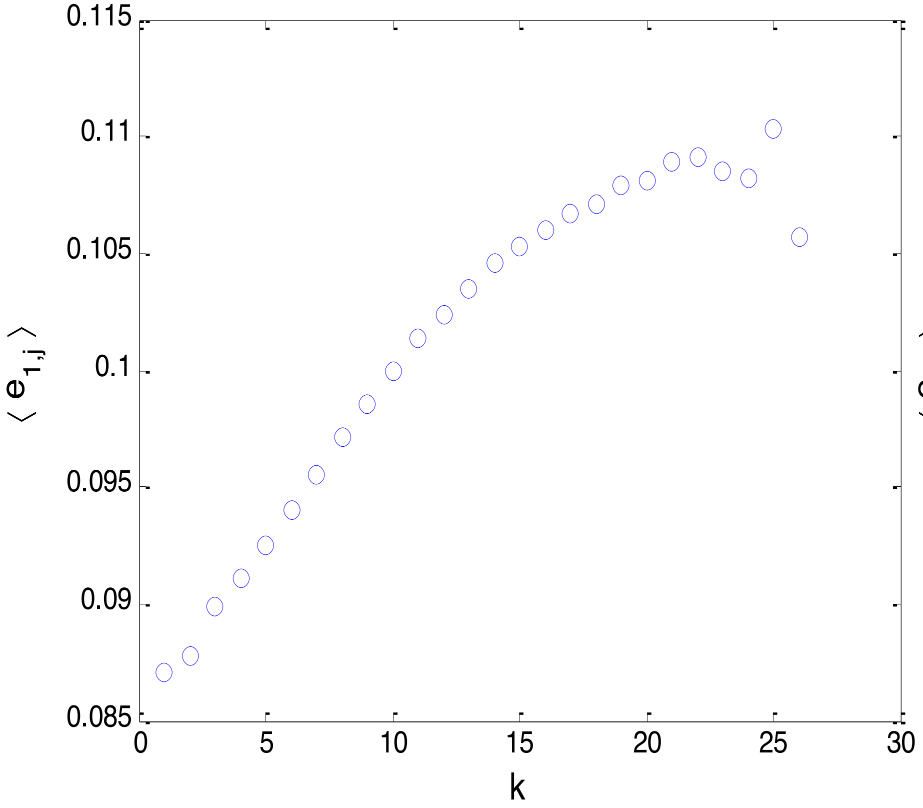
Numerical Test: Chaotic Rössler Network



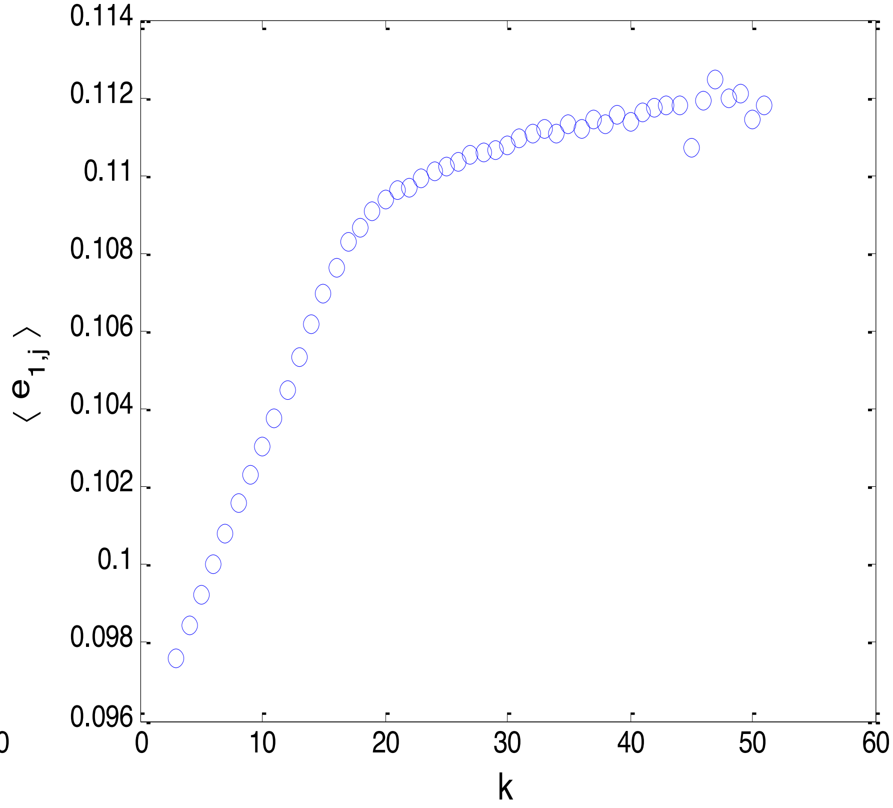
Random networks: p=0.1, N=100, ε = 0.5, δ = 1. T0=3000.

Kuramoto Network

$$\frac{\mathrm{d}\varphi_i}{\mathrm{d}t} = \omega_i + \varepsilon \sum_{j=1}^{N} A_{ij} \sin(\varphi_j - \varphi_i)$$



Random



Scale-Free

A New Phenomenon

$x_i(t)$ - A set of oscillatory signals measured from various points in network

- Noise is present

Then, calculate

$$\left\langle \left| \Delta x_i \right| \right\rangle_T = \left\langle \left| x_i(t) - \left\langle x(t) \right\rangle_E \right| \right\rangle_T$$

- **Time averaged fluctuation about mean field**
- **Scales inversely with square root of node degree**

1. One-to-one correspondence with node degree;
2. Universality

Setting

Oscillatory networked dynamical system

$$\dot{\mathbf{x}}_i(t) = F(\mathbf{x}_i) + c\sum_{j=1}^{N} G_{ij} H(\mathbf{x}_j) + \xi,$$

G_{ij} -- Coupling matrix (network connections)

$H(x)$ -- Coupling function

ξ -- a stochastic process (noise)

Node state $x_i(t)$	Node dynamics $F(x)$	noise ξ	Coupling strength c	Network topology
measurable	unknown	present	unknown	???

