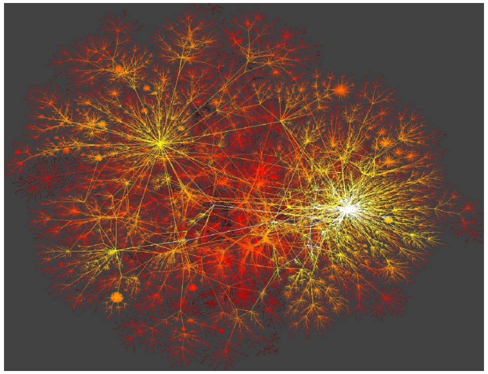
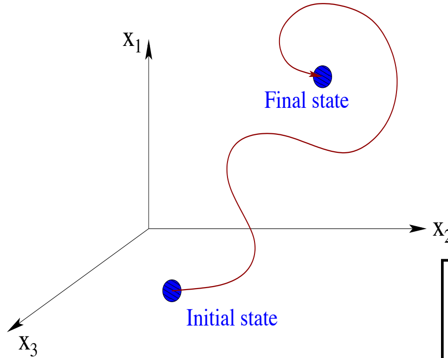
Linear Network Control





A dynamical system is controllable if it can be driven from **any** initial state to **any** desired final state **in finite time** by suitable choice of input control signals.

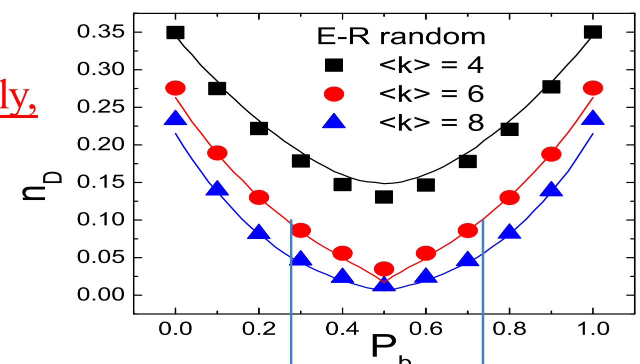
General Mathematical framework: **Kalman's Controllability Rank Condition**

Focus of existing works: **minimal number of signals** required to control the network

e.g., Structural controllability: Y.-Y. Liu, J.-J. Slotine, and A.-L. Barabasi, "Controllability of complex networks," *Nature* **473**, 167 (2011)
Exact controllability: Z.-Z. Yuan, C. Zhao, Z.-R. Di, W.-X. Wang, and Y.-C. Lai, "Exact controllability of complex networks," *Nat. Commun.* **4**, 2447 (2013)

Issue: controllability is mathematically well defined but physically, control may be difficult

In terms of **ENERGY**



High probability of energy divergence

Energy bounds: G. Yan, J. Ren, Y.-C. Lai, C. H. Lai, B. Li, "Controlling complex networks: how much energy is needed?" *Phys. Rev. Lett.* **108,** 218703 (2012).
Energy scaling: Y.-Z. Chen, L.-Z. Wang, W.-X. Wang, and Y.-C. Lai, "Energy scaling and reduction in controlling complex networks," *Roy. Soc. Open Sci.* **3**, 160064 (2016).
Physical controllability: L.-Z. Wang, Y.-Z. Chen, W.-X. Wang, and Y.-C. Lai, "Physical controllability of complex networks," *Sci. Rep.* **7**, 40198 (2017).

Optimal Control Input and Energy

$$\mathbf{u}_t = \mathbf{B}^T \cdot \exp[\mathbf{A}^T (T_f - t)] \cdot \mathbf{W}^{-1}_{T_f} \cdot \mathbf{v}_{T_f}, \text{ where}$$

$$\mathbf{W}_{T_f} \equiv \int_0^{T_f} \exp(\mathbf{A}t) \cdot \mathbf{B} \cdot \mathbf{B}^T \cdot \exp(\mathbf{A}^T t)\, dt \quad \text{and}$$

$$\mathbf{v}_{T_f} \equiv \mathbf{x}_{T_f} - \exp(\mathbf{A}T_f) \cdot \mathbf{x}_0$$

$$\text{Energy} \quad \varepsilon(T_f) \equiv \int_0^{T_f} \|\mathbf{u}_t\|^2 dt = \mathbf{v}^T_{T_f} \cdot \mathbf{W}^{-1}_{T_f} \cdot \mathbf{v}_{T_f}$$

W - Symmetric **Gramian matrix** (positive definite if system is controllable)

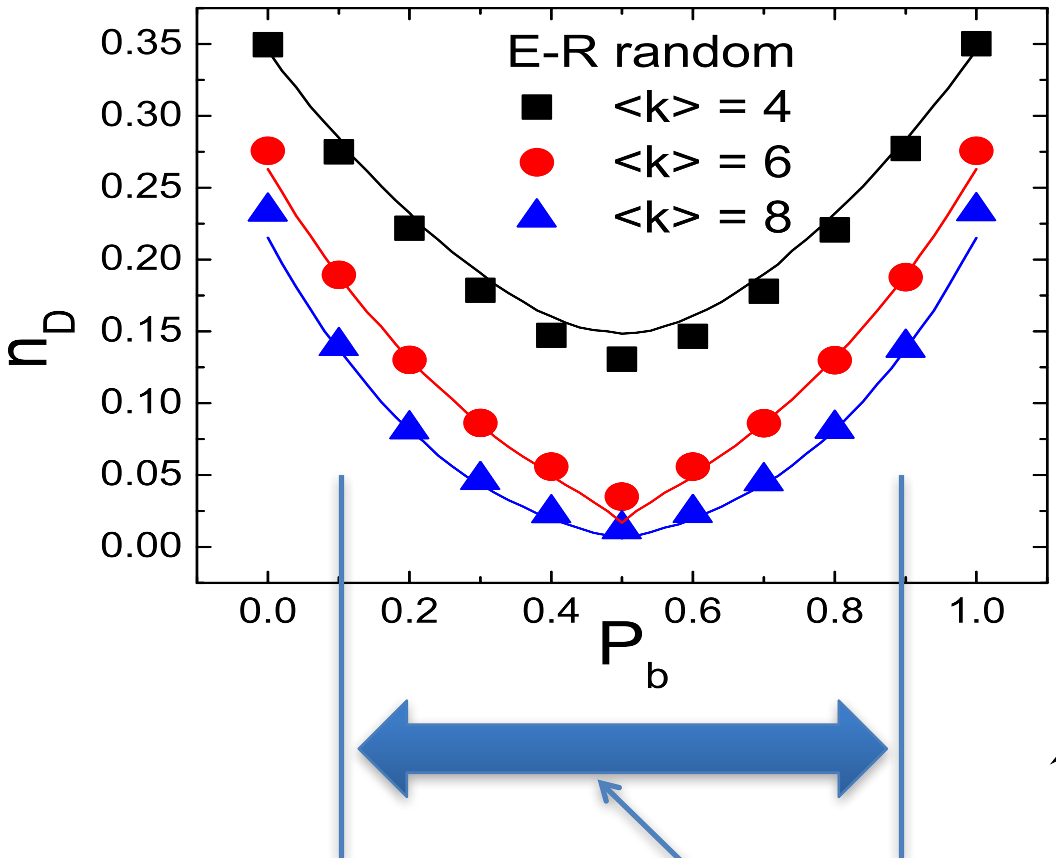
$$\mathbf{H}(T_f) \equiv \exp(-\mathbf{A}T_f) \cdot \mathbf{W}_{T_f} \cdot \exp(-\mathbf{A}^T T_f)$$

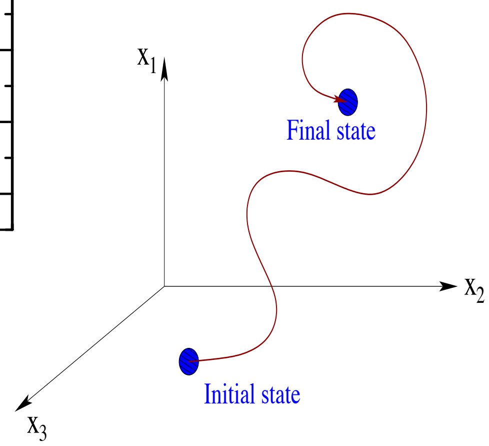
<u>Normalized energy cost</u>

$$E(T_f) \equiv \varepsilon(T_f)/\|\mathbf{x}_0\|^2 = \frac{\mathbf{x}_0^T \cdot \mathbf{H}^{-1} \cdot \mathbf{x}_0}{\mathbf{x}_0^T \cdot \mathbf{x}_0}$$

W. J. Rugh, *Linear System Theory* (2nd ed.) (Prentice-Hall, NJ, 1996)

Is Control Physically Achievable?





Our finding: High probability of divergence in required energy to achieve control

Physical Controllability

ε – measurement error or computer roundoff

Consider linear equation: $W \cdot X = Y$

C_W - condition number of W

Say $e_X = 10^{-k}$ - accuracy of solution of X

Then $e_x \geq C_W \cdot \varepsilon$ (Strang, *Linear Algebra and Its Applications*, AP, 1976)

$$\Rightarrow C_W \leq e_x / \varepsilon \equiv \bar{C}_W$$

$\Rightarrow$ A linear system can be physically controlled with accuracy e_X only for

