EEE 303, Spring 2004
Final Exam

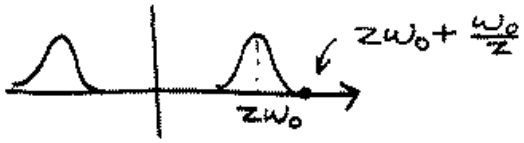
7:40-9:30am, Tuesday, 5/11/2004

Name **ID**

I Multiple-choice problems (20 points - 1 point each)

1. A continuous-time signal $x(t)$ has Nyquist rate ω_0. Define a new signal: $y(t) = x(t)\cos(2\omega_0 t)$. The Nyquist rate of $y(t)$ is

   $Y(j\omega) = \frac{1}{2}[X(j(\omega+2\omega_0)) + X(j(\omega-2\omega_0))]$

   (a) $2\omega_0$
   (b) $3\omega_0$
   (c) $4\omega_0$
   (d) $5\omega_0$

2. Let $y(t) = x_1(t) * x_2(t)$, where $x_1(t)$ and $x_2(t)$ are continuous-time, band-limited signals: $X_1(j\omega) = 0$ for $|\omega| > 1000\pi$ (Hz) and $X_2(j\omega) = 0$ for $|\omega| > 2000\pi$ (Hz), where $X_1(j\omega)$ and $X_2(j\omega)$ are the Fourier transforms of $x_1(t)$ and $x_2(t)$, respectively. Impulse-train sampling of $y(t)$ yields

$$y_p(t) = \sum_{n=-\infty}^{+\infty} y(nT)\delta(t-nT).$$

   To ensure a complete recovery of $y(t)$ from $y_p(t)$, the maximum value for the sampling period T is

   Maximum frequency of $y(t) = 1000\pi$
   Nyquist rate $= 2000\pi$
   $T_{max} = \frac{2\pi}{2000\pi} = 10^{-3}$ (s)

   (a) 10^{-3}(sec)
   (b) 2×10^{-3}(sec)
   (c) 3×10^{-3}(sec)
   (d) 4×10^{-3}(sec)

3. Consider impulse-train sampling of a continuous-time signal $x(t)$,

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT)\delta(t-nT),$$

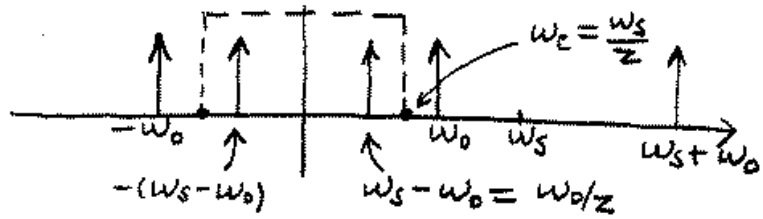
   where the sampling period is $T = 2.0$. Let $X(j\omega)$ and $X_p(j\omega)$ be the Fourier transforms of $x(t)$ and $x_p(t)$, respectively. If $X(j\omega)|_{\omega=0} = 1.0$, the value of $X_p(j\omega)|_{\omega=0}$ is

   $X_p(j0) = \frac{1}{T} X(j0) = 0.5$

   (a) 0.2
   (b) 0.5
   (c) 1.0
   (d) 2.0

4. Let $x(t) = \cos(\omega_0 t)$. Say we do impulse-train sampling at frequency ω_s to obtain $x_p(t)$ and then use a low-pass filter of cut-off frequency $\omega_c = \omega_s/2$, attempting to recover $x(t)$. If $\omega_s = 3\omega_0/2$ (undersampling), then in the following frequency range: $-\omega_c < \omega < \omega_c$, $X_p(j\omega)$, the Fourier transform of $x_p(t)$, consists of two impulses at $\omega_p > 0$ and $-\omega_p$, where $\omega_p < \omega_c$. The value of ω_p is