Average Fluctuation about Mean Field

$$\langle |\Delta x_i| \rangle_T = \left\langle \left| x_i(t) - \langle x(t) \rangle_E \right| \right\rangle_T$$

$\langle x(t) \rangle_E$ --- ensemble average over all nodes' states at time t

$\langle \ldots \rangle_T$ --- average over time .

Key issue: scaling law between $\langle |\Delta x_i| \rangle_T$ and k_i

Degree of node i

Will consider three different types of node dynamics:

- **Consensus dynamics**
- **Chaotic system**
- **Kuramoto system**

Example 1: Consensus dynamics

$$\dot{x}_i(t) = -c\sum_{j=1}^{N} A_{ij}[x_i(t) - x_j(t)] + \xi,$$

ξ --- Gaussian white noise of zero mean and variance σ^2

$$A_{ij} = \begin{cases} 1, & i \neq j \\ 0, & i = j \end{cases}$$ --- adjacency matrix

Rewrite

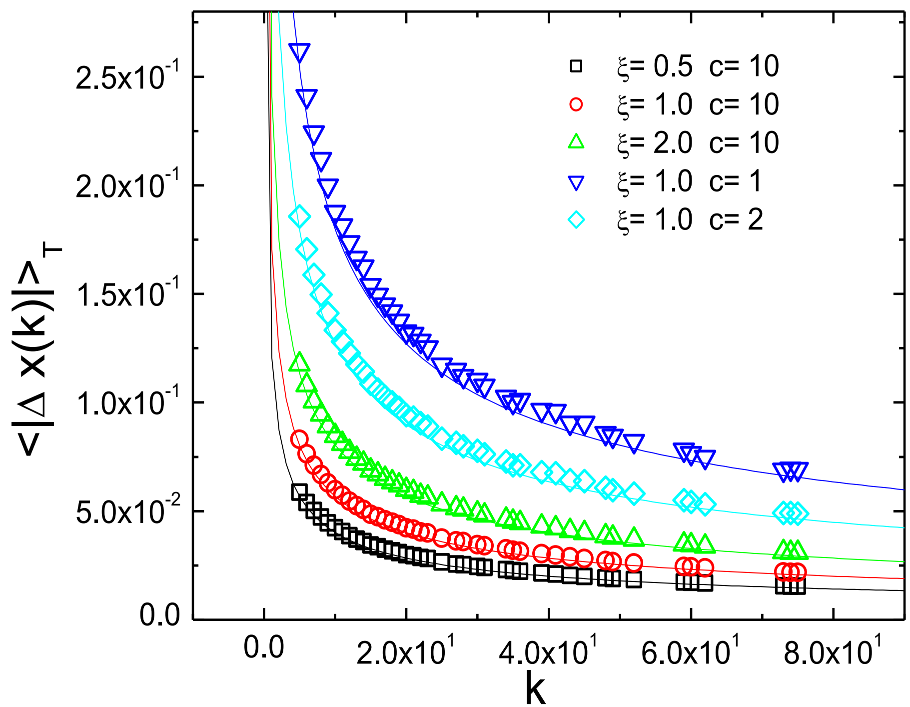
$$\dot{x}_i = -cLx + \xi,$$

L --- Laplacian Matrix defined as

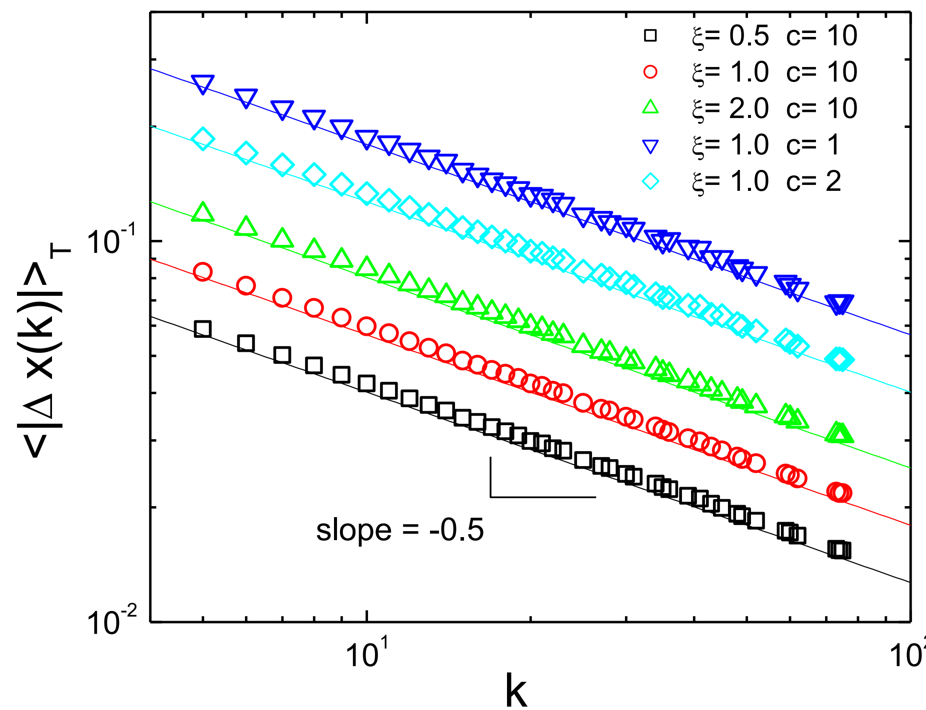
$$L_{ij} = \begin{cases} -1, & i \neq j \\ k_i, & i = j \end{cases}$$