$$C_W \leq \bar{C}_W$$

Physical controllability can be characterized by

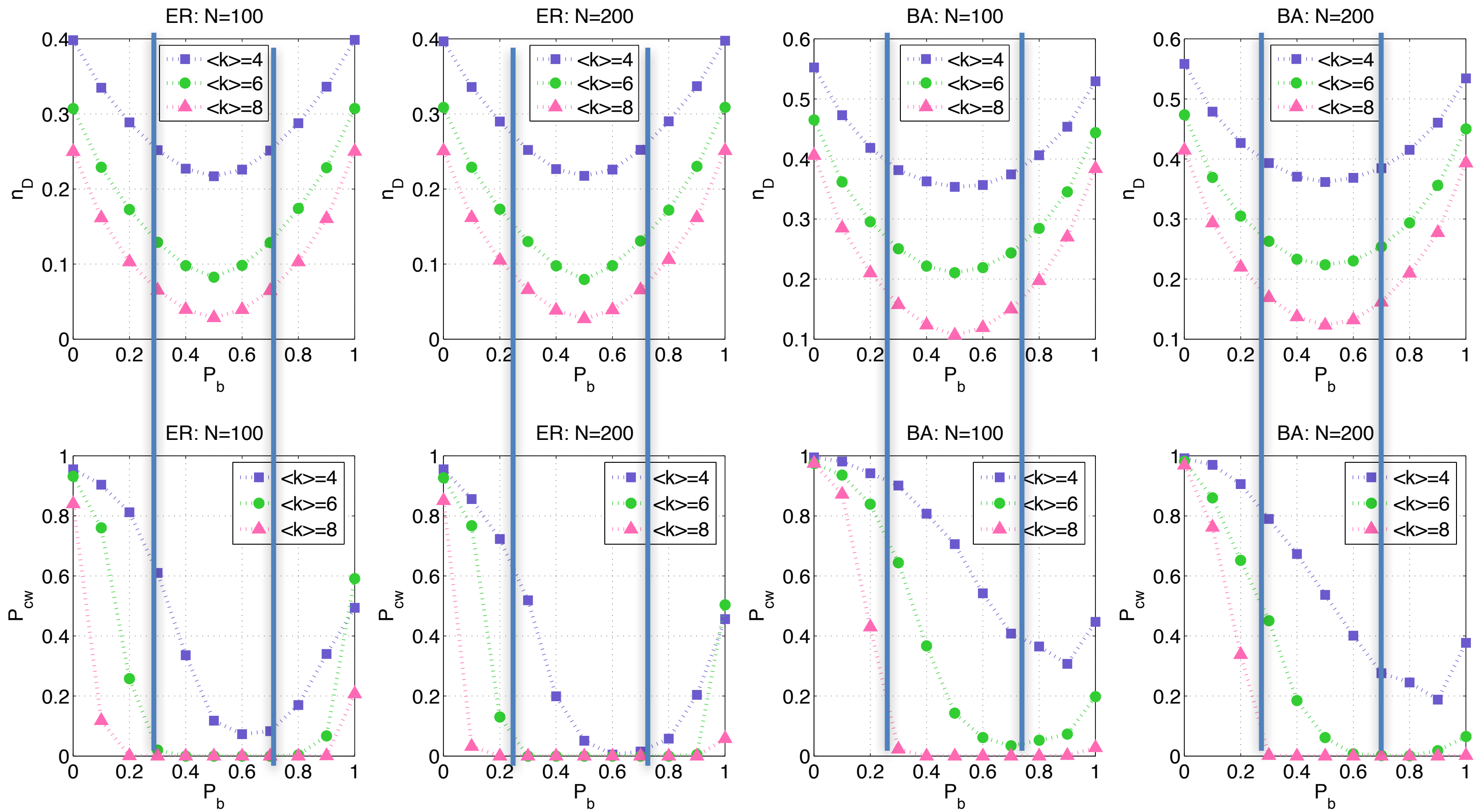
$P(\bar{C}_W)$ - probability that a network with its conditional number less than $\bar{C}_W$

$P(\bar{C}_W) <\approx 1$ - network is physically controllable

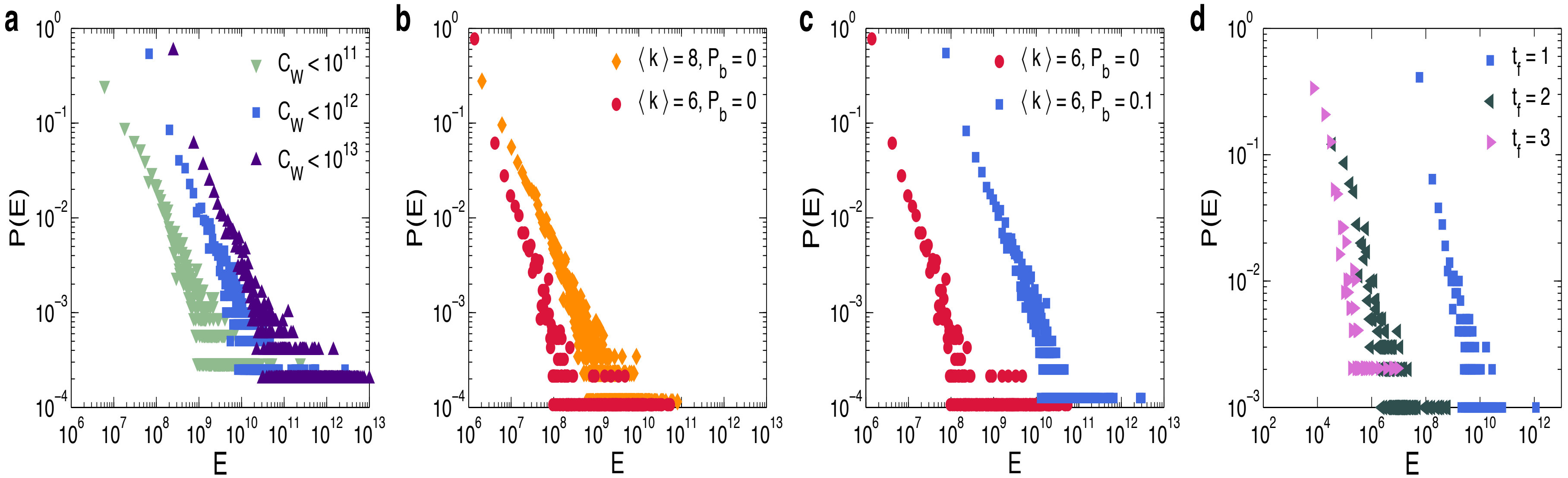
$P(\bar{C}_W) \cong 0$ - network is physically uncontrollable

L.-Z. Wang, Y.-Z. Chen, W.-X. Wang, and Y.-C. Lai, "Physical controllability of complex networks," *SREP* **7**, 40198 (2017).

Structural versus Physical Controllability



Energy Scaling for Physically Controllable Networks



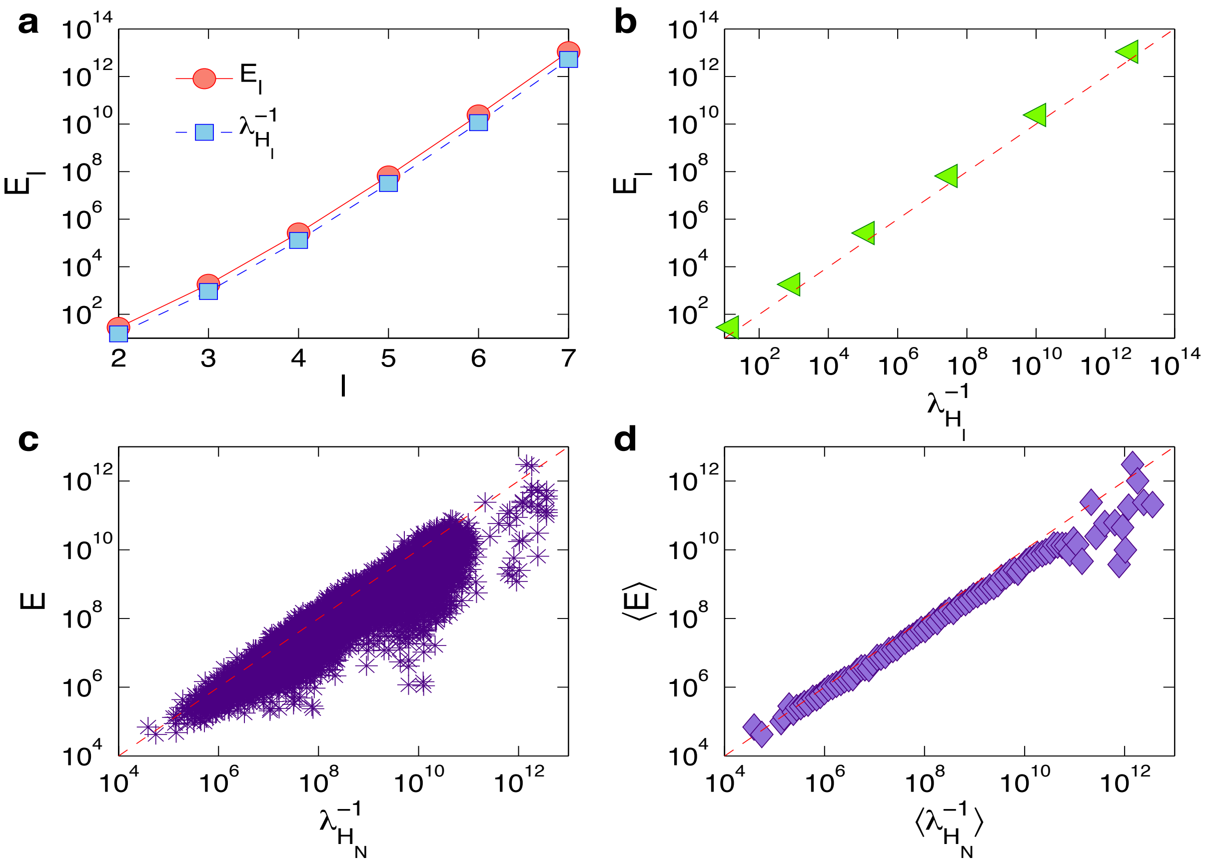
$$P(E) \propto E^{-\alpha}$$

There are networks that require an enormous amount of energy to be controlled!

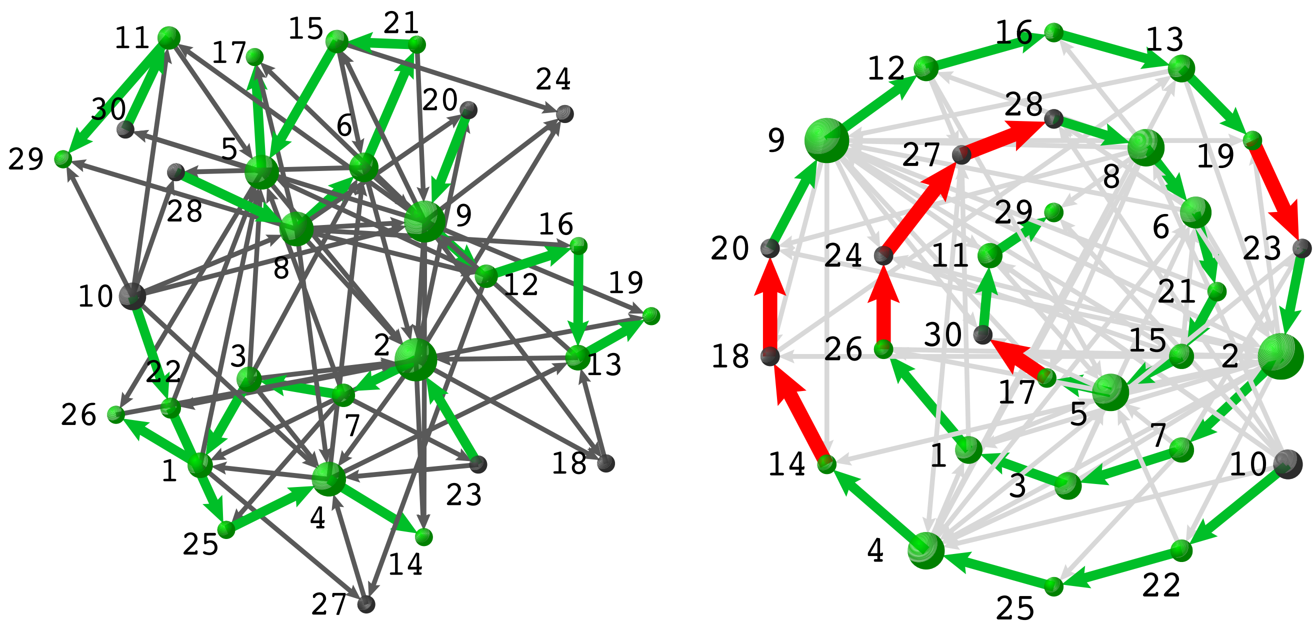
Control Energy: One-Dimensional Chain versus Random Networks

One-dimensional chain of length l : $\mathrm{E} \cong \lambda_H^{-1}$

(can be derived analytically)



Maximum Matching – An Example



(a) (b)

