Lecture 13 Exact Controllability of Complex Networks

(Popov–Belevitch–Hautus)

PBH rank condition : $\underset{\sim}{\dot{x}} = \overset{n\times n}{A} \cdot \underset{\sim}{x} + \overset{n\times m}{B} \cdot \overset{m\times 1}{\underset{\sim}{u}}$

is fully controllable if and only if

$\text{rank}\,(\lambda I_n - A,\, B) = n$

λ : any eigenvalue of A

* A - coupling matrix (not necessarily adj. matr.)

min. # of control signals required $= \min\{\text{rank}(B)\}$

Our result : (2013) $\boxed{N_D = \max_i \{\mu(\lambda_i)\}}$

geometric multiplicity of λ_i $\Leftarrow$ $\mu(\lambda_i) = \dim V_{\lambda_i} = n - \text{rank}(\lambda_i I_n - A)$

$\dim V_{\lambda_i}$: dimension of the eigenvec. space associated with λ_i

Say $A \cdot \underset{\sim}{v_i} = \lambda_i \underset{\sim}{v_i}$

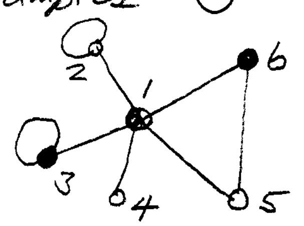
Dimension of $\underset{\sim}{v_i}$ space $=$ # of free components of $\underset{\sim}{v_i}$

$\lambda_i I_n - A$: $\left(\quad \right)\begin{pmatrix} v_i^1 \\ \vdots \\ v_i^n \end{pmatrix} = 0$

rank$(\lambda_i I_n - A)$ 1 • 1 $\neq 0$

How to identify a minimum set of drivers?

Examples ①



$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$ undirected

$\lambda_i \Rightarrow \max\{\mu\}$

Note Finding rank $(\lambda_i I_n - A)$ – elementary column transformations

$N_D =$ # of dependent rows

Locations of the dependent rows $\Rightarrow$ where control signals should be applied!

Say two dependent rows

$(A - \lambda I \mid B)$ makes $\Rightarrow \text{rank}(\lambda_i I_n - A, B) = n$

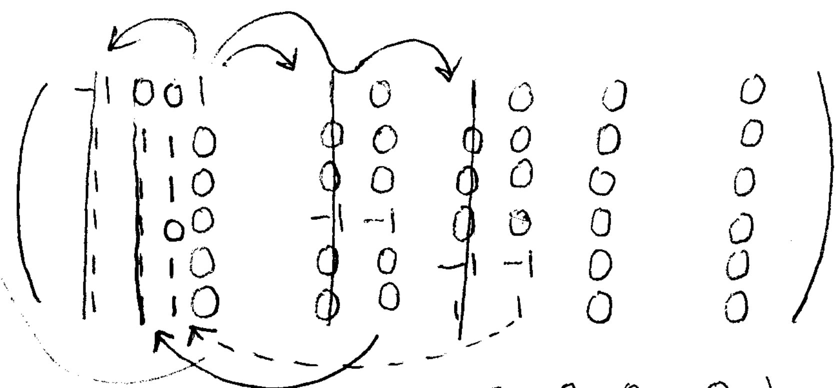
$$\begin{pmatrix} -1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}$$

$A - \lambda^M I$

Eigenvalues of A:

−1.9032
−1.0000
0.1939
1.0000 ← $\lambda^M = 1.0$
1.0000 ←
2.7093

$$\begin{matrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 2 \end{matrix}$$



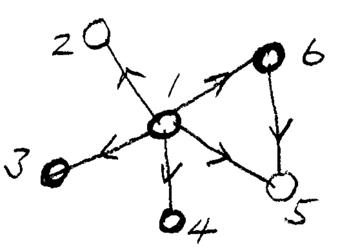
Finally

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \boxed{0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \boxed{0 \;\; 2 \;\; 0 \;\; -1 \;\; 0 \;\; 0} \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ u_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & u_2 \end{pmatrix}$$