Consensus Dynamics on Scale-Free Networks

Linear plot

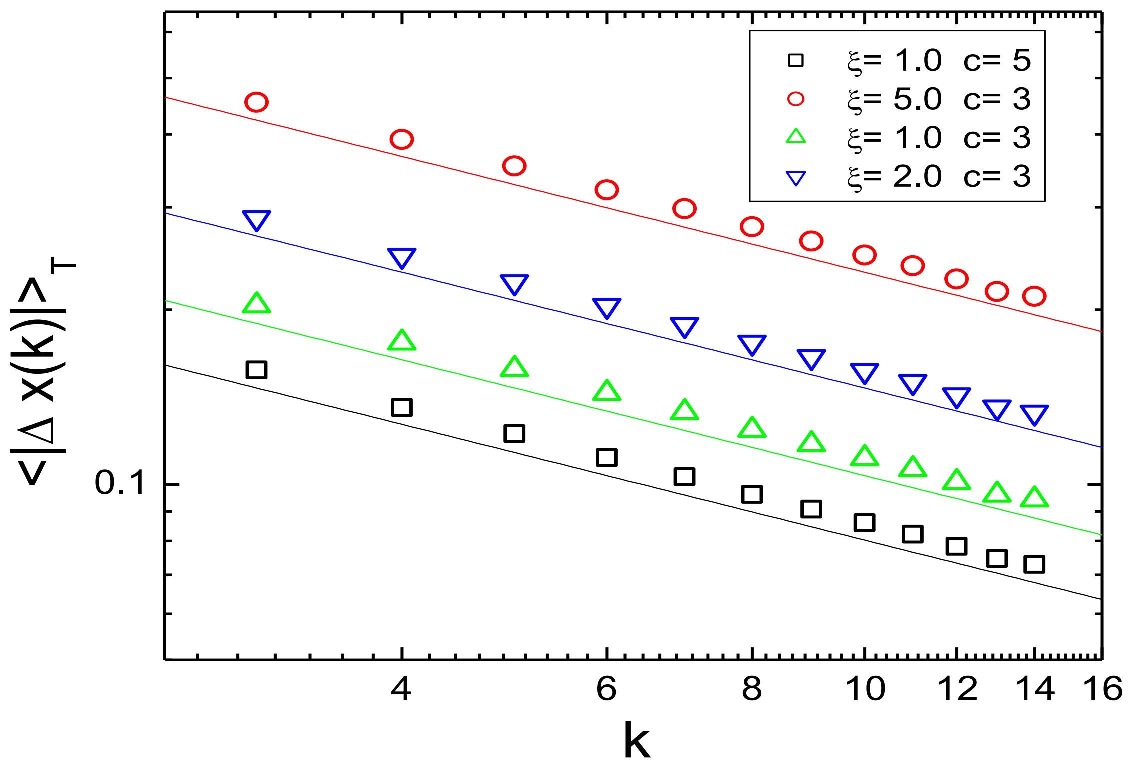


Logarithmic plot



Network size is 500. Results are obtained from a single network by choosing different variance of noise and coupling strengths. Data points are from simulations and lines are from theoretical predictions.

Consensus Dynamics on Small-World Networks



The phenomenon that $\left\langle \left| \Delta x_i \right| \right\rangle_T$ is proportional to $\frac{1}{\sqrt{k_i}}$ holds regardless of network structure

Analysis

$$\dot{x}_i(t) = -c\sum_{j=1}^{N} A_{ij}[x_i(t) - x_j(t)] + \xi,$$

Variational equation about consensus state:

$$\Delta\dot{x}_i = -ck_i\Delta x_i - c\sum_{j=1}^{N} A_{ij}\Delta x_j + \xi,$$

Δx_j **being random** ⟹ $\sum \approx 0$

we have

$$\Delta\dot{x}_i = -ck_i(\Delta x_i + \xi_i'),$$

where

$$\xi_i' = -\frac{\xi}{ck_i}.$$

Regarding ξ_i' as input and Δx_i as output, the system's transfer function is

$$H(s) = -\frac{ck_i}{s + ck_i}.$$

Variance of Δx_i

$$\left\langle \Delta x_i^2 \right\rangle_T = \int_{-\infty}^{\infty} S_{\Delta x_i}(f)df$$

$$= \int_{-\infty}^{\infty} |H(j2\pi f)|^2 S_{\xi_i'}(f)df$$

where $S_{\Delta x_i}(f)$ and $S_{\xi_i'}(f)$ are the power spectral density (PSD) for Δx_i and ξ_i', respectively. Since $S_{\xi_i'}(f)$ and $|H(j2\pi f)|^2$ are even functions of *f*, we have

$$\left\langle \Delta x_i^2 \right\rangle_T = 2\int_{0}^{\infty} |H(j2\pi f)|^2 S_{\xi_i'}(f)df.$$

Analysis (cont.)

Definition of PSD:

$$S_{\xi_i'}(f) = \int_{-\infty}^{\infty} \langle \xi_i'(t+\tau)\xi_i'(t)\rangle_T \exp(-j2\pi f\tau)d\tau = \left\langle \xi_i'(t)^2 \right\rangle_T = \frac{\sigma^2}{c^2 k_i^2}$$

σ^2 **-- variance of Gaussian noise**

Note that *H(s)* is the transfer function of a low-pass filter, we have

$$|H(j2\pi f)|^2 \approx U(0) - U\left(\frac{ck_i}{2\pi}\right),$$

$U(x)$ **-- unit step function**

$$\Rightarrow \quad \left\langle \Delta x_i^2 \right\rangle_T \approx 2\frac{\sigma^2}{c^2 k_i^2} \times \frac{ck_i}{2\pi} = \frac{\sigma^2}{\pi k_i c}.$$

Or $$\langle |\Delta x_i| \rangle_T \approx \sqrt{\left\langle \Delta x_i^2 \right\rangle} = \frac{\sigma}{\sqrt{\pi c}} \frac{1}{\sqrt{k_i}}.$$

W.-X. Wang, Q.-F. Chen, L. Huang, Y.-C. Lai, and M. A. F. Harrison, "Scaling of noisy fluctuations in complex networks and applications to network detection," *Phys. Rev. E* **80**, 016116 (2009).

