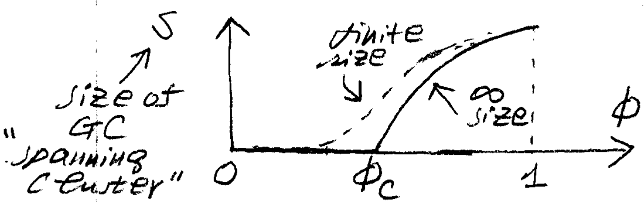
Lecture 18 Percolation

Percolation — removal of some nodes (together with their edges)



ϕ — occupation prob.

ϕ_c — percolation threshold

Applications: ① router failures $\phi_c \to$ small

② vaccination of individuals

$(1-\phi)$ — fraction of vaccinated

$\to \phi_c$ as large as possible

"knock-on effect": vaccinating one $\to$ others are safe

"herd immunity": vaccinating a small fraction $\to$ prevent spread of disease

<u>Analysis</u>

u — ave. probability that a vertex is not connected to GC via a particular neighbor

u^l — prob. that this node does not belong to GC

$$\underbrace{\sum p_l u^l}_{\text{Ave. prob. of node (any) not in GC}} = g_0(u)$$

$1 - g_0(u)$ — prob. that a random node $\in$ GC

Size of GC: $\quad S = \phi\,[1 - g_0(u)]$ $\quad \phi$ — fraction of existent nodes

$u = ?$

① i has been removed $\quad 1-\phi$

② i is present but $i \notin$ GC

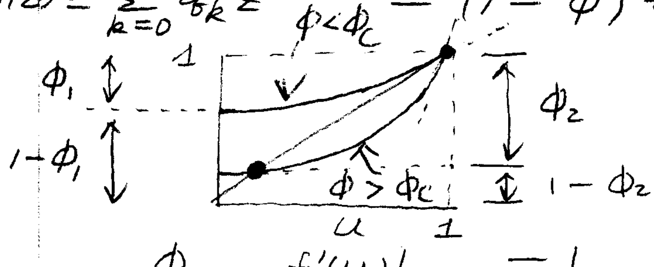
— i not connected to GC via any of its other neighbors (k)

u^k — none is not connected to GC

Total prob. of not connecting to GC: (node i) $\to (1-\phi) + \phi \cdot u^k$ $\quad k$ — excess degree of i

$$\Rightarrow \quad u = \sum_{k=0}^{\infty} q_k\,[(1-\phi) + \phi\, u^k]$$

$$g_1(z) \equiv \sum_{k=0}^{\infty} q_k z^k \qquad = (1-\phi) + \phi \sum_{k=0}^{\infty} q_k u^k = \underbrace{(1-\phi) + \phi g_1(u)}_{f(u)}$$



$f(1) = 1 - \phi + \phi = 1$

$f(0) = 1 - \phi$

ϕ_c: $\quad f'(u)|_{u=1} = 1 \quad \Rightarrow \quad \phi_c\, g_1'(1) = 1$

$$\phi_c = \frac{1}{g_1'(1)}$$

$$g_1'(z) = \sum k q_k z^{k-1}$$

$$g_1'(1) = \sum k q_k = \frac{1}{\langle k \rangle} \sum_{k=0}^{\infty} k(k+1) p_{k+1}$$

$$= \frac{1}{\langle k\rangle} \sum_{k=0}^{\infty} k(k-1)\, p_k = \frac{\langle k^2\rangle - \langle k\rangle}{\langle k\rangle}$$

$$\Rightarrow \quad \phi_c = \frac{\langle k\rangle}{\langle k^2\rangle - \langle k\rangle}$$

Say want to reduce ϕ_c
$\Rightarrow$ making $\langle k^2\rangle \gg \langle k\rangle$
– SF networks

Random Nets: $p_k = e^{-c}\frac{c^k}{k!}$
$\langle k\rangle = c \quad \langle k^2\rangle = c(c+1)$
$\Rightarrow \phi_c = 1/c$

say $c = 4$
$\Rightarrow \phi_c = 0.25$
$\Rightarrow$ >75% nodes failing
$\Rightarrow$ GC is gone

