Lecture 12 Controllability — Dynamics based Centrality

Linear dynamical network

$$\underset{\sim}{\dot{X}}(t) = \underset{n\times n}{A} \cdot \underset{\sim}{X}(t) + \underset{n\times m}{B} \cdot \underset{m\times 1}{\underset{\sim}{u}(t)}$$

m — # of control signals

$\underset{\sim}{X}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$ — one-dim. linear nodal dynamics $\qquad \underset{\sim}{u}(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{pmatrix}$

Controllability (Linear):

A dynamical system is <u>controllable</u> if it can be driven from any initial state to any desired final state in finite time by suitable choice of input control signals.

Kalman's rank condition

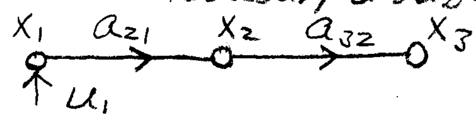
$$C \equiv (B, A\cdot B, A^2\cdot B, \cdots, A^{n-1}\cdot B)_{n\times nm}$$

— controllability matrix

$$\boxed{\text{rank}(C) = n} \iff \boxed{\text{System is controllable}}$$

necessary & sufficient condition for contro.

Examples:

① 

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \underbrace{\begin{pmatrix} b_1 \\ 0 \\ 0 \end{pmatrix}}_{B} u$$

$$\dot{x}_1(t) = b_1\, u(t) \Rightarrow x_1(t) = \int b_1 u(t)\, dt$$

$$\dot{x}_2(t) = a_{21}\, x_1(t) \Rightarrow x_2(t) = \int a_{21}\, x_1(t)\, dt$$

$$\dot{x}_3(t) = a_{32}\, x_2(t) \Rightarrow x_3(t) = \int a_{32}\, x_2(t)\, dt$$

$$C = (B, A\cdot B, A^2\cdot B) = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & a_{21} & 0 \\ 0 & 0 & a_{32}a_{21} \end{pmatrix}$$

$$\text{rank}(C) = 3 \rightarrow \text{Controllable}$$

How to design the control signals?

In general, Given $\underset{\sim}{X}(0)$, $\underset{\sim}{X}(t_f)$, t_f — finite control time

$$\underset{\sim}{u}(t) = B^T \cdot \exp[A^T \cdot (t_f - t)] \cdot W_{t_f}^{-1} \cdot \underset{\sim}{v}_{t_f}$$

$$W_{t_f} \equiv \int_0^{t_f} \exp(At)\cdot B \cdot B^T \cdot \exp(A^T t)\, dt$$

$$\underset{\sim}{v}_{t_f} \equiv \underset{\sim}{X}(t_f) - \exp(A t_f)\cdot \underset{\sim}{X}(0)$$

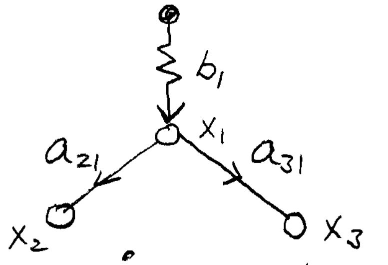
Gramian matrix (symmetric) $\qquad H(t_f) = \exp(-A t_f)\cdot W_{t_f} \cdot \exp(-A^T t_f)$

Control energy cost (normalized) $\qquad E(t_f) = \dfrac{\int_0^{t_f} \| \underset{\sim}{u}(t) \|^2 dt}{\| \underset{\sim}{X}(0) \|^2} = \dfrac{\underset{\sim}{X}_0^T \cdot H^{-1} \cdot \underset{\sim}{X}_0}{\underset{\sim}{X}_0^T \cdot \underset{\sim}{X}_0}$

— Linear systems theory

(e.g., W. J. Rugh, Prentice Hall, 1996)

(2)



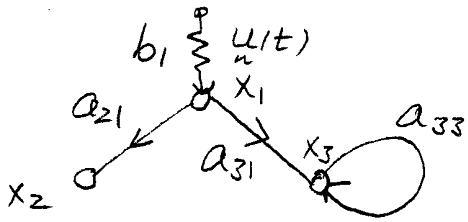
$$\Rightarrow \mathbb{C} = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & a_{21} & 0 \\ 0 & a_{31} & 0 \end{pmatrix} \Rightarrow \begin{array}{l} \text{rank}(\mathbb{C}) \\ = 2 < 3 \end{array}$$

$\Rightarrow$ uncontrollable

$$\dot{x}_1(t) = b_1 u_1(t)$$

no longer independent
$$\begin{cases} \dot{x}_2(t) = a_{21}\, x_1(t) \\ \dot{x}_3(t) = a_{31}\, x_1(t) \end{cases} \Rightarrow \frac{x_2(t)}{x_3(t)} = \frac{a_{21}}{a_{31}}$$
— "stuck" in an invariant subspace

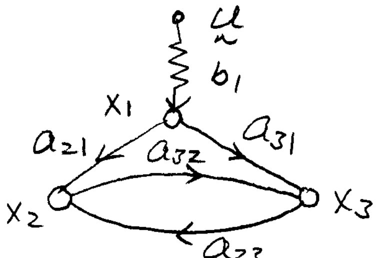
(3)



$$\mathbb{C} = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & a_{21} & 0 \\ 0 & a_{31} & a_{33}a_{31} \end{pmatrix}$$