(a) $\omega_0/2$
(b) $\omega_0/3$
(c) $\omega_0/4$
(d) $\omega_0/6$



5. Let $x(t) = \cos(\omega_0 t)$. Say we do impulse-train sampling at frequency ω_s to obtain $x_p(t)$ and then use a low-pass filter of cut-off frequency $\omega_c = \omega_s/2$ in an attempt to recover $x(t)$. If $\omega_s = 3\omega_0/2$ (undersampling), then $x_r(t)$, the output signal of the low-pass filter, is

(a) $\cos(\omega_0 t)$
(b) $\sin(\omega_0 t)$
(c) $\cos(\omega_0 t/2)$
(d) $\sin(\omega_0 t)/2$

$$X_r(j\omega) = \frac{1}{2}\left[\delta\left(\omega - \frac{\omega_0}{2}\right) + \delta\left(\omega + \frac{\omega_0}{2}\right)\right]$$

6. Given a continuous-time signal: $x(t) = (9e^{-6t} - 8e^{-5t})u(t)$. The ROC (Region of Convergence) of $X(s)$, the Laplace transform of $x(t)$, is

(a) $Re\{s\} > -6$
(b) $Re\{s\} > -5$
(c) $Re\{s\} > -9$
(d) $Re\{s\} > +8$

$\{Re\{s\} > -6\} \cap \{Re\{s\} > -5\}$
$= Re\{s\} > -5$

7. The Laplace transform of the impulse response function $h(t)$ of a causal LTI system is given by

$$H(s) = A\frac{(s-1)^2}{(s+1)(s-2)},$$

where $A \neq 0$ is a finite constant. Let $H(j\omega)$ be the Fourier transform of $h(t)$. Which of the following is true?

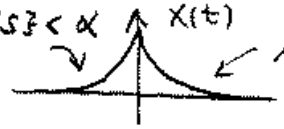
(a) $H(j\omega)$ exists if A is small enough.
(b) $H(j\omega)$ exists regardless of the value of A.
(c) $H(j\omega)$ does not exist regardless of the value of A.
(d) It is impossible to construct such an LTI system in laboratory.

ROC: $Re\{s\} > 2$
— does not contain the $j\omega$ axis

8. Let $x(t) = e^{-\alpha|t|}$ be a two-sided, continuous-time signal, where $\alpha > 0$ is a finite constant. The ROC of $X(s)$ is

(a) $Re\{s\} > 0$
(b) $-\alpha < Re\{s\} < 0$
(c) $0 < Re\{s\} < +\alpha$
(d) $-\alpha < Re\{s\} < +\alpha$

ROC: $Re\{s\} < \alpha$



ROC: $Re\{s\} > -\alpha$

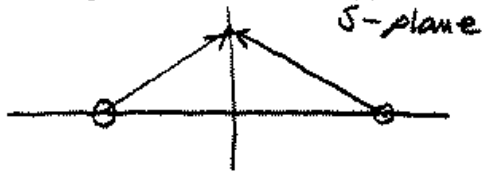
9. The Laplace transform $X(s)$ of a continuous-time signal $x(t)$ has no pole and only a zero at $s = -a$, where $a > 0$ is a finite, real constant. Another continuous-time signal $x_1(t)$ satisfies the following conditions: (i) $x_1(t) \neq x(t)$ and (ii) $|X_1(j\omega)| = |X(j\omega)|$, where $X(j\omega)$ and $X_1(j\omega)$ are the Fourier transforms of $x(t)$ and $x_1(t)$, respectively. Then $X_1(s)$, the Laplace transform of $x_1(t)$, has

(a) a zero at $s = +a$.

(b) a zero at $s = -a$.

(c) a pole at $s = +a$.

(d) a pole at $s = -a$.