Green edges – maximum matching

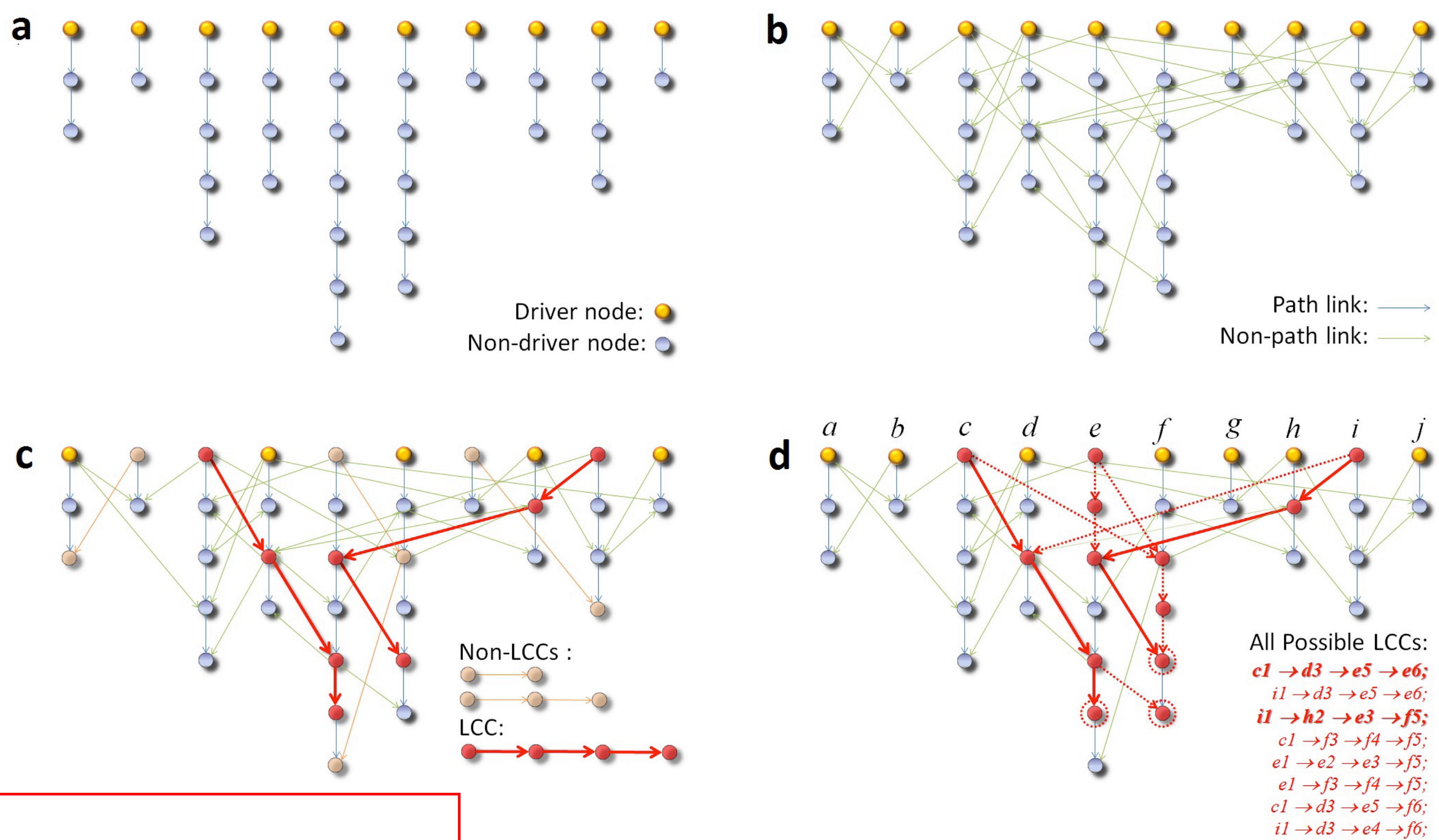
Green nodes – matched nodes

Gray nodes – unmatched nodes – control nodes

Red edges – added edges to realize perfect matching

W.-X. Wang, X. Ni, Y.-C. Lai, and C. Grebogi, Phys. Rev. E 85, 025115 (2012)

Elements for Power-Law Energy Scaling – Longest Control Chains (LCCs)



$$E \approx m \cdot E_L$$

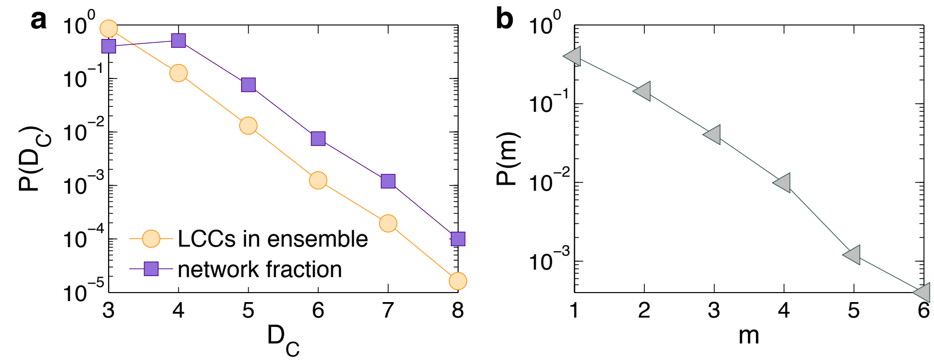
E_L - Energy required to control an LCC

m - # of LCCs of identical length (degeneracy)

Power-Law Energy Scaling - Theory

Control diameter D_C = length of LCC

Distribution of D_C : $P(D_C) \propto e^{-b \cdot D_C}$



Exponential dependence of E_L on D_C

$$E_L \propto e^{\beta \cdot D_C}$$

$$\Rightarrow P(E_L) \propto E_L^{-(1 + b/\beta)}$$

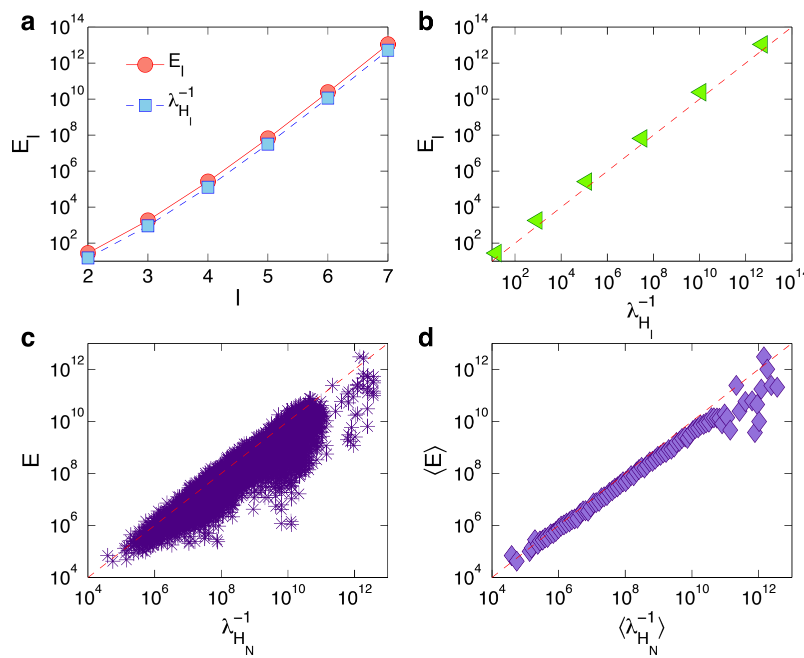
$E \approx m \cdot E_L \Rightarrow$ Cumulative distribution of E

$$F(E) = P(m \cdot E_L < E) =$$

$$\int_0^\infty \left[\int_0^{E/E_L} P(E_L, m) \cdot dm \right] dE_L$$

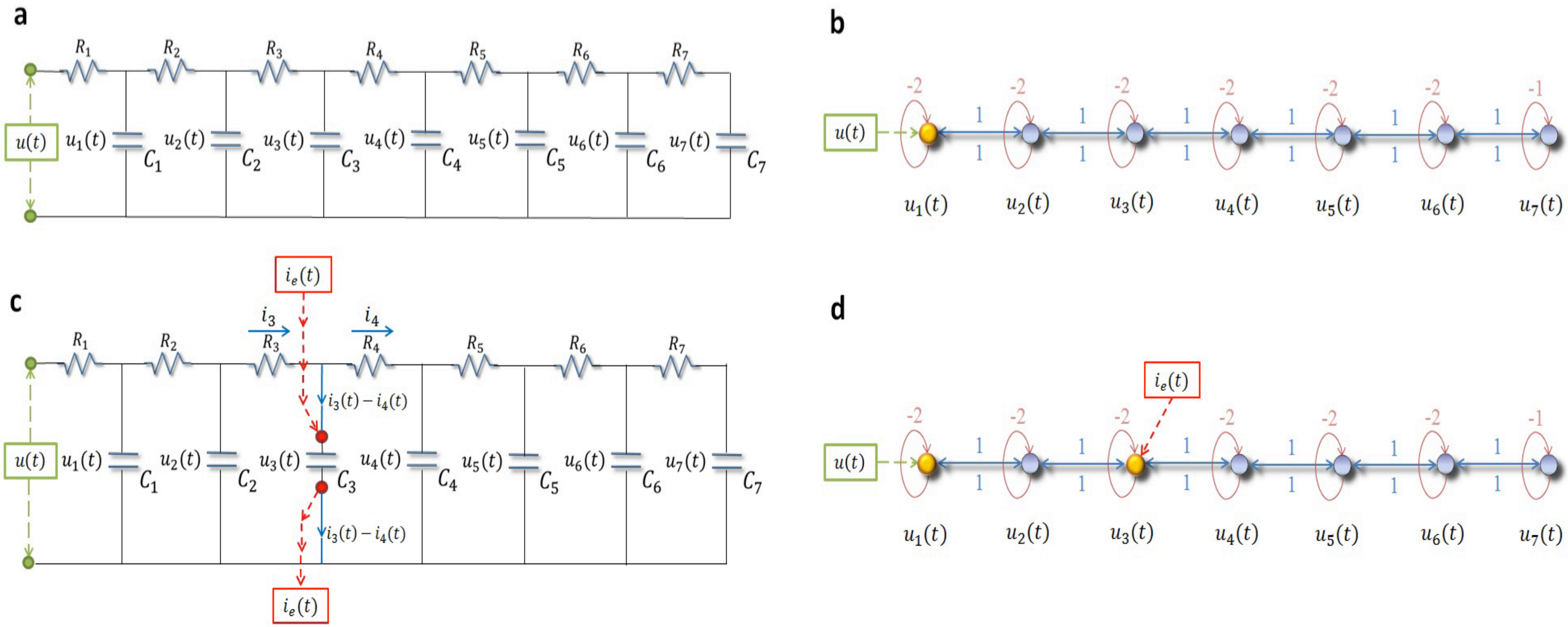
$$\approx \int_0^\infty P(E_L) \left[\int_0^{E/E_L} P(m) dm \right] dE_L$$