Example 2: Chaotic Oscillator Network

$$\begin{cases} \dot{x}_i = -(y_i - z_i) + c\sum_{j=1}^{N} A_{ij}(x_j - x_i) + \xi, \\ \dot{y}_i = x + 0.2y_i + c\sum_{j=1}^{N} A_{ij}(y_j - y_i), \\ \dot{z}_i = 0.2 + z_i(x_i - 9.0) + c\sum_{j=1}^{N} A_{ij}(z_j - z_i), \end{cases}$$

Noiseless system can be expressed as

$$\dot{X}_i = F(X) - cLX$$

$$\Rightarrow \quad \Delta\dot{X}_i = DF(X)\Delta X - cLDH(X)\Delta X$$

Chaotic Oscillator Network: Heuristic Analysis

Looking at x-component

$$\Delta \dot{x}_i = DF(x_i)\Delta x_i - ck_i \Delta x_i + c\sum_{j=1}^{N} A_{ij}\Delta x_j + \xi.$$

Δx_j **being random** ⟹ $\sum \approx 0$

For large coupling strength c and nodes with high degree k

⟹ $DF(x_i)\Delta x_i << ck_i \Delta x_i$

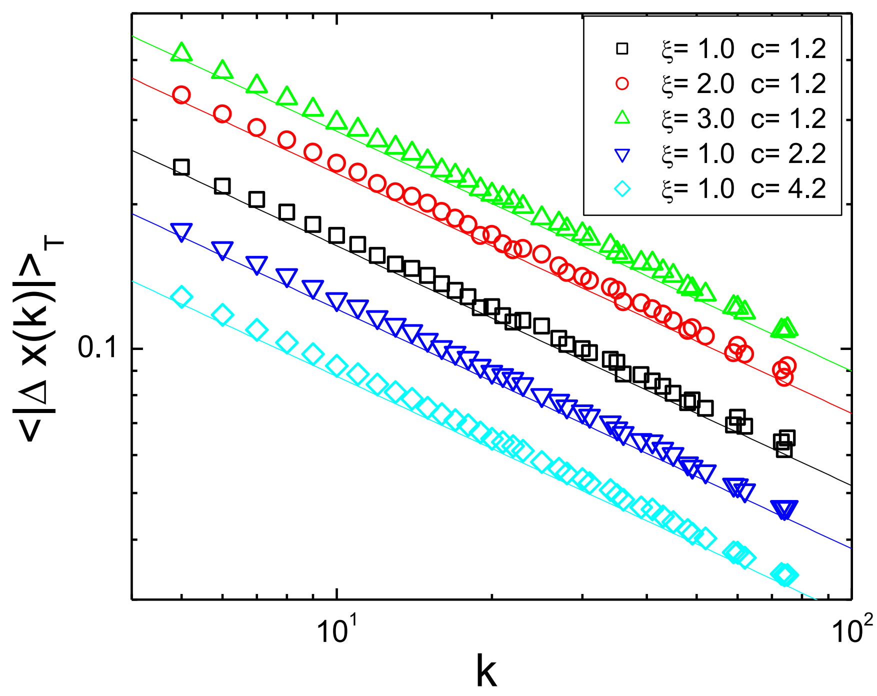
⟹ $\Delta \dot{x}_i = -ck_i \Delta x_i + \xi$

----- same equation as for consensus dynamics

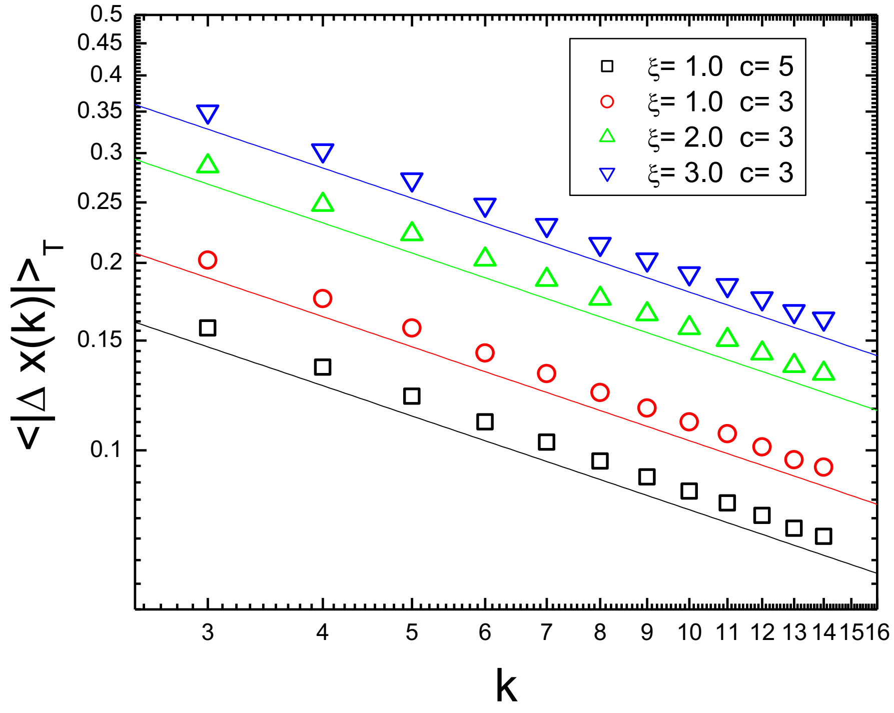
⟹ $$\langle |\Delta x(k)| \rangle_T \approx \frac{\sigma}{\sqrt{\pi c}} \frac{1}{\sqrt{k}}$$

