Lecture 17  Configuration model & Excess degree distribution

your friends have more friends than you do on average — why?

1. \( k_i \) — Given \( \Sigma k_i = 2m \)

Choosing two stubs at random & linking them creates all 2m stubs have been lined up \( \Rightarrow \) Ensemble of networks \( G(n,m) \)

Recall \( P_k \) — probability that a node chosen at random has degree \( k \)

Following an edge \( \Rightarrow \) (2m-1) stubs to reach

Prob. of ending at a node of degree \( k \) = \( \frac{k}{(2m-1)} \)

Consequence?

Ave. degree of neighbor = \( \Sigma k \frac{P_k}{\langle k \rangle} \) = \( \frac{k^2}{\langle k \rangle} \) — more likely to reach a high-degree node by following any one of the \( k \) edges

\[ <k^2> - 2nd \text{ moment} \]

Reason: when going through the nodes & averaging the degrees of the neighbors of each node, many of these neighbors appear in more than one average (average over a node of degree \( k \))

High-degree node \( \rightarrow \) over represented

Excess degree \( k-1 \)

\( \Rightarrow \frac{q_k}{\langle k \rangle} = \frac{(k+1)P_{k+1}}{\langle k \rangle} \)

Generating functions

\[ g_0(z) = \sum_k P_k z^k \]

\[ g_1(z) = \frac{1}{\langle k \rangle} \sum_{k=0}^{\infty} (k+1)P_{k+1} z^k = \frac{1}{\langle k \rangle} \sum_{k=0}^{\infty} kP_k z^{k-1} \]

\[ g_1(z) = \frac{g_0'(z)}{g_0(z)} \]

2nd neighbors

\( \Rightarrow \frac{\# \text{ of 2nd neighbors}}{\# \text{ of degree of 1st neighbors}} \)
Prob. of having exactly $k$ neighbors given that there are $l$ first neighbors.

$$P_k^{(2)} = \frac{\delta_{k,l}}{\delta_0} \prod_{r=1}^{\infty} q_r^{\delta_{0,r}} \delta(k, \sum_{r=1}^{\infty} \delta_{0,r})$$

$$P^{(2)}(k|l) = \sum_{\delta_0=0}^{\infty} \cdots \sum_{\delta_{l-1}=0}^{\infty} \prod_{r=1}^{\infty} q_r^{\delta_{0,r}} \delta(k, \sum_{r=1}^{\infty} \delta_{0,r})$$

$$Z_k^{(2)} = \sum_{\delta_0=0}^{\infty} \cdots \sum_{\delta_{l-1}=0}^{\infty} \delta(k, \sum_{r=1}^{\infty} \delta_{0,r}) \prod_{r=1}^{\infty} q_r^{\delta_{0,r}}$$

$$G_E \quad \theta_2(z) = \sum_{\delta_0=0}^{\infty} \prod_{r=1}^{\infty} q_r^{\delta_{0,r}} \delta(k, \sum_{r=1}^{\infty} \delta_{0,r})$$

$$= \sum_{\delta_0=0}^{\infty} \sum_{\delta_1=0}^{\infty} \cdots \sum_{\delta_{k-1}=0}^{\infty} \prod_{r=1}^{\infty} q_r^{\delta_{0,r}} \delta(k, \sum_{r=1}^{\infty} \delta_{0,r}) \delta_0^k \delta_1^j \cdots \delta_{k-1}^l$$

$$= \delta_k^{(2)} z_k = \cdots = \sum_{\delta_0=0}^{\infty} \theta_2(\sum_{r=1}^{\infty} \delta_{0,r}) \delta_0^k \delta_1^j \cdots \delta_{k-1}^l$$

Generalizing:

$$P_k^{(3)} : \quad \theta_3(z) = \theta_0(\theta_2(\theta_1(z))) = \theta_2(\theta_1(z))$$

$$\theta_0(z) = \theta_{d-1}(\theta_1(z))$$

What is the ave. member of neighbors at distance $d$?

$$d = 2 : \quad \theta_2(z) \big| z = 1 = \theta_0(\theta_1(z)) \theta_1(\theta_1(z)) \big| z = 1$$

$$\theta_1(z) = \theta_0'(z) / \theta_0'(1)$$

$$\Rightarrow C_2 = \frac{<k^2>-<k>}{<k>^2}$$

Arbitrary value of $d$

$$C_d = \left( \frac{C_2}{C_1} \right)^{d-1} \cdot C_1$$

$$C_2 > C_1 \Rightarrow \text{Giant component}$$

or

$$<k^2> - <k> > <k> \Rightarrow <k^2> > 2<k>$$

**Clustering Coefficient $C$**

$$C = \frac{\sum_{k_i=0}^{\infty} q_{k_i}^2 q_{k_0}^2 k_i k_0}{2m} = \frac{1}{2m} \left( \sum_{k=0}^{\infty} k q_{k}^2 \right)^2$$

$$= \frac{1}{2m<k^2>^2} \left[ \sum_{k=0}^{\infty} k(k+1) q_{k+1} \right]^2 = \frac{1}{2m<k^2>^2} \left[ \sum_{k=0}^{\infty} (k-1) k q_k \right]^2$$

$$= \frac{1}{\frac{<k^2>^2}{<k>^3}} \left( <k^2> - <k> \right)^2$$

Random Net: Poisson $\rightarrow$ $C \sim \frac{1}{n} \rightarrow 0$

Scale Free Net: Power-law degree dist.

$$C \rightarrow \frac{<k^2>}{n} \rightarrow \text{constant}$$

if $<k^2> \text{ diverges}$

($d < 3$)