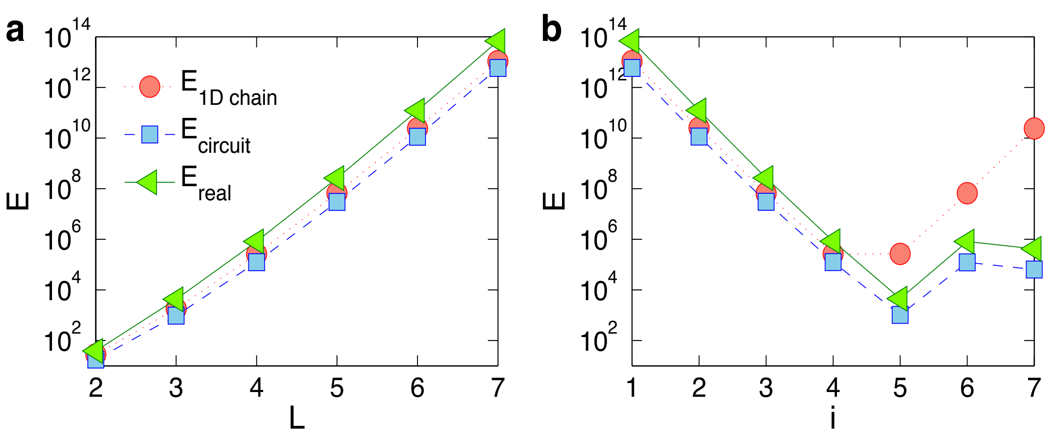
$$P(m) \propto e^{-gm}$$



$$\text{Finally} \quad P(E) = \frac{dF(E)}{dE} \propto E^{-(1+b/\beta)}$$

Energy Reduction in a Linear Circuit Network





Y.-Z. Chen, L.-Z. Wang, W.-X. Wang, and Y.-C. Lai, "Energy scaling and reduction in controlling complex networks," *Royal Society Open Science* **3**, 160064 (2016).

Nonlinear Dynamical Systems 101

Nonlinear dynamical system:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}),$$

where $\mathbf{f}(\mathbf{x})$ is a nonlinear vector function of $\mathbf{x}$ which cannot be written in the form $\mathbf{A} \cdot \mathbf{x}$.

Example from mechanics: particle motion in a 1D potential field with the potential function

$$V(x) = \frac{x^2}{2} - \frac{x^4}{4}$$

Local minima or maxima:

$$\frac{dV}{dx} = x - x^3 = 0 \rightarrow x = 0, \pm 1$$

Newton's equation of motion (assuming unit mass)

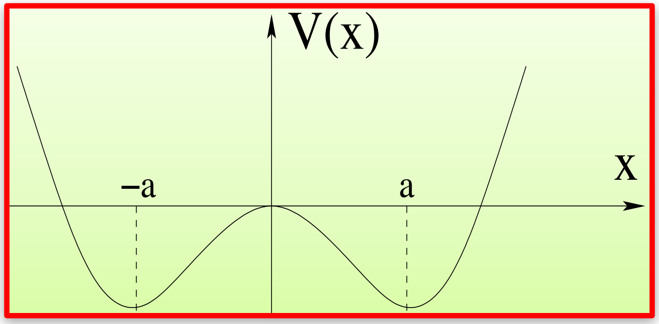
$$\frac{dv}{dt} = -\alpha v - \frac{dV}{dx} = -\alpha v - x + x^3$$

where $\alpha > 0$ is the friction coefficient.

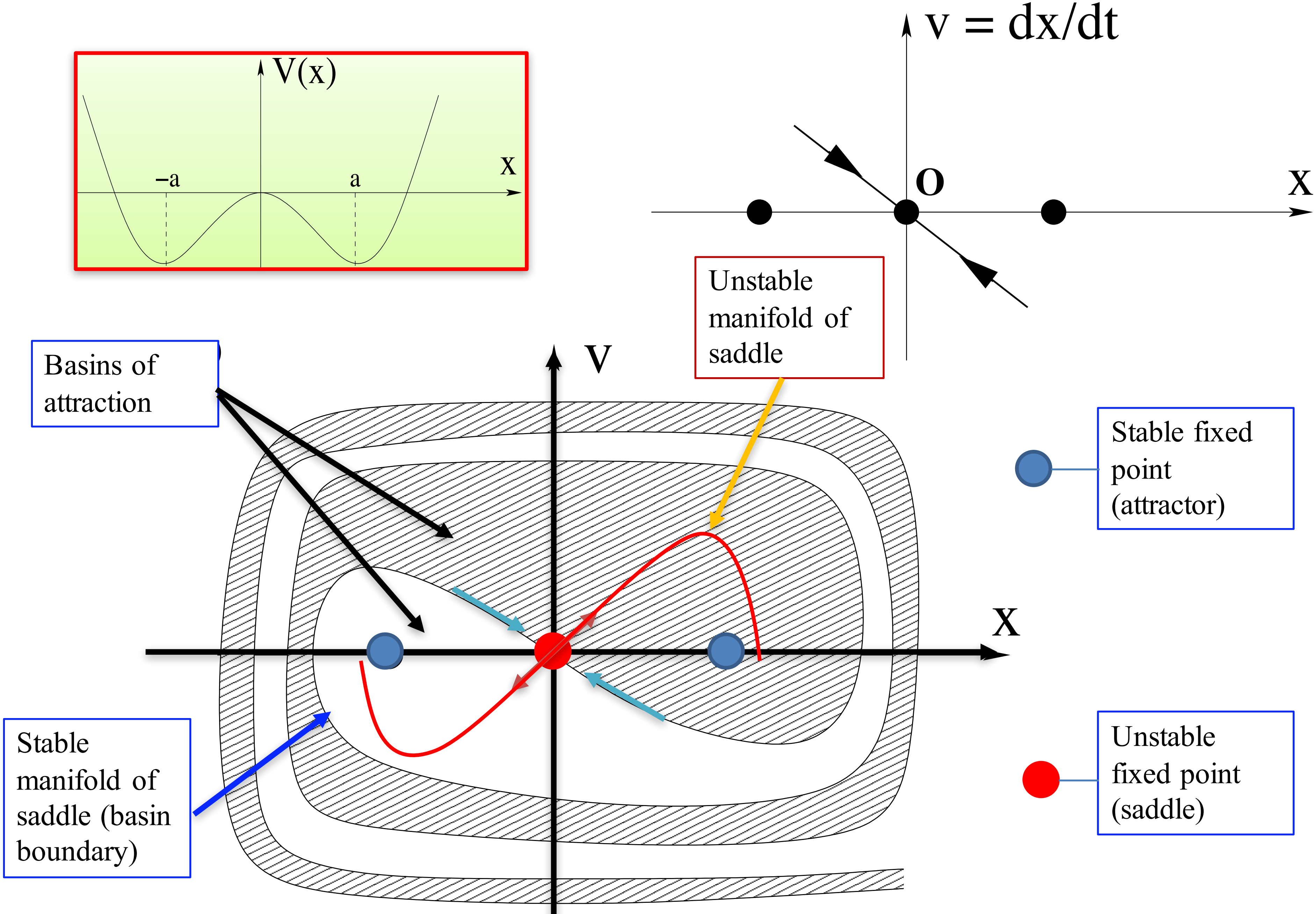
Letting $x_1(t) \equiv x(t)$ and $x_2(t) \equiv v(t)$, one has

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = -\alpha x_2 - x_1 + x_1^3$$



Nonlinear Dynamical Systems 101 – Phase Space Behaviors

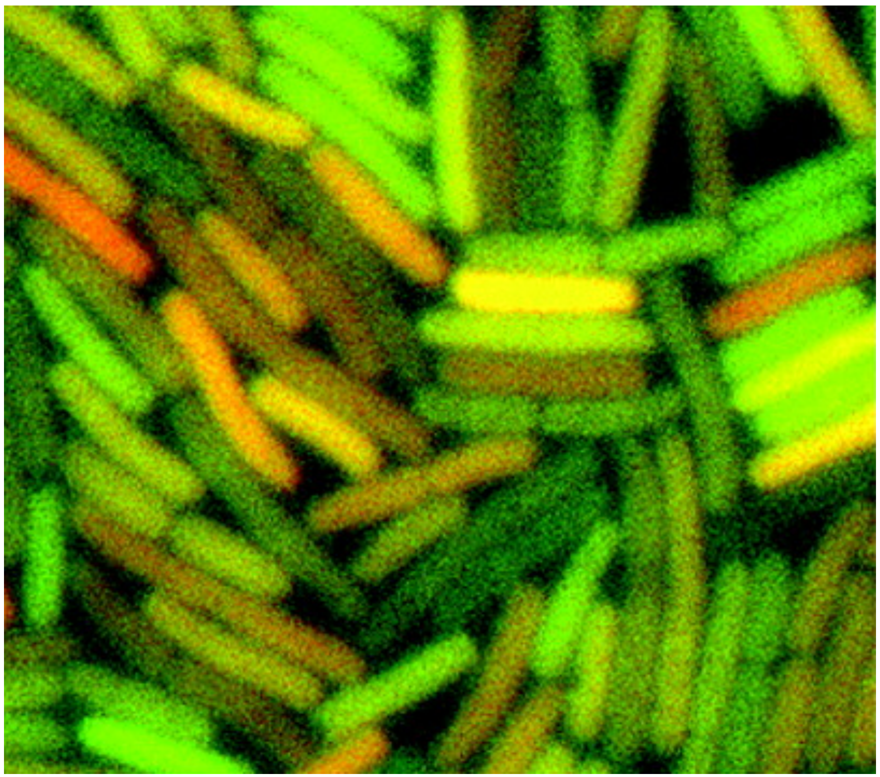


Synthetic Biology

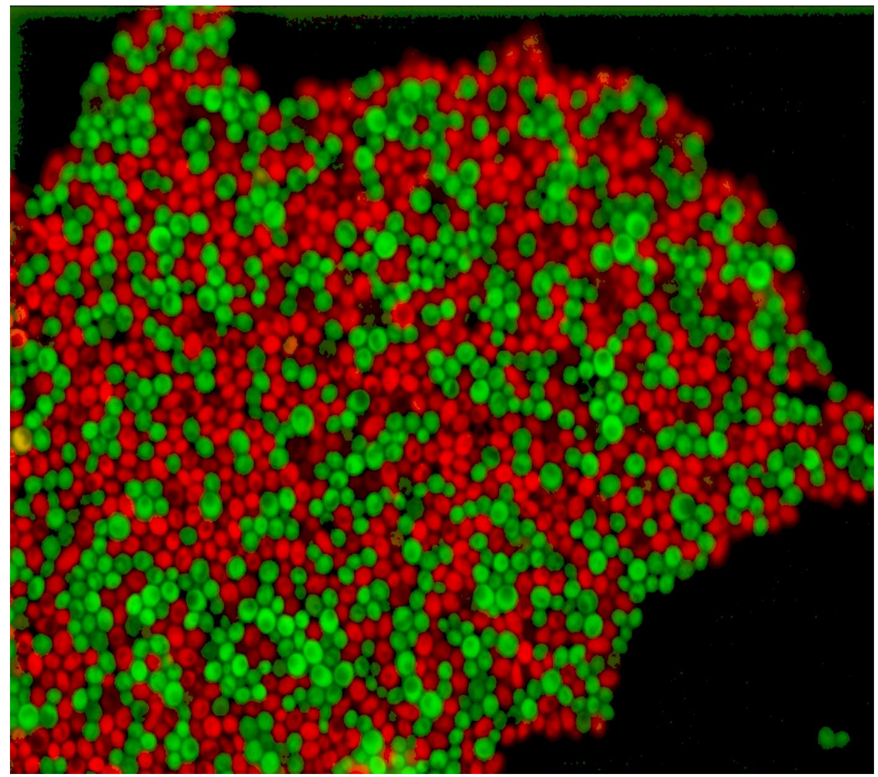
- Synthetic Biology applies engineering principles to build controllable and predictable cellular networks and behaviors
- Biotech applications: bio-energy, vaccines, petrochemical substitutes, etc.