Numerical Verification

Scale-free



Small-world



Example 3: Kuramoto Network

$$\dot{\theta}_i = \omega_i + c\sum_{j=1}^{N} A_{ij}\sin(\theta_j - \theta_i) + \xi$$

⇓

Natural frequency of oscillator *i*

Near synchronization state $\theta_1 = \theta_2 = \cdots = \theta_N$ ⇒ $\sin(\theta_j - \theta_i) \approx \theta_j - \theta_i$

⇒ $$\dot{\theta}_i \approx \omega_i + c\sum_{j=1}^{N} A_{ij}(\theta_j - \theta_i) + \xi$$

$$= -ck_i\theta_i + c\sum_{j=1}^{N} A_{ij}\theta_j + \omega_i + \xi$$

Variational equation

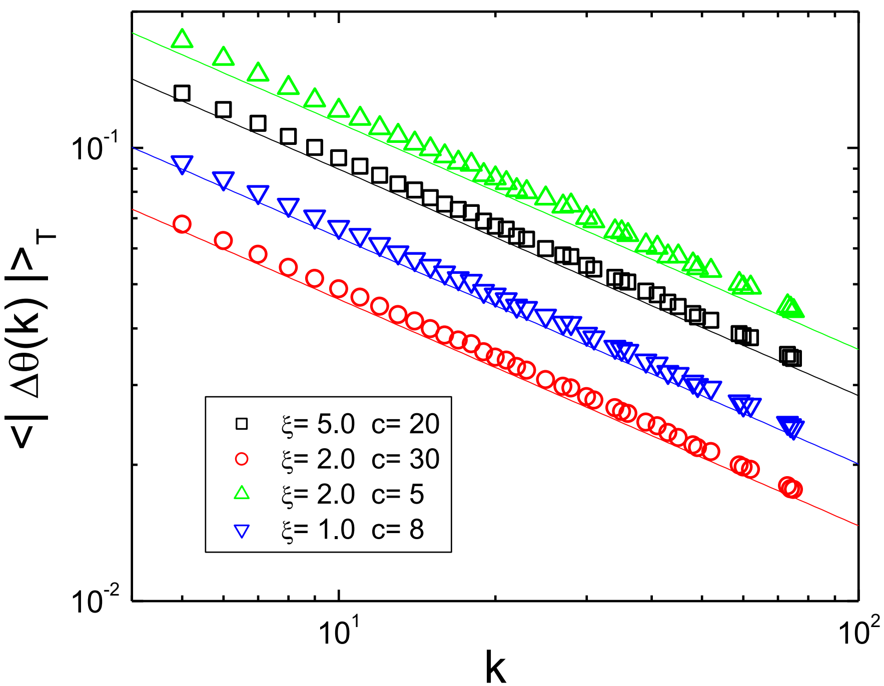
$$\Delta\dot{\theta}_i = -ck_i\Delta\theta_i + c\sum_{j=1}^{N} A_{ij}\Delta\theta_j + \xi \quad \approx -ck_i\Delta\theta_i + \xi$$

--- same as that for consensus dynamics!

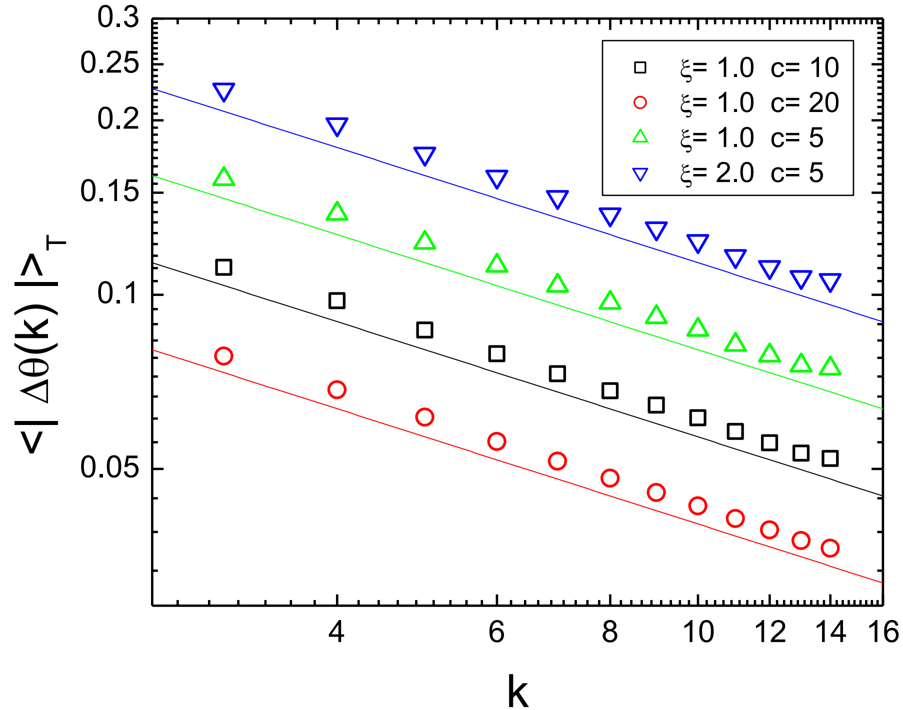
⇒ $$\langle|\Delta\theta(k)|\rangle_T \approx \frac{\sigma}{\sqrt{\pi c}}\frac{1}{\sqrt{k}}$$