Reflection:

$|X_1(j\omega)| = |X(j\omega)|$

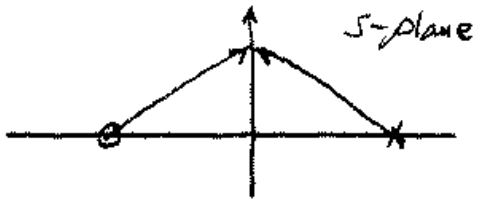
10. The Laplace transform $X(s)$ of a continuous-time signal $x(t)$ has no pole and only a zero at $s = -a$, where $a > 0$ is a finite, real constant. Another continuous-time signal $x_2(t)$ satisfies the following conditions: (i) $x_2(t) \neq x(t)$ and (ii) the phase angles of $X_2(j\omega)$ and $X(j\omega)$ are equal, where $X(j\omega)$ and $X_2(j\omega)$ are the Fourier transforms of $x(t)$ and $x_2(t)$, respectively. Then $X_2(s)$, the Laplace transform of $x_2(t)$, has

(a) a zero at $s = -a$.

(b) a zero at $s = +a$.

(c) a pole at $s = -a$.

(d) a pole at $s = +a$.



Reflection + conversion

$\measuredangle X_2(j\omega) = \measuredangle X(j\omega)$

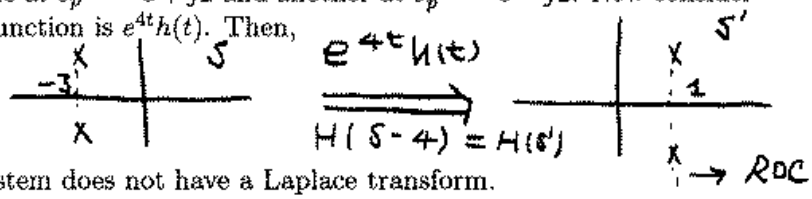
11. The Laplace transform $H(s)$ of the impulse response function $h(t)$ of a causal LTI system has two poles that constitute a complex-conjugate pair: one at $s_p = -3 + j2$ and another at $s_p^* = -3 - j2$. Now consider a new LTI system whose impulse response function is $e^{4t}h(t)$. Then,

(a) the new LTI system is stable.

(b) the new LTI system is unstable.

(c) the impulse response of the new LTI system does not have a Laplace transform.

(d) the impulse response of the new LTI system has a Fourier transform.



12. The Laplace transform of a continuous-time signal $x(t)$ is given by

$$X(s) = \frac{1}{(s+1)(s+2)}, \quad Re\{s\} > -1.$$

The ROC of the Laplace transform of

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

is

(a) $Re\{s\} > 0$

(b) $Re\{s\} > -1$

(c) $Re\{s\} < -1$

(d) $Re\{s\} > -2$

$\{Re\{s\} > -1\} \cap \{Re\{s\} > 0\}$

$= Re\{s\} > 0$

3

13. A first-order causal system is given by the following impulse response function:

$$h(t) = \frac{1}{\tau} e^{-t/\tau} u(t),$$

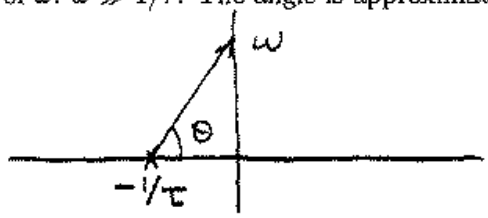
where $\tau > 0$ is a real constant. From a pole-zero plot of $H(s)$, the Laplace transform of $h(t)$, a competent EEE303 student can determine the phase angle of $H(j\omega)$, the frequency response of the system, for sufficiently large value of ω: $\omega \gg 1/\tau$. The angle is approximately

(a) π

(b) $-\pi$

(c) $\pi/2$

(d) $-\pi/2$



$\measuredangle H(j\omega) = -\theta$

$\rightarrow -\frac{\pi}{2}$

if $\omega \gg \frac{1}{\tau}$

14. A first-order causal system is given by the following impulse response function:

$$h(t) = \frac{1}{\tau} e^{-t/\tau} u(t),$$

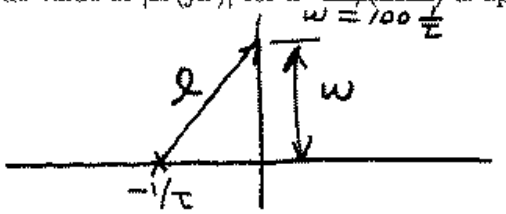
where $\tau > 0$ is a real constant. From a pole-zero plot of $H(s)$, the Laplace transform of $h(t)$, a competent EEE303 student can determine the behavior of $|H(j\omega)|$, the magnitude of the frequency response function of the system. If a dB scale (*i.e.*, $20 \log_{10} |H(j\omega)|$) is utilized, the student finds that the value of $|H(j0)|$ is zero dB. Then, the value of $|H(j\omega)|$ for $\omega = 100\frac{1}{\tau}$ is approximately

(a) -10 dB.