Cell Fates

Cell-cell variability studied using synthetic gene networks



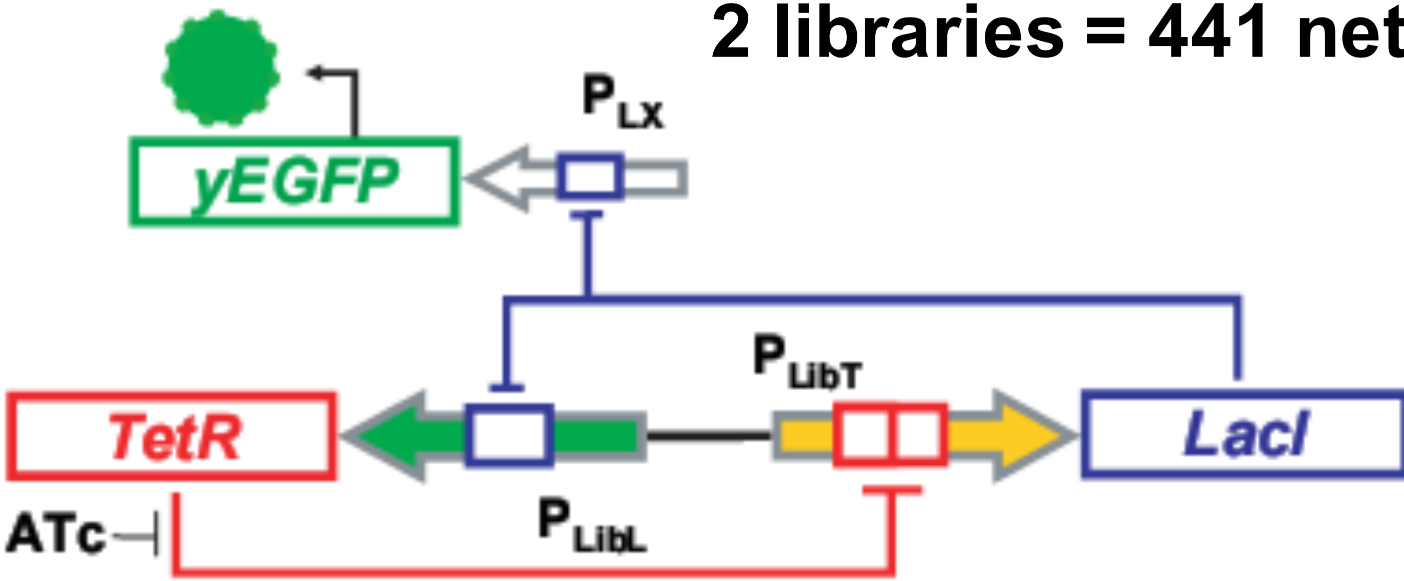
Elowitz *et al*, 2002



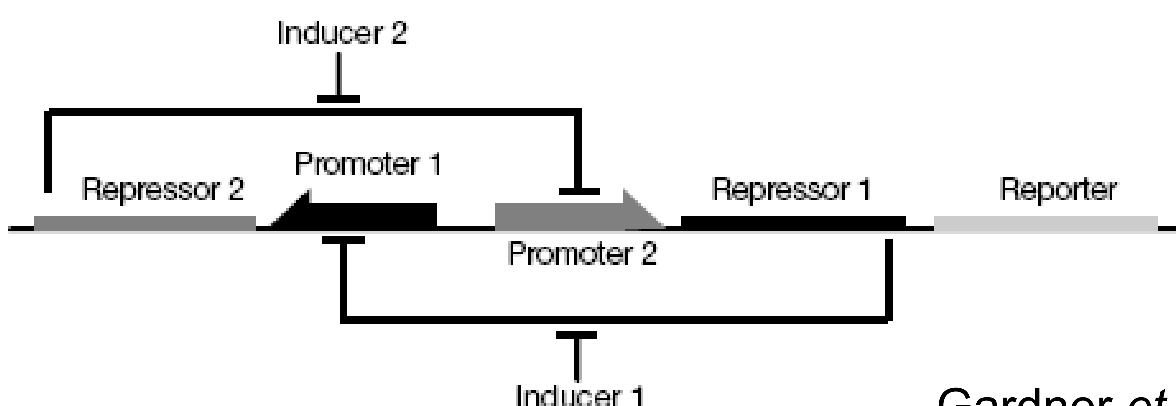
Wu *et al*, 2013

Yeast Toggle Switch

2 libraries = 441 networks

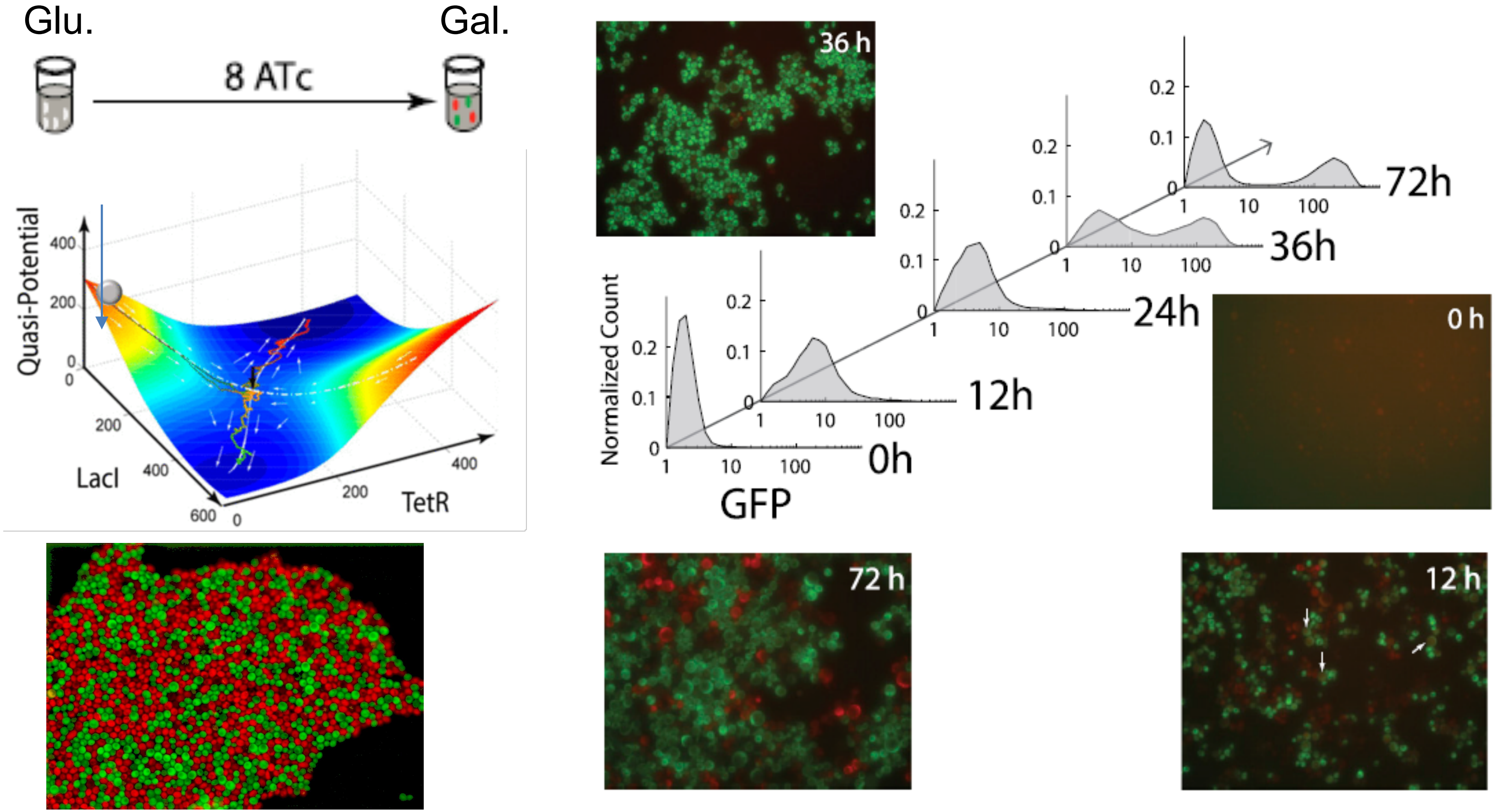


Ellis & **Wang** *et al*,
Nature Biotech, 2009



Gardner *et al.*, Nature, 2000

Experimental Result: Multistability



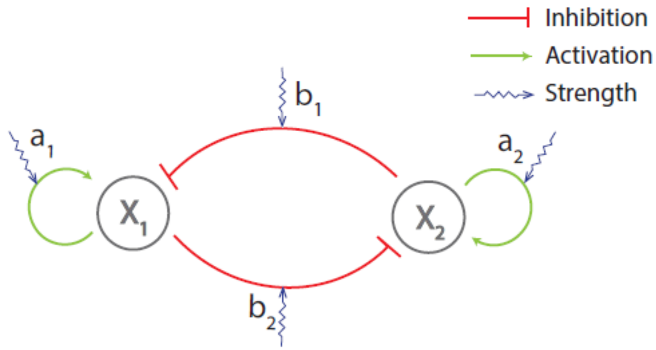
- M. Wu, R.-Q. Su, X.-H. Li, T. Ellis, Y.-C. Lai, and X. Wang, "Engineering of regulated stochastic cell fate determination," *PNAS* **110**, 10610-10615 (2013).
- F.-Q Wu, R.-Q. Su, Y.-C. Lai, and X. Wang, "Engineering of a synthetic quadrastable gene network to approach Waddington landscape and cell fate determination," *eLIFE* **6**, e23701 (2017).

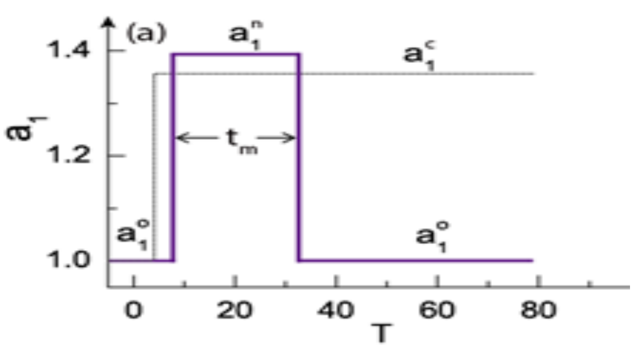
Gene Regulatory Networks

- Directed connections between nodes:
  - Inhibition or excitation
- Coupling strengths can be adjusted by, e.g., application of drugs
- Only stable steady states can be observed.

How to drive the system from an initial attractor ***to a desirable attractor*** by changing some ***experimentally accessible parameters*** ?