Scaling Law: Numerical Support

Scale-free network

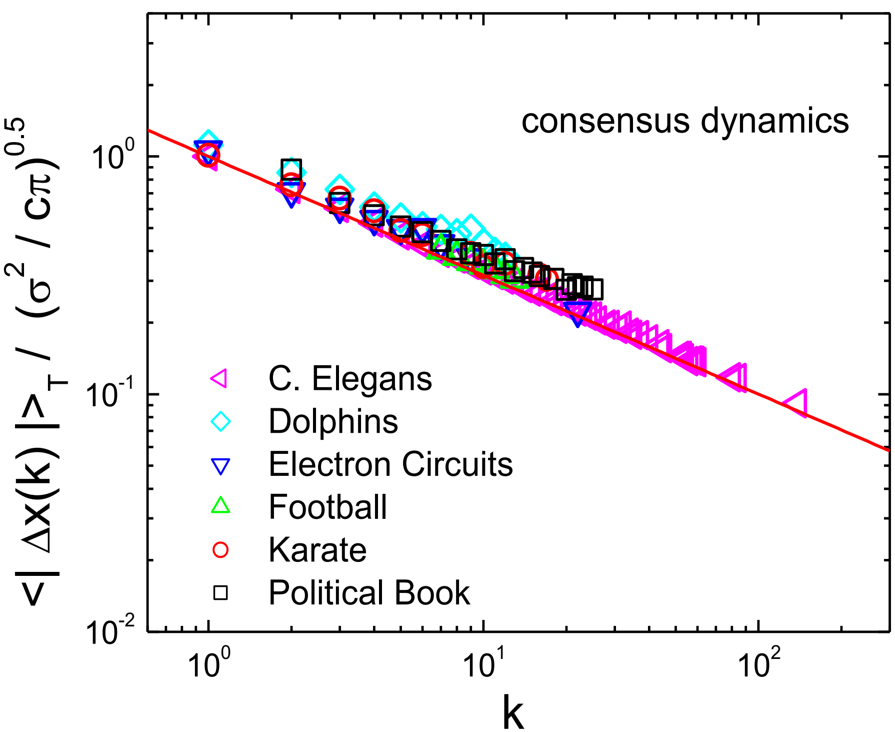


Small-world network



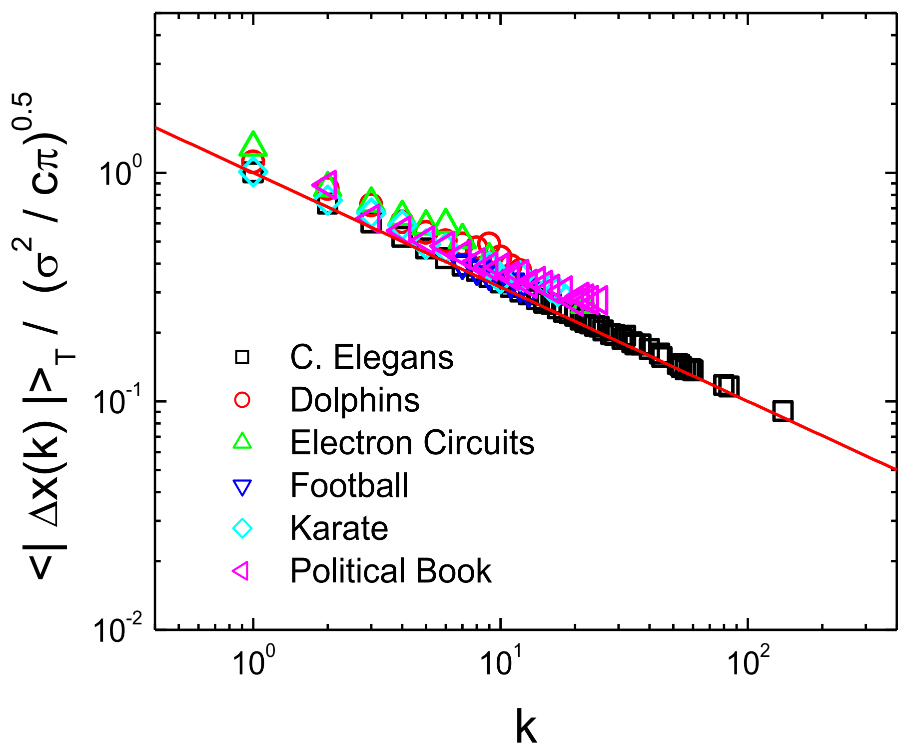
Oscillatory Dynamics on Real-World Networks