Linearly dependent on others

control signals to nodes 3 and 6

② A directed network



$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\underbrace{\lambda = 0, 0, 0, 0, 0, 0}_{\text{eigenvalue degeneracy} = 6} \qquad A - \lambda I \qquad \mu(\lambda) = 4 < 6 \equiv \delta(\lambda)$$

eigenvalue degeneracy = 6 ⇒ algebraic multiplicity

Special results

(a) Undirected networks

$$N_D = \max\{\delta(\lambda_i)\}$$ — maximum algebraic multiplicity of λ_i

(b) Large sparse networks

$$N_D = \max\{1, N - \text{rank}(A)\}$$

(b) Densely connected networks

$$N_D = \max\{1, N - \text{rank}(w I_N + A)\}$$

↑ identical link weight

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ARTICLE

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Exact controllability of complex networks

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Controlling complex networks is of paramount importance in science and engineering. Despite the recent development of structural controllability theory, we continue to lack a framework to control undirected complex networks, especially given link weights. Here we introduce an exact controllability paradigm based on the maximum multiplicity to identify the minimum set of driver nodes required to achieve full control of networks with arbitrary structures and link-weight distributions. The framework reproduces the structural controllability of directed networks characterized by structural matrices. We explore the controllability of a large number of real and model networks, finding that dense networks with identical weights are difficult to be controlled. An efficient and accurate tool is offered to assess the controllability of large sparse and dense networks. The exact controllability framework enables a comprehensive understanding of the impact of network properties on controllability, a fundamental problem towards our ultimate control of complex systems.