Applying temporal perturbations to parameters a_1, a_2, b_1 or b_2





$$\dot{x_1} = a_1 \cdot \frac{x_1^n}{s^n + x_1^n} + b_1 \cdot \frac{s^n}{s^n + x_2^n} - k \cdot x_1$$

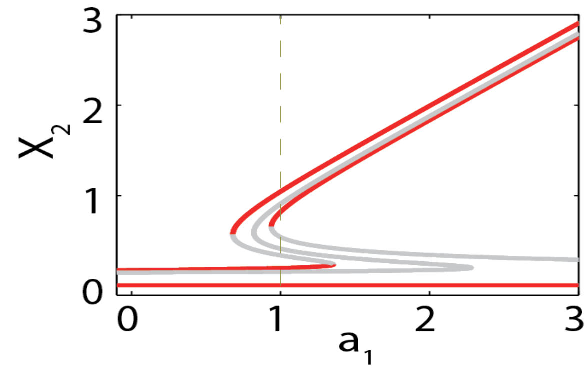
$$\dot{x_2} = a_2 \cdot \frac{x_2^n}{s^n + x_2^n} + b_2 \cdot \frac{s^n}{s^n + x_1^n} - k \cdot x_2$$

How Control is Done

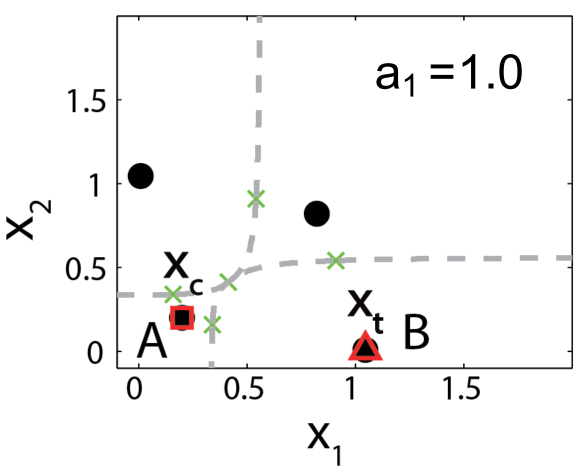
$$\dot{x}_1 = a_1 \cdot \frac{x_1^n}{s^n + x_1^n} + b_1 \cdot \frac{s^n}{s^n + x_2^n} - k \cdot x_1$$

$$\dot{x}_2 = a_2 \cdot \frac{x_2^n}{s^n + x_2^n} + b_2 \cdot \frac{s^n}{s^n + x_1^n} - k \cdot x_2$$

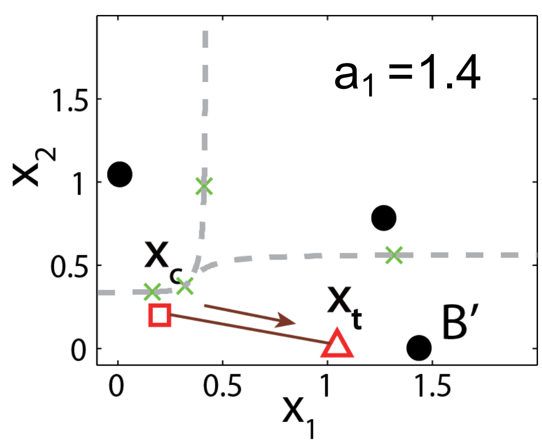
a_2=1; b_1 , b_2=0.2, k=1.1, s=0.5, n=4



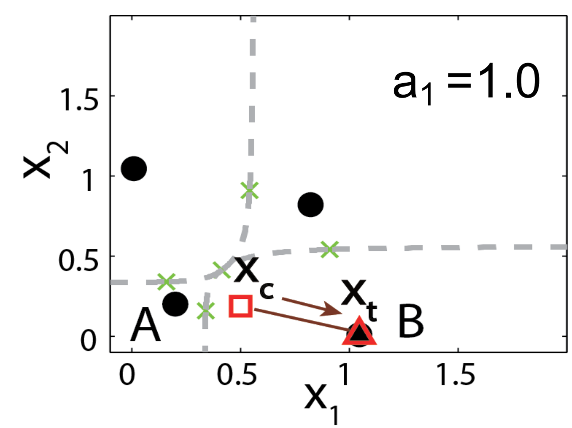
4 attractors



Basin of original attractor **A** is absorbed into that of attractor **B**

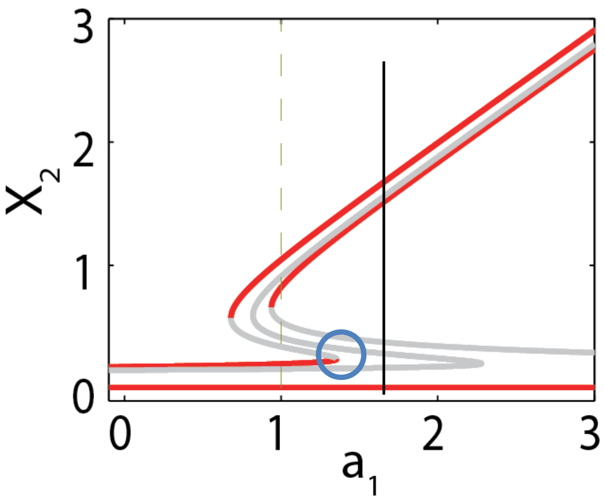


When system is in basin of attractor **B**, remove perturbation to a_1



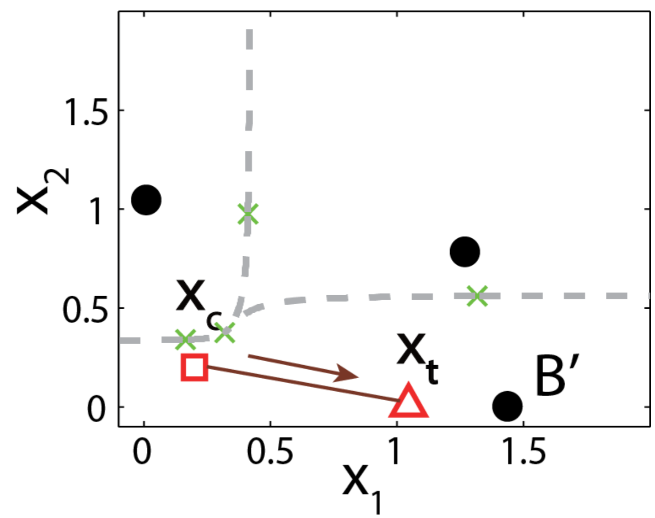
Control Strength and Duration

How large is the required parameter perturbation?



Parameter a_1 has to pass a critical point, say, $a_1 > 1.36$

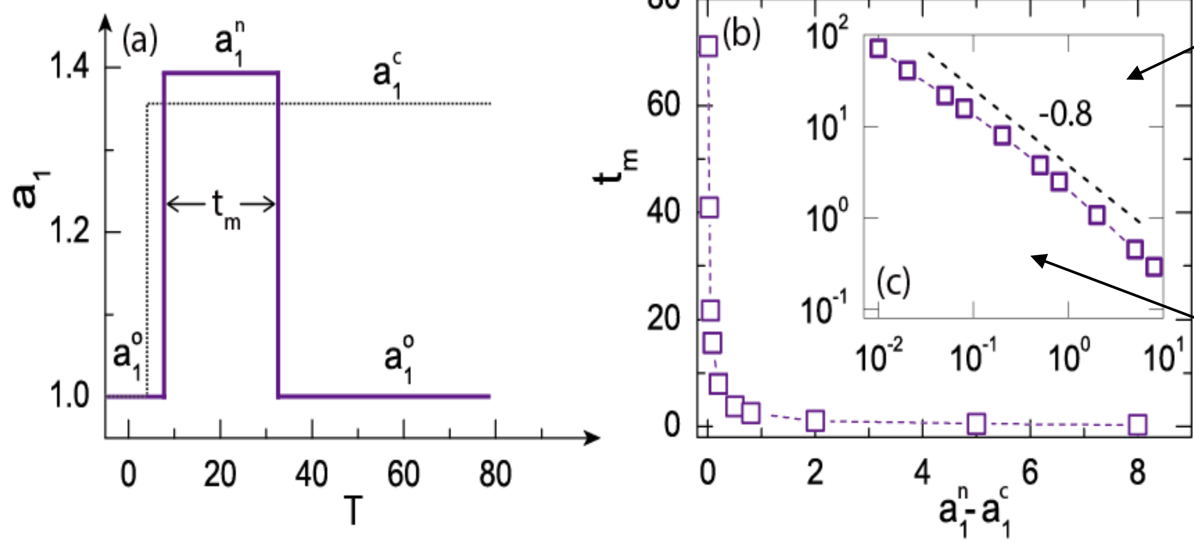
How long should the perturbation be maintained?



Perturbation has to stay “on” until x_t crosses basin boundary

Scaling between Control Strength and Duration

Minimal control duration t_m versus perturbation to a_1

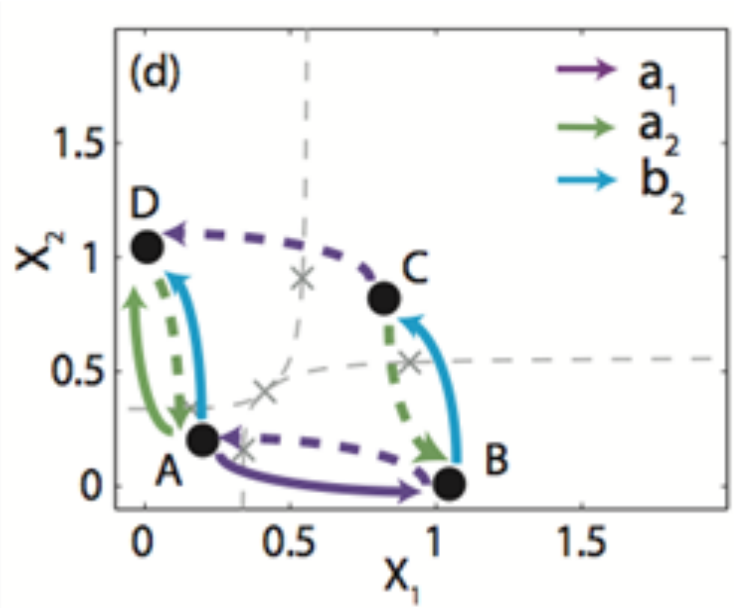


controllable region

uncontrollable region

Insofar as the combination of $[a_1, t_m]$ is in the controllable region, the system can be driven to the desirable attractor