Consensus dynamics

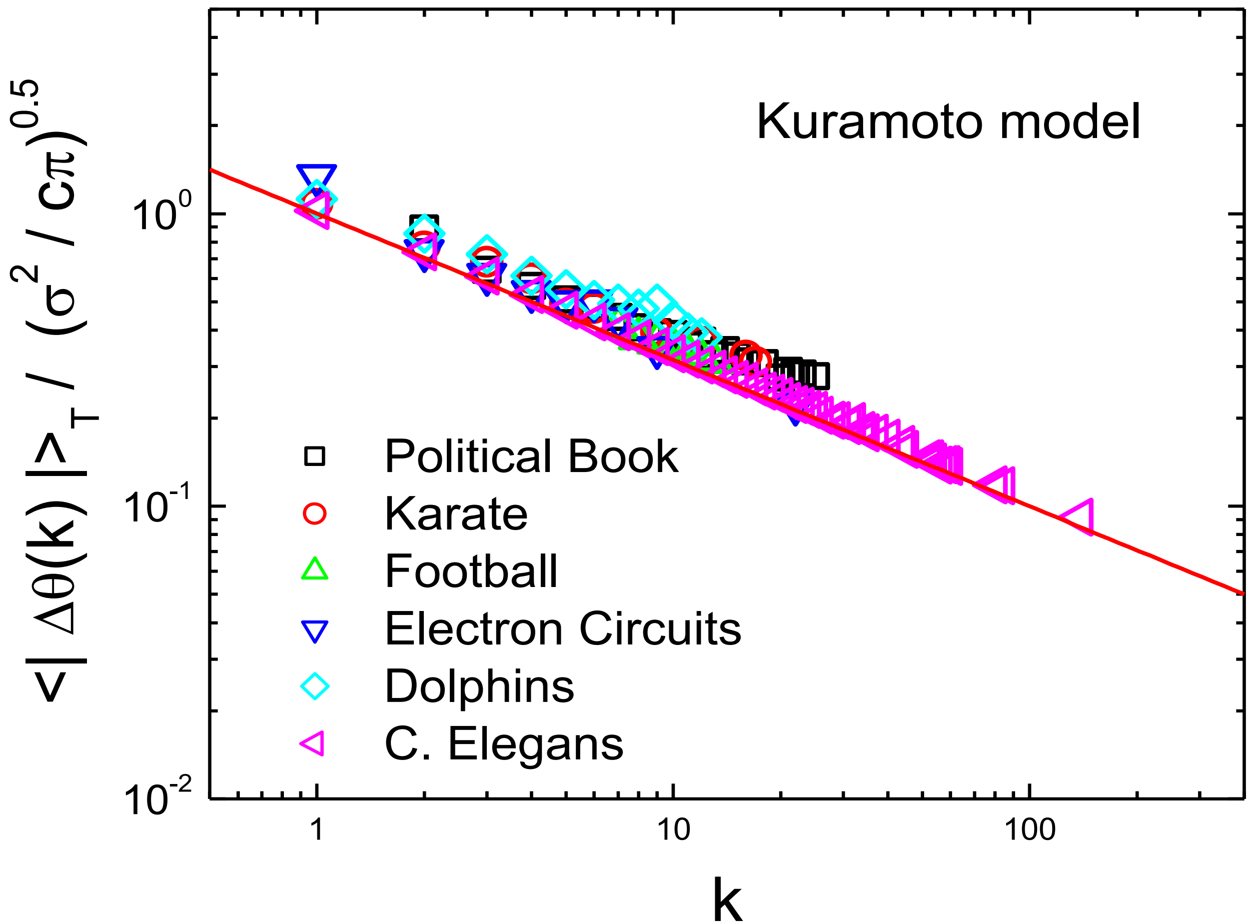


Chaotic dynamics

Rössler system



Kuramoto Dynamics on Real-World Networks



Rescaled results on all considered real-world networks converge to

$$y = \frac{1}{\sqrt{x}}$$

Heterogeneous Coupling – Weighted Networks

$$c' = c k_i^{\alpha}$$

Consensus dynamics

$$\dot{x}_i(t) = -c k_i^{\alpha} \sum_{j=1}^{N} A_{ij} [x_i(t) - x_j(t)] + \xi$$

$$\Rightarrow \Delta \dot{x}_i = -c k_i^{1+\alpha} \Delta x_i - c k_i^{\alpha} \sum_{j=1}^{N} A_{ij} \Delta x_j + \xi \quad \approx -c k_i^{1+\alpha} \Delta x_i + \xi$$

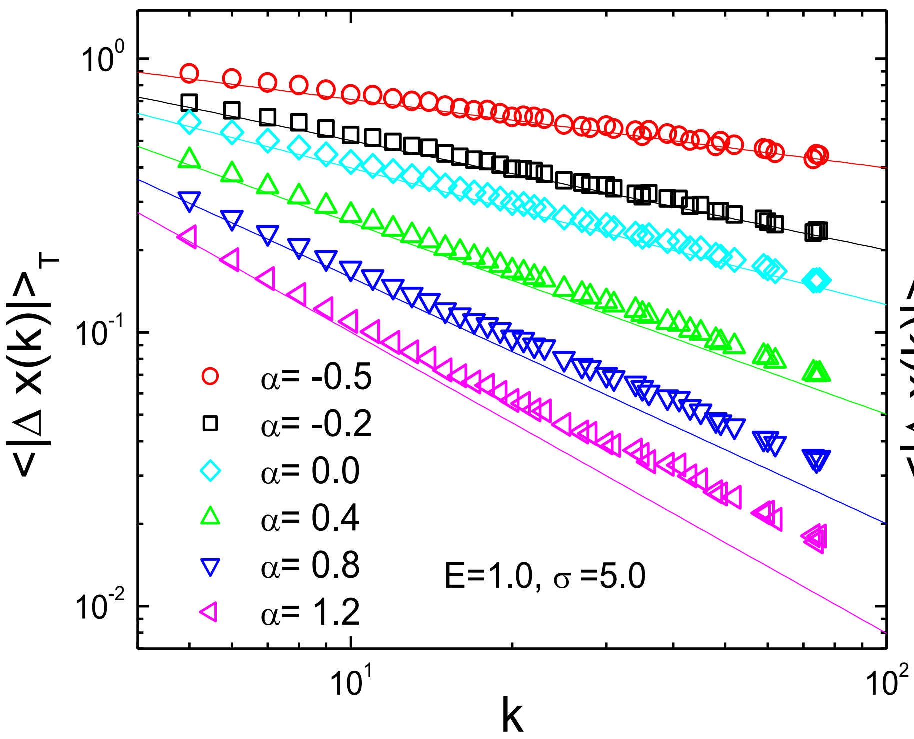
A number of algebraic steps leads to

$$\langle |\Delta x| \rangle_T \approx \frac{\sigma}{\sqrt{\pi c}} k^{-\frac{1+\alpha}{2}}$$