(b) -20 dB.

(c) -30 dB.

(d) -40 dB.



$|H(j\omega)| \sim \frac{1}{\ell} = \frac{1}{\sqrt{\omega^2 + (\frac{1}{\tau})^2}}$

For $\omega = 100\frac{1}{\tau}$

$|H(j\omega)| \cong (100)^{-1} |H(j0)|$

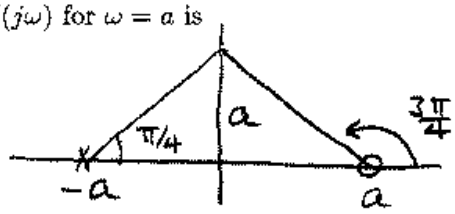
15. The frequency response of an all-pass system satisfies: $|H(j\omega)| = 1$. The corresponding Laplace transform of the impulse response function has a pole at $x = -a$ and a zero at $x = a$, where $a > 0$ is a real constant. The phase angle of $H(j\omega)$ for $\omega = a$ is

(a) $\pi/2$.

(b) $\pi/4$.

(c) $-\pi/4$.

(d) $-\pi/2$.



$\measuredangle H(j\omega) = \frac{3\pi}{4} - \frac{\pi}{4}$

$= \frac{\pi}{2}$

16. Consider the following discrete-time signal $x[n]$ of finite duration: $x[n] = a^n$ for $0 \le n \le N-1$ and $x[n] = 0$ otherwise, where $a > 0$ is a real constant. The ROC of the z-transform of $x[n]$ is

(a) entire z-plane including $z = 0$ and $z = +\infty$.

(b) entire z-plane including $z = 0$ but excluding $z = +\infty$.

(c) $|z| > a$

(d) entire z-plane including $z = +\infty$ but excluding $z = 0$.

$\sum_{n=0}^{N-1} (az^{-1})^n$

$= \frac{z^N - a^N}{z^{N-1}(z-a)}$

$z = a$ is not a pole

but $z = 0$ is.

4

17. The ROC of the discrete-time signal $x[n] = \delta[n+1]$ is

$$\sum \delta[n+1]\, z^{-n} = z$$

(a) entire z-plane including both $z = 0$ and $z = +\infty$.

(b) entire z-plane including $z = 0$ but excluding $z = +\infty$.

(c) entire z-plane excluding $z = 0$ but including $z = +\infty$.

(d) entire z-plane excluding both $z = 0$ and $z = +\infty$.

18. A discrete-time LTI system is characterized by the following impulse response function:

$$h[n] = (1/2)^n u[n].$$

An EEE 303 student can figure out the z-transform of $h[n]$ immediately. From that, by staring at the complex z-plane for two minutes, he or she gets a good idea about the behavior of $H(e^{j\omega})$, the frequency response function of the LTI system. At frequency $\omega = \pi/3$, the phase angle of $H(e^{j\omega})$ is

$$\sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n z^{-n} = \frac{z}{z - 1/2}, \quad |z| > \frac{1}{2}$$

(a) $\pi/3$.

(b) $-\pi/3$.

(c) $-\pi/6$.