Illustration of Exact Controllability Theory

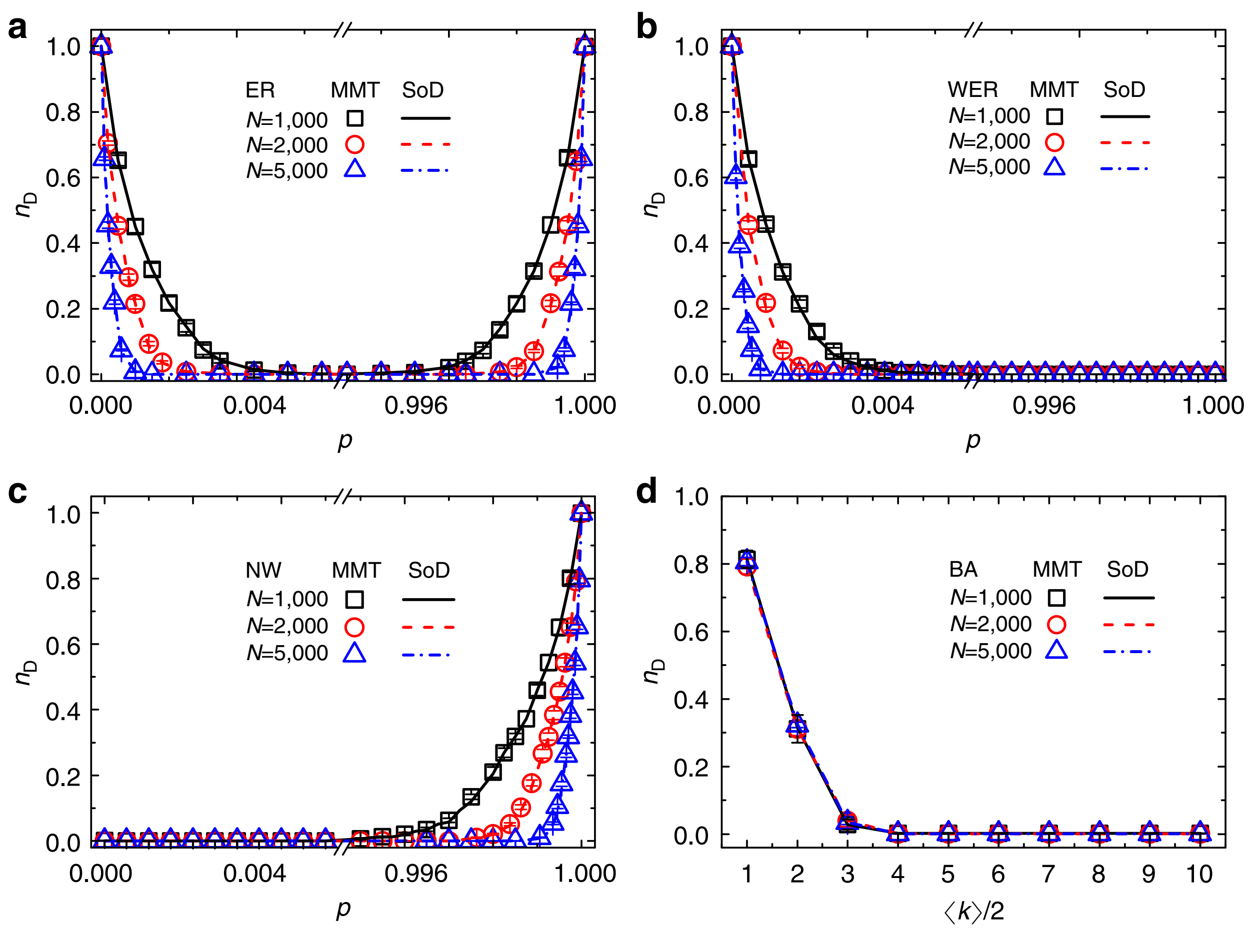
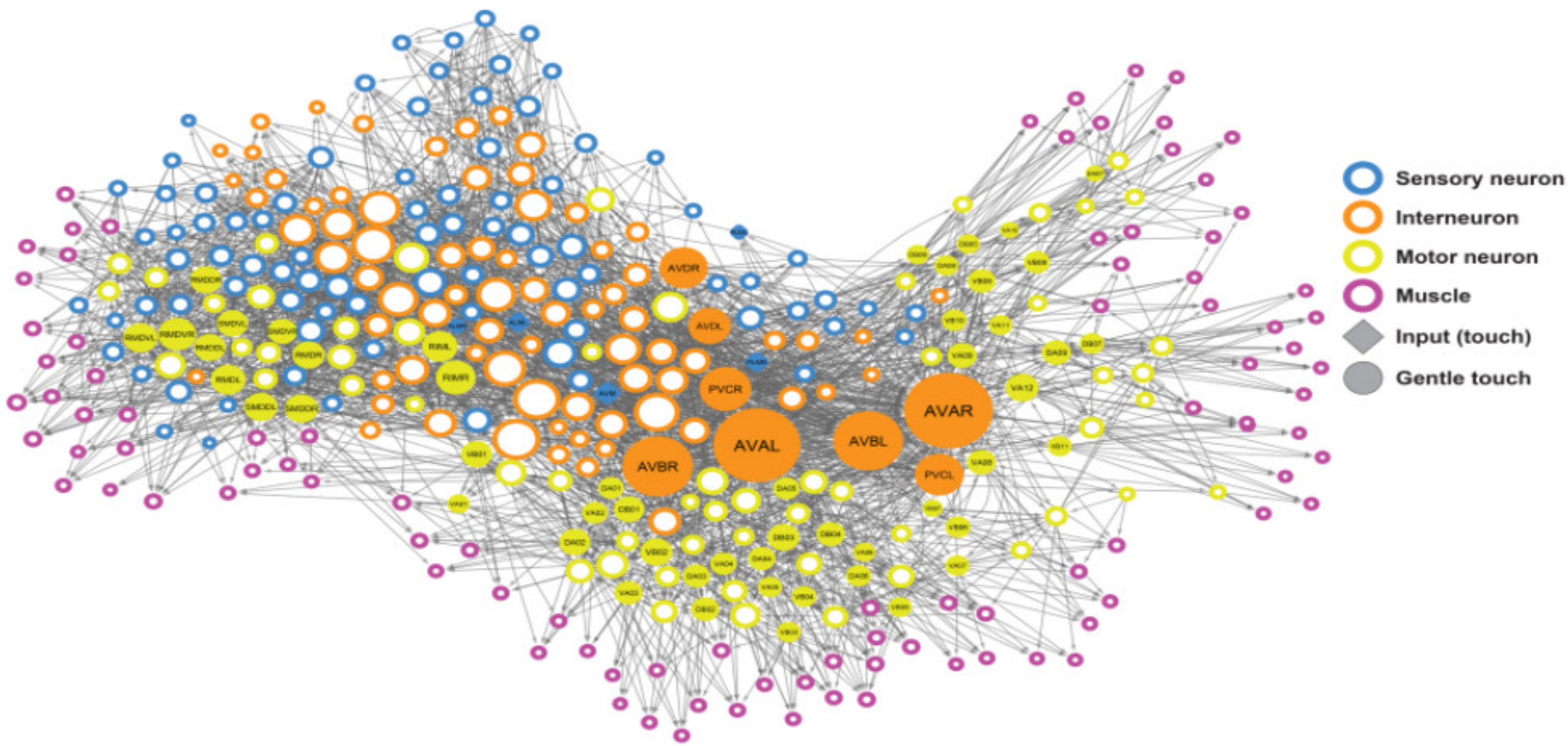