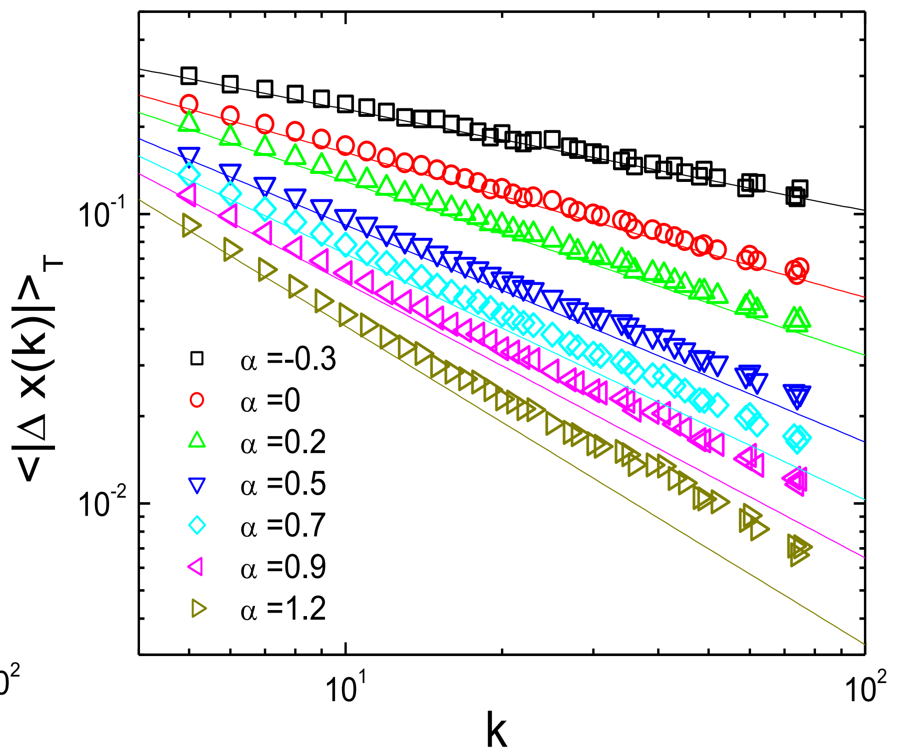
The same scaling law holds for Rössler and Kuramoto dynamics.

Numerical Support

Consensus dynamics

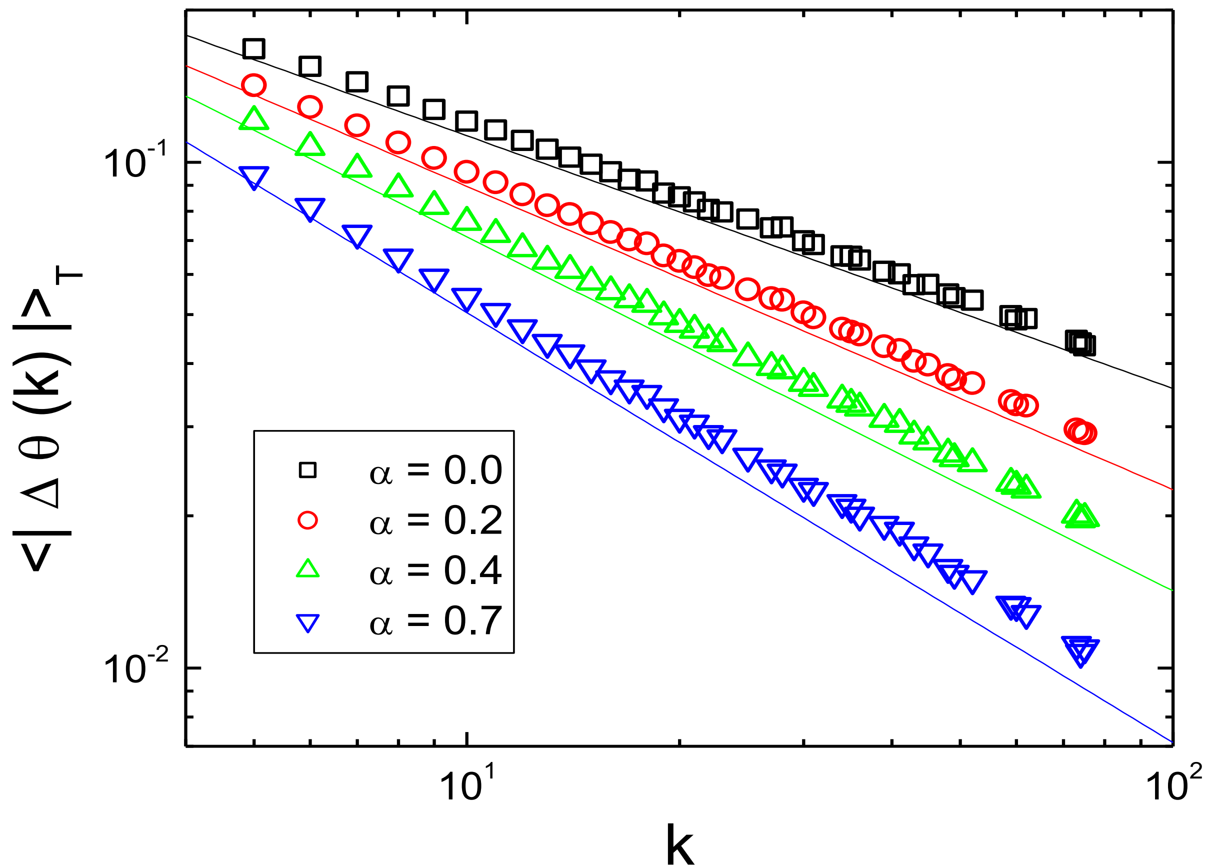


Rössler dynamics



Weighted scale-free networks

Kuramoto Dynamics on Weighted Scale-Free Networks



In all cases, there is robust one-to-one correspondence between time-averaged fluctuations about mean field and node degree!

Conclusions

- Developed two methods for detecting complex networks from time series:

(1) Principal component analysis;

(2) Time-averaged fluctuation about mean field.

- **Universal** scaling law:

  The fluctuation scales inversely with square root of node degree.