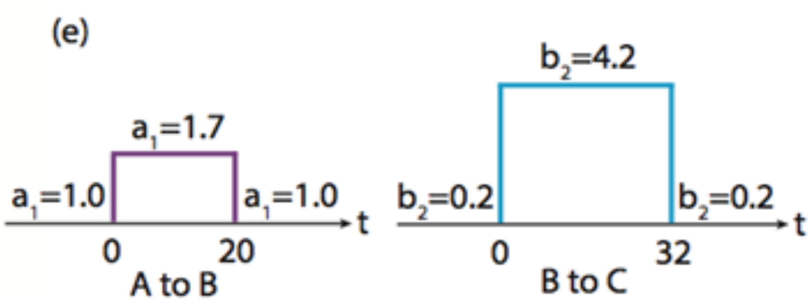
Attractor Network



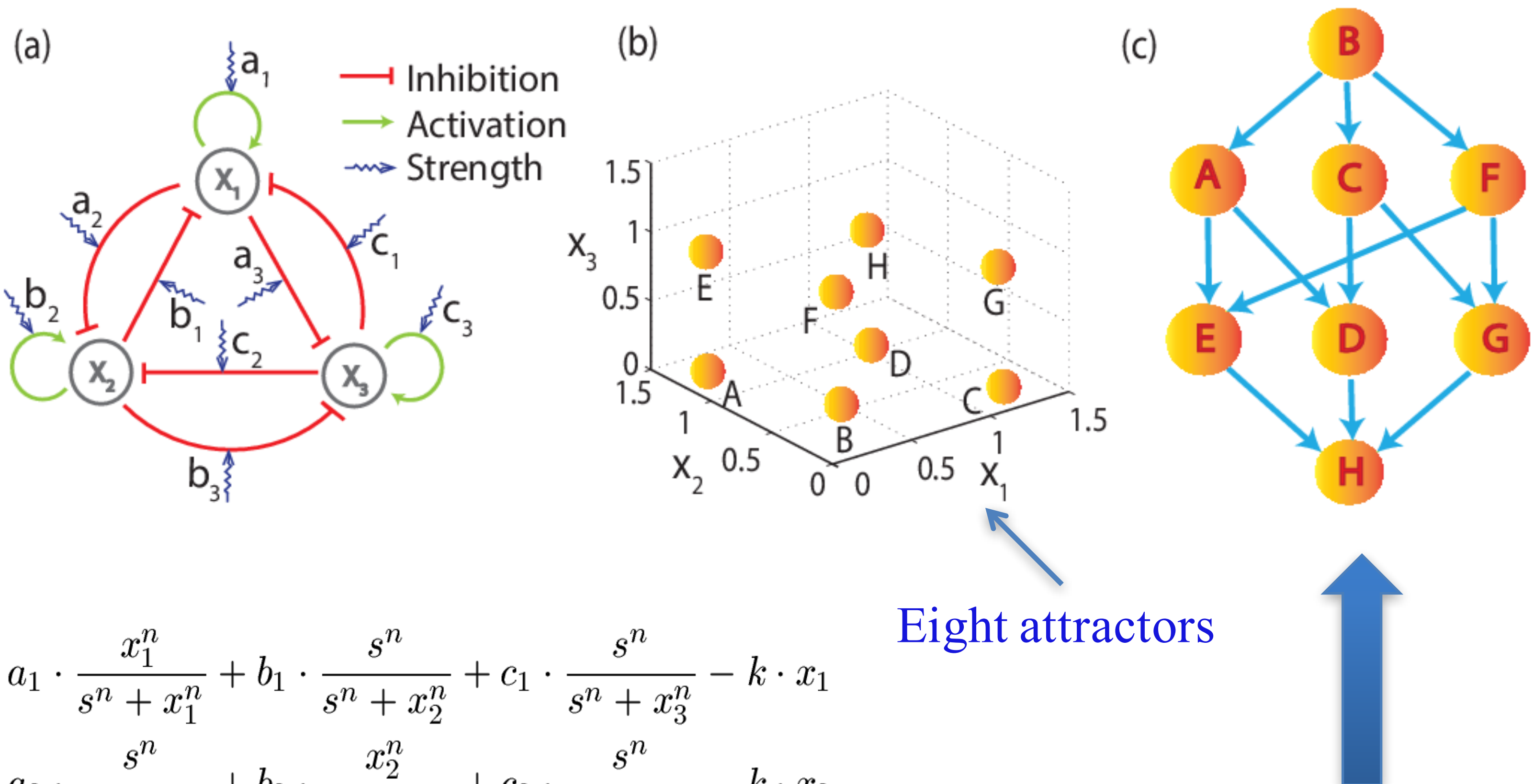
Each node in the attractor network represents one attractor. If perturbation of any accessible parameter can drive the system from attractor **A** to attractor **B**, we say there is a directed link from **A** to **B**.

Y.-C. Lai, "Controlling complex, nonlinear dynamical networks," *Nat. Sci. Rev.* **1**, 339-341 (2014).

If the attractor network is strongly connected, we can drive the system from any undesirable attractor to a desirable one, e.g., from A to C:



A Three-Gene Network



Eight attractors

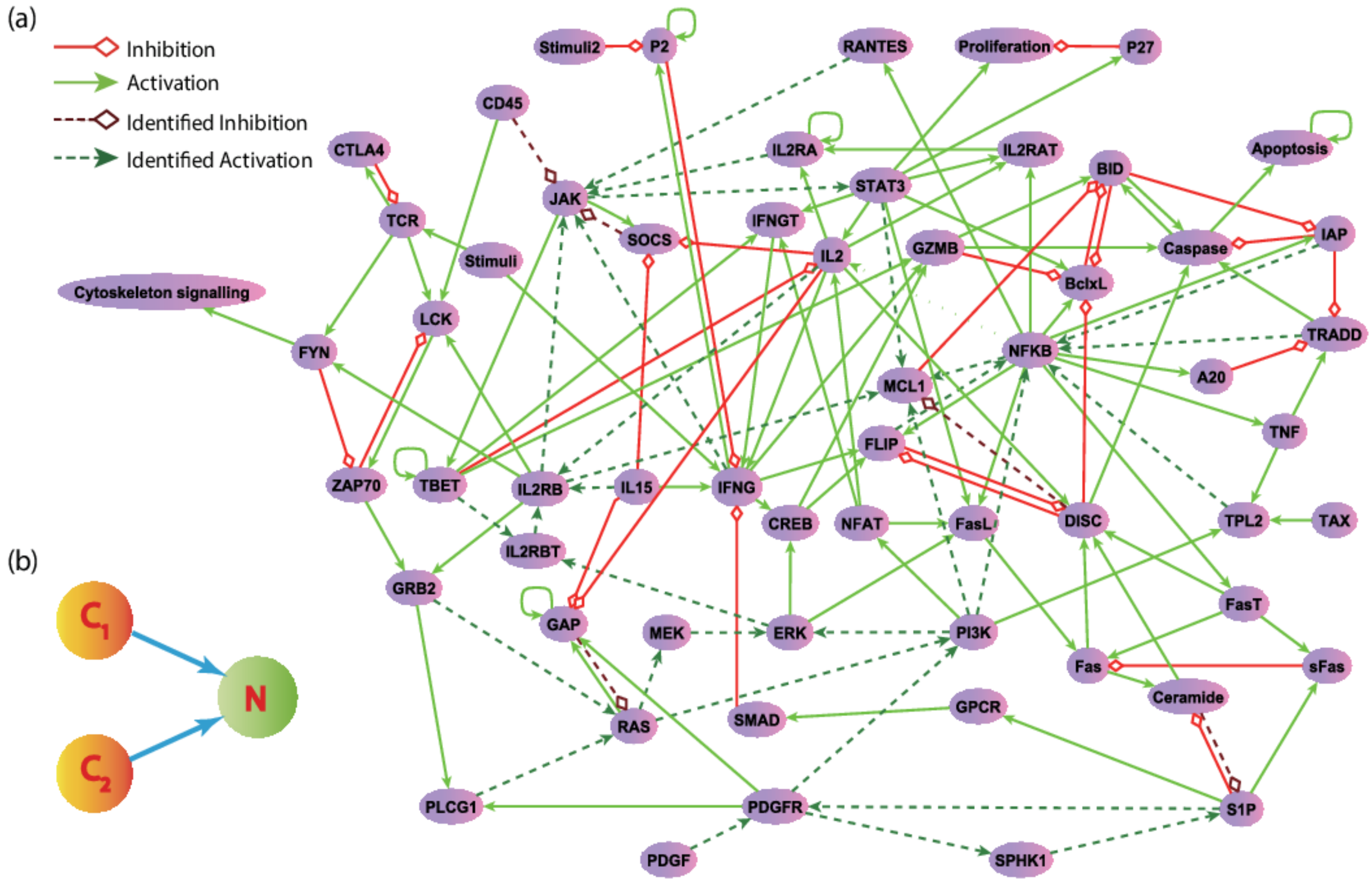
Attractor network

$$\dot{x_1} = a_1 \cdot \frac{x_1^n}{s^n + x_1^n} + b_1 \cdot \frac{s^n}{s^n + x_2^n} + c_1 \cdot \frac{s^n}{s^n + x_3^n} - k \cdot x_1$$

$$\dot{x_2} = a_2 \cdot \frac{s^n}{s^n + x_1^n} + b_2 \cdot \frac{x_2^n}{s^n + x_2^n} + c_2 \cdot \frac{s^n}{s^n + x_3^n} - k \cdot x_2$$

$$\dot{x_3} = a_3 \cdot \frac{s^n}{s^n + x_1^n} + b_3 \cdot \frac{s^n}{s^n + x_2^n} + c_3 \cdot \frac{x_3^n}{s^n + x_3^n} - k \cdot x_3$$

Application: T-LGL Survival Network

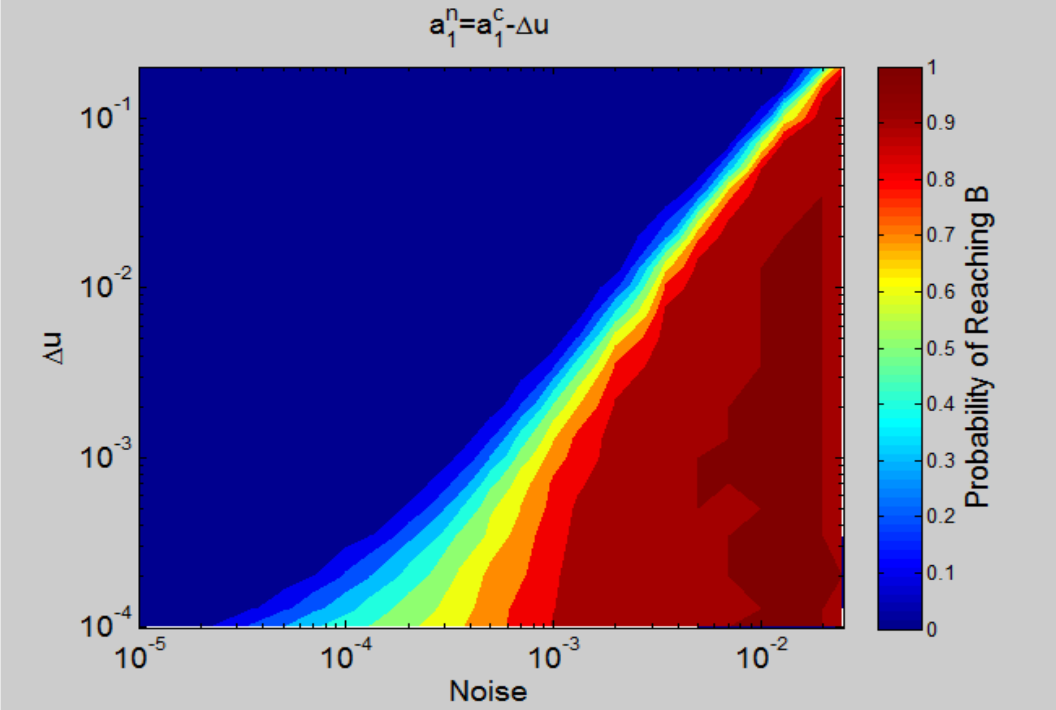


- R. Zhang, M. V. Shah, J. Yang, S. B. Nyland, X. Liu, J. K. Yun, and T. P. Loughran. Network model of survival signaling in large granular lymphocyte leukemia. *PNAS*, **105**:16308–16313, 2008.
- A. Saadatpour, R.-S. Wang, A. Liao, X. Liu, T. P. Loughran, I. Albert, and R. Albert. Dynamical and structural analysis of a t cell survival network identifies novel candidate therapeutic targets for large granular lymphocyte leukemia. *PloS Comp. Biol.*, **7**:e1002267, 2011.