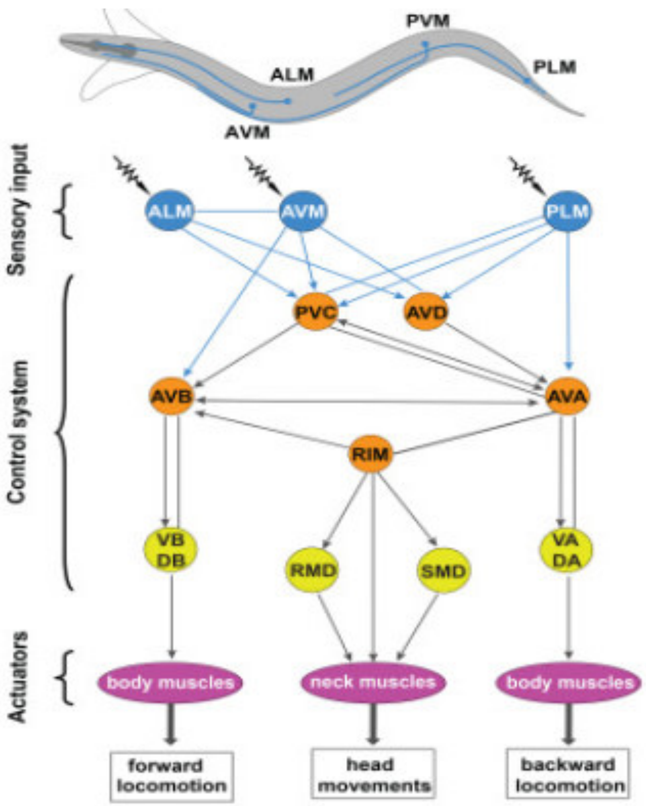
Figure 2 | Exact controllability of undirected networks. Exact controllability measure n_D as a function of the connecting probability p for (**a**) unweighted ER random networks and (**b**) ER random networks with random weights assigned to links (WER). (**c**) n_D versus the probability p of randomly adding links for Newman–Watts small-world networks. (**d**) n_D versus half of the average degree $\langle k \rangle/2$ for Barabási–Albert scale-free networks. All the networks are undirected and their coupling matrices are symmetric. The data points are obtained from the MMT equation (4) and the error bars denote the s.d., each from 20 independent realizations. The curves (SoD) are the theoretical predictions of equations (5) and (6) for sparse and dense networks, respectively. The representative network sizes used are $N = 1{,}000$, 2,000 and 5,000.