SF $\quad k^{-\alpha} \quad$ Internet $\alpha \approx 2.5$
$\langle k\rangle$ finite $\quad \langle k^2\rangle \to \infty$
$\Rightarrow \quad \phi_c = 0 \quad \Rightarrow$ GC always exists no matter how many nodes are removed
Absurd! $\leftarrow$

$\langle k^2\rangle$ is never ∞ for any finite network

Consider a specific form of $g_1(u)$ with the requirement: $\quad g_1(u) = 1 - C_0(1-u)^{\beta}$, $\quad 0<\beta<1$

$g_1'(1) = \frac{\langle k^2\rangle - \langle k\rangle}{\langle k\rangle} \to \infty$

$g_1'(1) \Rightarrow \infty \qquad \Rightarrow g_1'(u) = C_0\beta(1-u)^{\beta-1}$

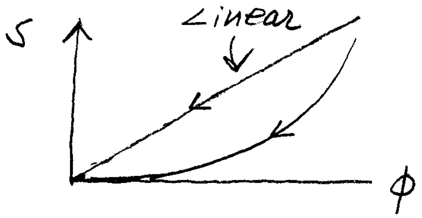
$$\Rightarrow \quad u = 1 - \phi + \phi\, g_1(u)$$
$$= 1 - C_0\,\phi\,(1-u)^{\beta}$$
$$1 - u = (C_0\phi)^{1/(1-\beta)}$$

Now consider S $\qquad g_0(u) \cong g_0(1) + g_0'(1)\,(u-1) \quad$ (for $u \approx 1$)
$$= 1 + \langle k\rangle\,(u-1)$$

$$S = \phi\,[1 - g_0(u)] \approx \phi\,\langle k\rangle\,(1-u) \sim \phi^{1+\frac{1}{1-\beta}}$$

$$\Rightarrow \quad S \sim \phi^{\frac{2-\beta}{1-\beta}} \qquad \frac{2-\beta}{1-\beta} > 1 \quad (0<\beta<1)$$

As $\phi \to 0$, $\quad S \to 0$ faster than linear behavior for SF networks



$\Rightarrow$ GC exists yes
but its size is exceedingly small for small values of ϕ

Disease spreading: $\quad$ threshold $\to 0$

Explosive Percolation in Random Networks

Science **323**, 1454 (2009)

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Networks in which the formation of connections is governed by a random process often undergo a percolation transition, wherein around a critical point, the addition of a small number of connections causes a sizable fraction of the network to suddenly become linked together. Typically such transitions are continuous, so that the percentage of the network linked together tends to zero right above the transition point. Whether percolation transitions could be discontinuous has been an open question. Here, we show that incorporating a limited amount of choice in the classic Erdös-Rényi network formation model causes its percolation transition to become discontinuous.

A selection rule is classified as "bounded-size" if its decision depends only on the sizes of the components containing the four end points of $\{e_1,e_2\}$ and, moreover, it treats all sizes greater than some (rule-specific) constant K identically. For example, a bounded-size rule with $K = 1$ due to Bohman and Frieze (BF) (*3*), the first selection rule to be analyzed, proceeds as follows: If e_1 connects two components of size 1, it is selected; otherwise, e_2 is selected. So, in Fig. 1B, e_2 would be selected. Bounded-size rules, in general, are amenable to rigorous mathematical analysis, and in (*3*, *4*) it was proven that such rules are capable both of delaying and of accelerating percolation. In contrast, unbounded-size rules seem beyond the reach of current mathematical techniques. A crucial point is that the percolation transition is strongly conjectured to be continuous for all bounded-size rules (*4*). This conjecture is supported both by numerical evidence and mathematical considerations, though a fully rigorous argument has remained elusive.