(d) $-\pi/2$.

$$\frac{\pi}{3} - \frac{\pi}{2} = -\frac{\pi}{6}$$

19. A discrete-time LTI system is characterized by the following impulse response function:

$$h[n] = (1/2)^n u[n].$$

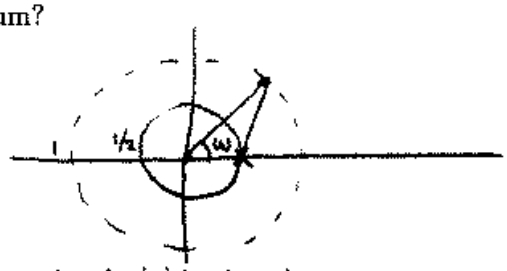
An EEE 303 student can figure out the z-transform of $h[n]$ immediately. From that, by staring at the complex z-plane for two minutes, he or she gets a good idea about the behavior of $H(e^{j\omega})$, the frequency response function of the LTI system. At what value of ω does $|H(e^{j\omega})|$, the magnitude of the frequency response function, reach minimum?

(a) $\omega = \pi$.

(b) $\omega = 3\pi/2$.

(c) $\omega = \pi/2$.

(d) $\omega = 0$.



$$|H(e^{j\omega})| \sim \frac{1}{1 + \frac{1}{4} - \cos\omega}$$

$\rightarrow$ minimum if $\omega = \pi$

20. The z-transform of a discrete time signal $x[n]$ is given by

$$X(z) = 8z^2 + 1 + 9z^{-3}.$$

The signal $x[n]$ consists of

$$X[n] = 8\,\delta[n+2] + \delta[n] + 9\,\delta[n-3]$$

(a) three impulses at $n = -2$, $n = 0$, and $n = 3$, respectively.

(b) three impulses at $n = 2$, $n = 0$, and $n = -3$, respectively.

(c) three impulses at $n = 8$, $n = 1$, and $n = -9$, respectively.

(d) three impulses at $n = -8$, $n = -1$, and $n = 9$, respectively.

II (4 points) The z-transform of a discrete-time signal $x[n]$ is given by

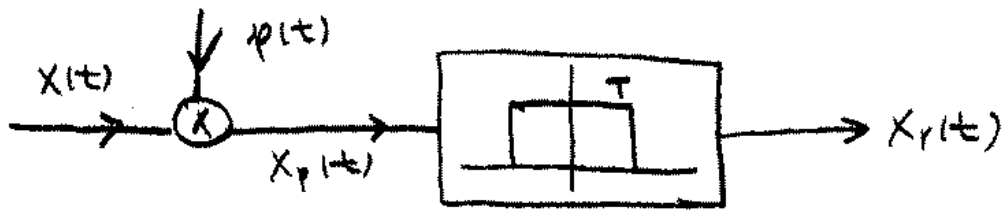
$$X(z) = \ln(1 + az^{-1}), \quad |z| > |a|.$$

Find $x[n]$.

$$X(z) = \ln(1+az^{-1}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} a^n z^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} \underbrace{\frac{(-1)^{n+1}}{n} a^n u[n-1]}_{x[n]} z^{-n}$$

III (6 points) The idea of sampling and reconstruction of a continuous-time signal $x(t)$ can be represented by the following diagram that you have seen many times:

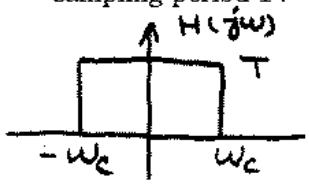


(1) (1 point) Write down the impulse response function $h(t)$ of the ideal low-pass filter in the diagram.

(2) (2 points) Obtain an explicit expression for $x_r(t) = x_p(t) * h(t)$ and then simplify it for the case of $\omega_c = \omega_s/2$, where ω_s is the sampling frequency.

(3) (3 points) For $\omega_c = \omega_s/2$, show that $x_r(kT) = x(kT)$ for any integer k, regardless of the choice of the sampling period T.