Neuronal Network of C-Elegans



Yan, Vértes, Towlson, Chew, Walker, Schafer, Barabási, Nature (2017)

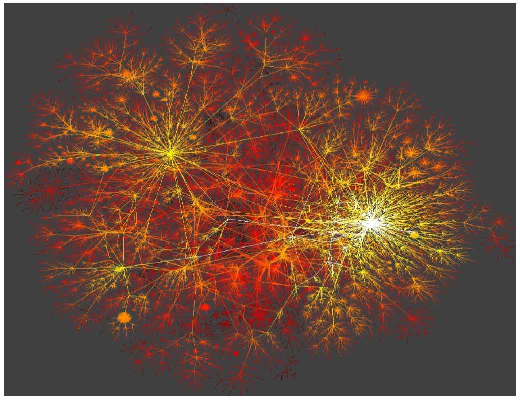
Discovery of a New Neuron Class

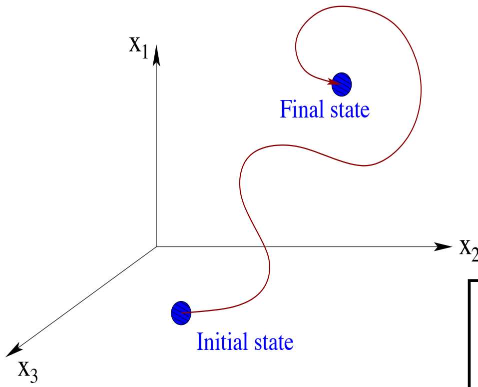


Control	Predicted Neuron Classes	Experimental Facts
Control Muscles	DA, DB	loss of backward/forward locomotion
	DD	uncoordinated motion
	AVA	uncoordinated motion
	VA, VB, VD, AS	likely loss of locomotion
	PDB	verified by new experiments
Control Motor Neurons	AVA, AVB	uncoordinated motion
	AVD	loss of reversal response
	PVC	loss of reversal response

Yan, Vértes, Towlson, Chew, Walker, Schafer, Barabási, Nature (2017)

Difficulty with Linear Network Control





A dynamical system is controllable if it can be driven from **any** initial state to **any** desired final state **in finite time** by suitable choice of input control signals.

General Mathematical framework: **Kalman's Controllability Rank Condition**

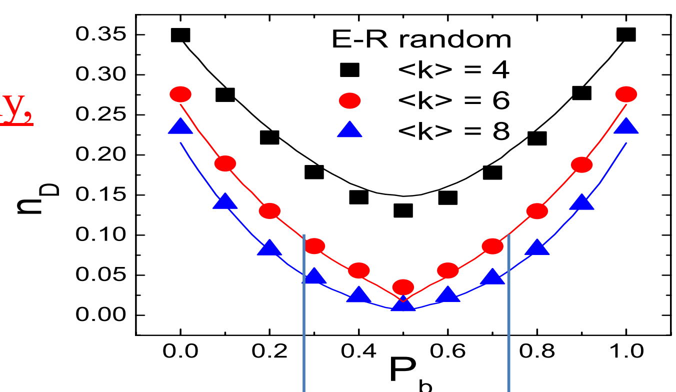
Focus of existing works: **minimal number of signals** required to control the network

e.g., Structural controllability: Y.-Y. Liu, J.-J. Slotine, and A.-L. Barabasi, "Controllability of complex networks," *Nature* **473**, 167 (2011)

Exact controllability: Z.-Z. Yuan, C. Zhao, Z.-R. Di, W.-X. Wang, and Y.-C. Lai, "Exact controllability of complex networks," *Nat. Commun.* **4**, 2447 (2013)

Issue: controllability is mathematically well defined but physically, control may be difficult

In terms of **ENERGY**



High probability of energy divergence

Energy bounds: G. Yan, J. Ren, Y.-C. Lai, C. H. Lai, B. Li, "Controlling complex networks: how much energy is needed?" *PRL* **108,** 218703 (2012).
Energy scaling: Y.-Z. Chen, L.-Z. Wang, W.-X. Wang, and Y.-C. Lai, "Energy scaling and reduction in controlling complex networks," *Roy. Soc. Open Sci.* **3**, 160064 (2016).
Physical controllability: L.-Z. Wang, Y.-Z. Chen, W.-X. Wang, and Y.-C. Lai, "Physical controllability of complex networks," *SREP* **7**, 40198 (2017).