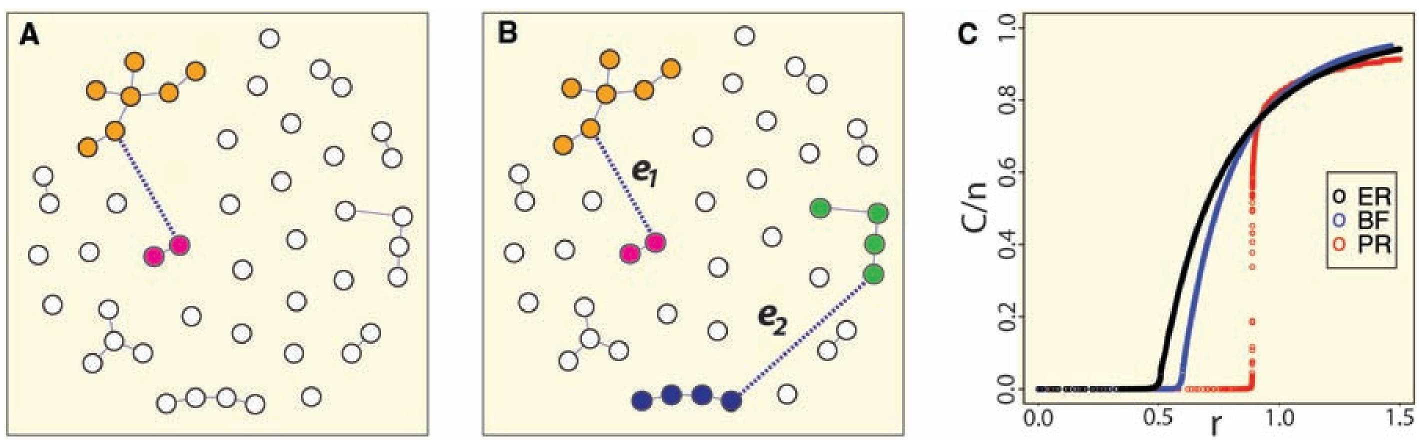

Fig. 1. Network evolution. (**A**) Under the Erdös-Rényi (ER) model, in each step two vertices are chosen at random and connected by an edge (shown as the dashed line). In this example, two components of size 7 and 2 get merged. (**B**) In models with choice, two random edges $\{e_1,e_2\}$ are picked in each step yet only one is added to the network based on some selection rule, whereas the other is discarded. Under the product rule (PR), the edge selected is the one minimizing the product of the sizes of the components it merges. In this example, e_1 (with product $2 \times 7 = 14$) would be chosen and e_2 discarded (because $4 \times 4 = 16$). In contrast, the rule selecting the edge minimizing the sum of the component sizes instead of the product would select e_2 rather than e_1. (**C**) Typical evolution of C/n for ER, BF (a bounded size rule with $K = 1$), and PR, shown for $n = 512{,}000$.

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The "weak" interdependence of infrastructure systems produces mixed percolation transitions in multilayer networks

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Previous studies of multilayer network robustness model cascading failures via a node-to-node percolation process that assumes "strong" interdependence across layers–once a node in any layer fails, its neighbors in other layers fail immediately and completely with all links removed. This assumption is not true of real interdependent infrastructures that have emergency procedures to buffer against cascades. In this work, we consider a node-to-link failure propagation mechanism and establish "weak" interdependence across layers via a tolerance parameter α which quantifies the likelihood that a node survives when one of its interdependent neighbors fails. Analytical and numerical results show that weak interdependence produces a striking phenomenon: layers at different positions within the multilayer system experience distinct percolation transitions. Especially, layers with high super degree values percolate in an abrupt manner, while those with low super degree values exhibit both continuous and discontinuous transitions. This novel phenomenon we call mixed percolation transitions has significant implications for network robustness. Previous results that do not consider cascade tolerance and layer super degree may be under- or over-estimating the vulnerability of real systems. Moreover, our model reveals how nodal protection activities influence failure dynamics in interdependent, multilayer systems.