Controllability of Nonlinear Networks

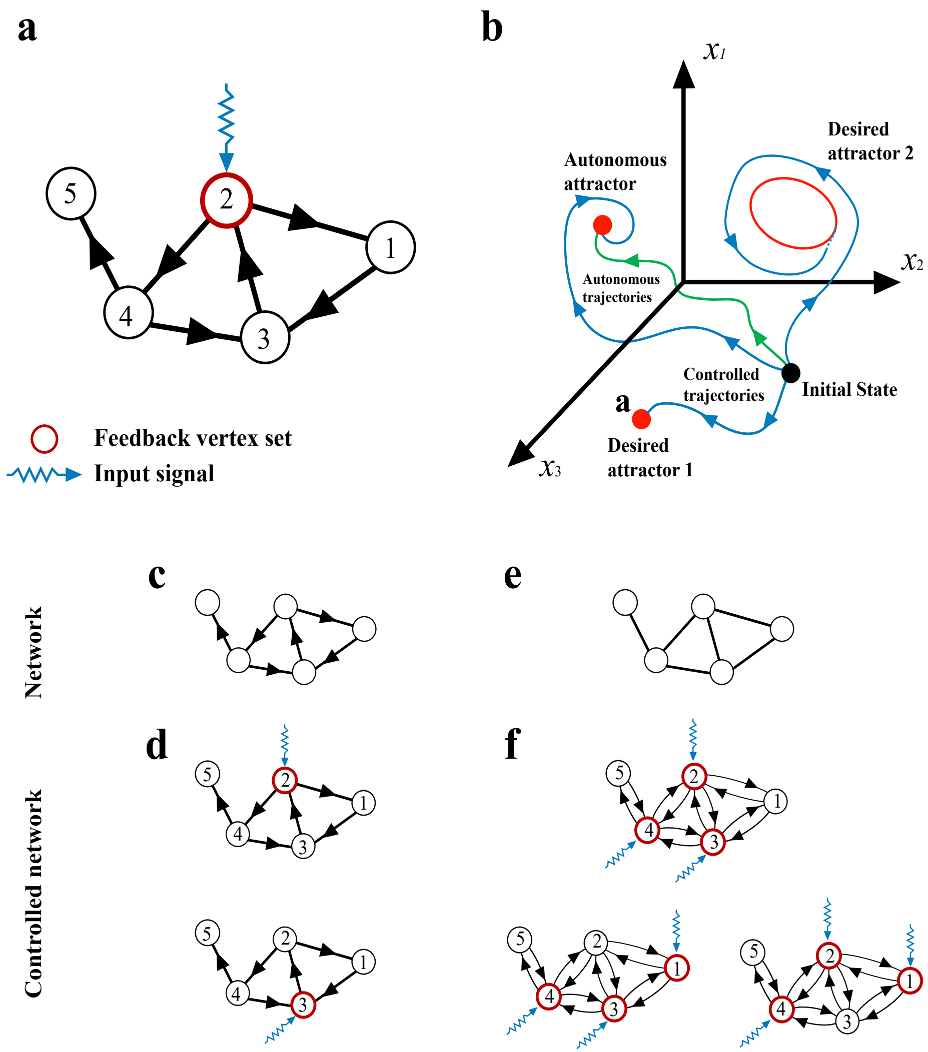
- Controllability of a nonlinear networks can be determined by the properties of its attractor network: The stronger the attractor network is connected, the more controllable the original network

- Noise-enhanced controllability



L.-Z. Wang, R.-Q. Su, Z.-G. Huang, X. Wang, W.-X. Wang, C. Grebogi, and Y.-C. Lai, "A geometrical approach to control and controllability of complex nonlinear dynamical networks," *Nat. Commun.* **7**, 11323 (2016).

FVS Based Control (Ongoing)

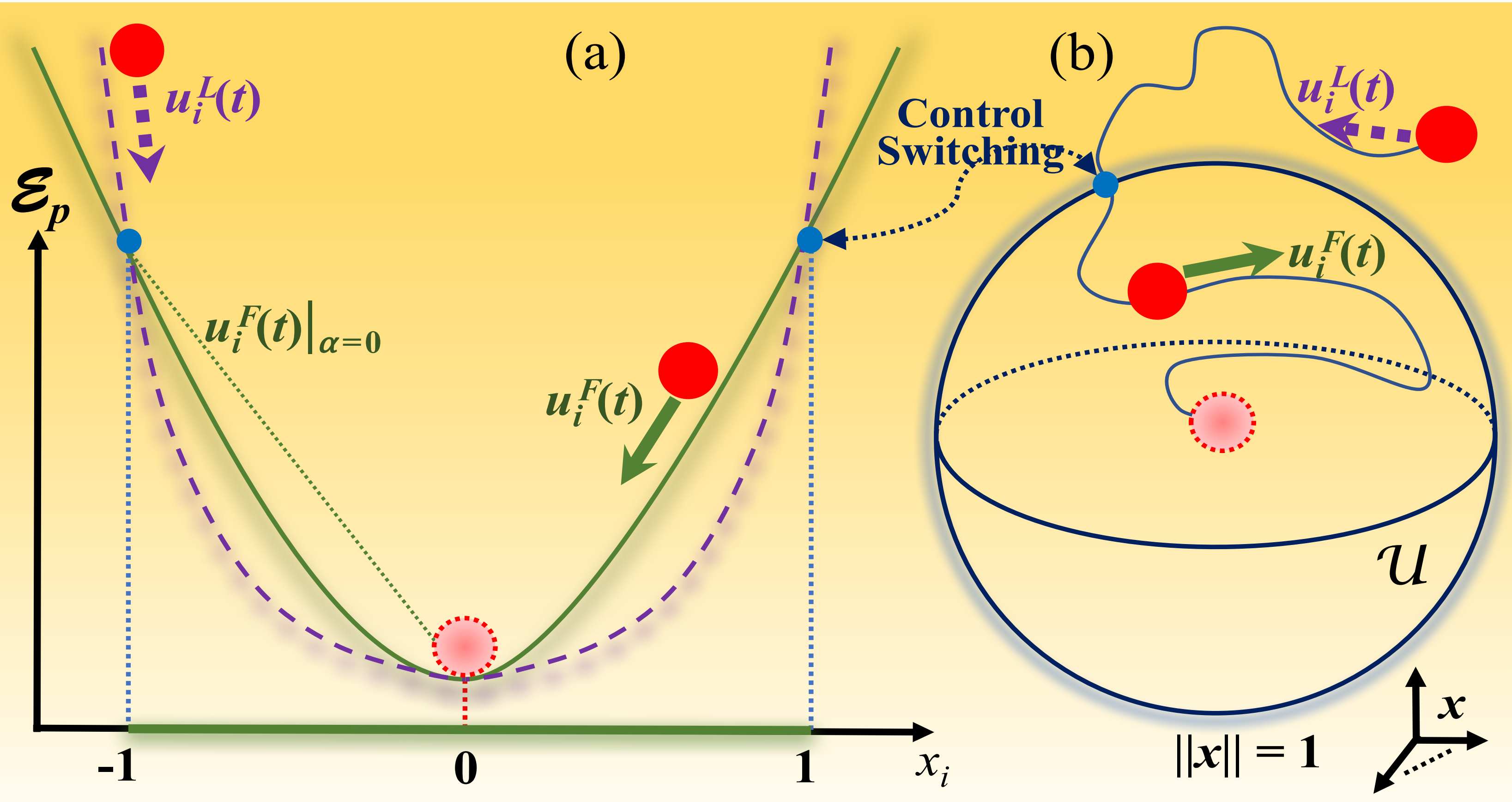


FVS (Feedback Vertex Set):

- A. Mochizuki et al., 2013
- H.-J. Zhou, 2013

Joint work with Beijing Normal colleagues:
Z.-S. Shen, W.-X. Wang, H.-J. Zhou, Z.-Y. Gao, A. Mochizuki, and Y.-C. Lai, "Control paradigm for nonlinear dynamical networks," preprint (2017).

Closed-Loop Control of Complex Nonlinear Dynamical Networks



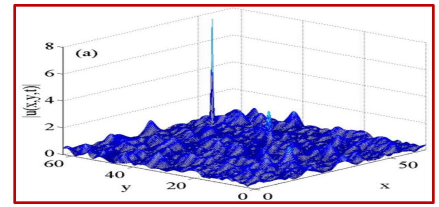
Joint work with Fudan colleagues:

Y.-Z. Sun, S.-Y. Leng, Y.-C. Lai, C. Grebogi, and W. Lin, *Phys. Rev. Lett.* **119**, 198301 (2017)

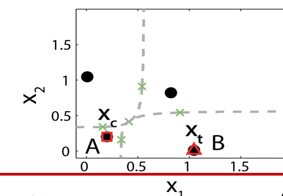
Take Home Message: Controlling Nonlinear Networks - DIVERSITY

1. Lack of a general mathematical control/controllability framework
2. Extremely **diverse** nonlinear dynamical behaviors require a diverse array of control methodologies:

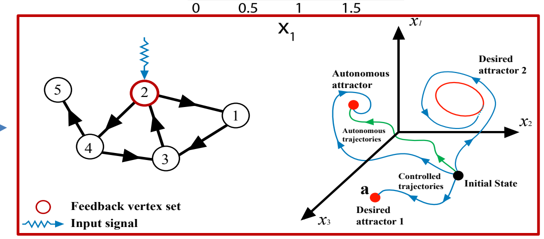
- Controlling **collective dynamics**, e.g., Y.-Z. Chen, Z.-G. Huang, and Y.-C. Lai, "Controlling extreme events on complex networks," *SREP* **4**, 6121 (2014)



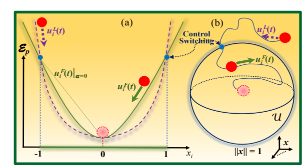
- Controlling **destinations** (attractors), e.g., L.-Z. Wang, R.-Q. Su, Z.-G. Huang, X. Wang, W.-X. Wang, C. Grebogi, and Y.-C. Lai, "A geometrical approach to control and controllability of complex nonlinear dynamical networks," *Nat. Commun.* **7**, 11323 (2016).



- Control principle based on feedback vertex set – ongoing work



- Closed-loop control, Y.-Z. Sun, S.-Y. Leng, Y.-C. Lai, C. Grebogi, and W. Lin, *Phys. Reve. Lett.* **119**, 198301 (2017)



- Predicting and controlling tipping point in complex mutualistic networks

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