$\text{rank}(\mathbb{C}) = 3 \Rightarrow$ controllable

Structural controllability



$$\mathbb{C} = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & a_{21} & a_{23}a_{31} \\ 0 & a_{31} & a_{32}a_{21} \end{pmatrix}$$

$\text{rank}(\mathbb{C}) = 3$ except for $\underbrace{\dfrac{a_{21}}{a_{31}} = \dfrac{a_{23}a_{31}}{a_{32}a_{21}}}_{\text{pathological condition}}$

$\boxed{\text{Structurally controllable:}}$

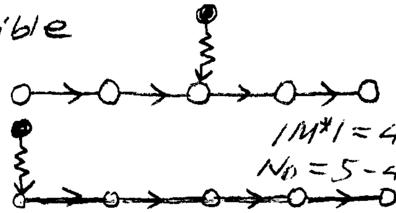
can be removed by arb. small pert. to link weights $\leftarrow$

Lin's theorem (1974)

A linear control system (A, B) is structurally controll. if and only if the structured matrix $[A; B]$ is irreducible and has no dilations.

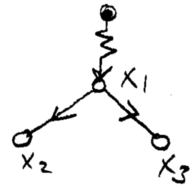
reducible

irreducible



$|M^*| = 4$
$N_D = 5 - 4 = 1$

Dilations:
subgraphs in which there are more "slaves" than "masters"



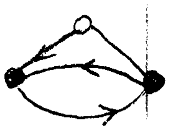
Maximum matching

Matching: an edge subset M in which no two edges share a common starting node or a common ending node

$|M^*| = 1$
$N_D = n - |M^*| = 2$

$$N_D = n - \underbrace{|M^*|}_{\text{size of maximum matching}}$$

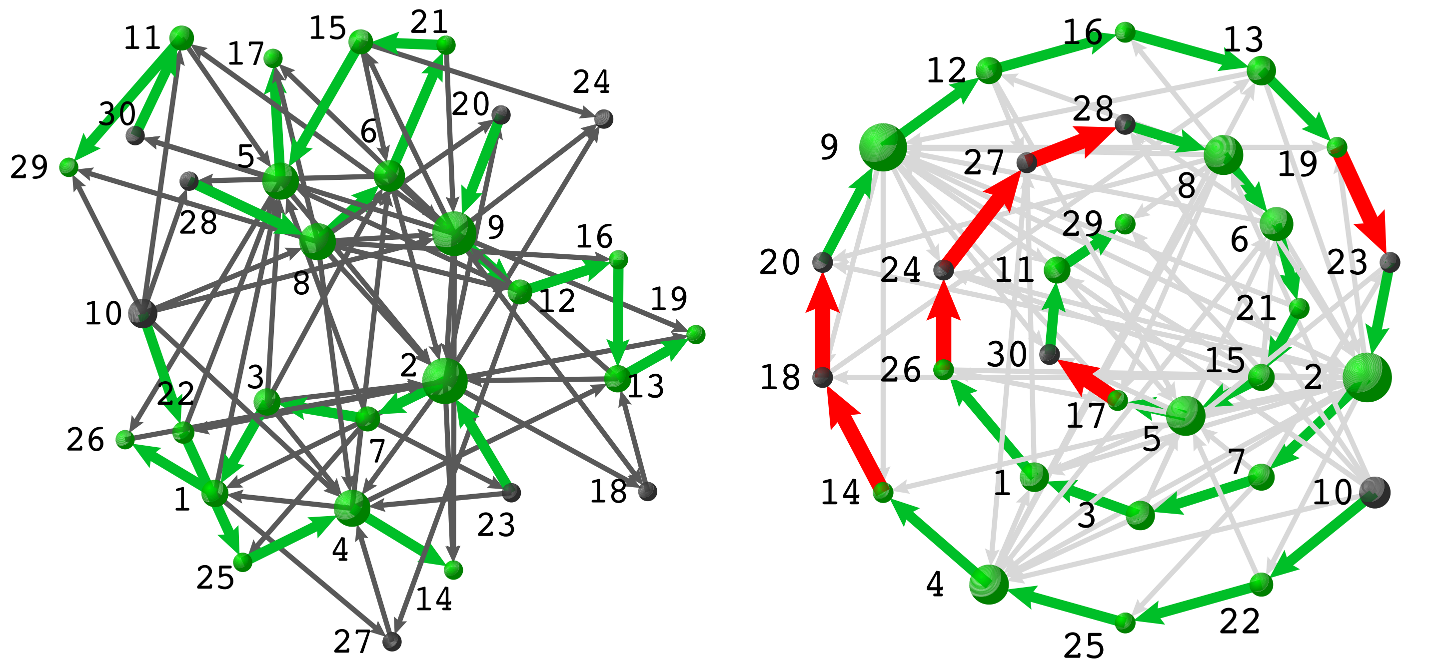
$|M^*| = 2$
$N_D = n - |M^*| = 1$



$|M^*| = 2$
$N_D = n - |M^*| = 1$

Centrality: set of nodes to which control signals should be applied $\rightarrow$ more "important"

Maximum Matching – An Example



(a) (b)

Green edges – maximum matching

Green nodes – matched nodes

Gray nodes – unmatched nodes – control nodes

Red edges – added edges to realize perfect matching

W.-X. Wang, X. Ni, Y.-C. Lai, and C. Grebogi, Phys. Rev. E 85, 025115 (2012)