(1)



$$h(t) = T \frac{\sin \omega_c t}{\pi t}$$

(2)

$$\begin{aligned}
x_r(t) &= x_p(t) * h(t) \\
&= \left[\sum_{n=-\infty}^{+\infty} x(nT)\, \delta(t-nT)\right] * h(t) \\
&= \sum_{n=-\infty}^{+\infty} x(nT)\, h(t-nT) \\
&= \sum_{n=-\infty}^{+\infty} x(nT) \frac{T \sin[\omega_c (t-nT)]}{\pi (t-nT)} \\
\omega_c = \frac{\omega_s}{2} = \frac{\pi}{T} \quad &= \sum_{n=-\infty}^{+\infty} x(nT) \frac{T}{\pi} \frac{\sin\left[\frac{\pi}{T}(t-nT)\right]}{t-nT}
\end{aligned}$$

(3)

$$x_r(kT) = \sum_{n=-\infty}^{+\infty} x(nT) \underbrace{\frac{\sin (k-n)\pi}{(k-n)\pi}}_{\begin{cases} 0, & k \neq n \\ 1, & k = n \end{cases}} = x(kT)$$

IV (10 points) This problem concerns the *actual* design of a second-order Butterworth filter. The square magnitude of the frequency response function of such a filter is given by

$$|B(j\omega)|^2 = \frac{1}{1 + (j\omega/j\omega_c)^{2N}},$$

where N is the order of the filter (a positive integer). Assume the impulse response function $b(t)$ is real. Let $B(s)$ be the Laplace transform of the impulse response function $b(t)$ of the filter.

(1) (1 point) What does the condition that $b(t)$ is real mean for $B(j\omega)$ and $B(s)$?

(2) (1 point) Let $G(s) = B(s)B(-s)$. Obtain an explicit expression for $G(s)$.

(3) (2 point) Draw the pole-zero diagram of $G(s)$ for $N = 2$.

(4) (2 point) From the above diagram, obtain a pole-zero diagram of $B(s)$ - you need to justify your choice.

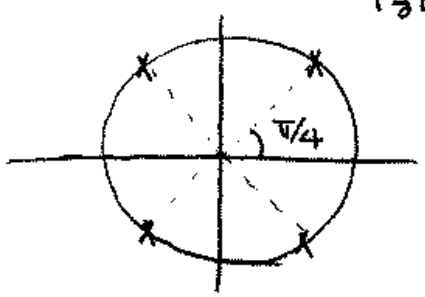
(5) (2 points) Write down an explicit expression of $B(s)$ for $N = 2$.

(6) (2 points) Obtain the ordinary differential equation that describes the second-order Butterworth filter as an LTI system.

(1) $B^*(s) = B(s^*)$, $B^*(j\omega) = B(-j\omega)$

(2) $G(s) = \dfrac{1}{1+\left(\frac{s}{j\omega_c}\right)^{2N}}$

(3)

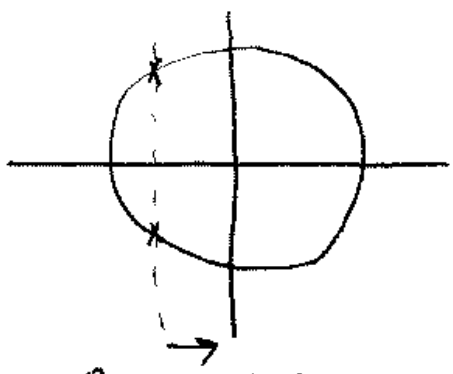


$N = 2$

Pole: $s_p = \omega_c e^{j\pi\frac{2k+1}{2N} + j\frac{\pi}{2}}$

$k = 0, \pm 1, \cdots$

(4)



ROC contains $j\omega$ axis

$\Rightarrow B(j\omega)$ exists

(5)

$$B(s) = \frac{\omega_c^2}{(s - \omega_c e^{j\frac{3\pi}{4}})(s - \omega_c e^{j\frac{5\pi}{4}})}$$

(6)

$$B(s) = \frac{\omega_c^2}{(s + \omega_c e^{-j\frac{\pi}{4}})(s + \omega_c e^{j\pi/4})}$$

$$= \frac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2}$$

$$\frac{d^2y}{dt^2} + \sqrt{2}\omega_c \frac{dy}{dt} + \omega_c^2 y(t) = \omega_c^2 x(t)$$

8