1. A continuous-time signal $x(t)$ has Nyquist rate $\omega_0$. Define a new signal: $y(t) = x(t) \cos(2\omega_0 t)$. The Nyquist rate of $y(t)$ is

(a) $2\omega_0$
(b) $3\omega_0$
(c) $4\omega_0$
(d) $5\omega_0$

2. Let $y(t) = x_1(t) x_2(t)$, where $x_1(t)$ and $x_2(t)$ are continuous-time, band-limited signals: $X_1(j\omega) = 0$ for $|\omega| > 1000\pi$ (Hz) and $X_2(j\omega) = 0$ for $|\omega| > 2000\pi$ (Hz), where $X_1(j\omega)$ and $X_2(j\omega)$ are the Fourier transforms of $x_1(t)$ and $x_2(t)$, respectively. Impulse-train sampling of $y(t)$ yields

$$y_p(t) = \sum_{n=-\infty}^{+\infty} y(nT) \delta(t - nT).$$

To ensure a complete recovery of $y(t)$ from $y_p(t)$, the maximum value for the sampling period $T$ is

(a) $10^{-3}$ (sec)
(b) $2 \times 10^{-3}$ (sec)
(c) $3 \times 10^{-3}$ (sec)
(d) $4 \times 10^{-3}$ (sec)

3. Consider impulse-train sampling of a continuous-time signal $x(t)$,

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT) \delta(t - nT),$$

where the sampling period is $T = 2.0$. Let $X(j\omega)$ and $X_p(j\omega)$ be the Fourier transforms of $x(t)$ and $x_p(t)$, respectively. If $X(j\omega)|_{\omega=0} = 1.0$, the value of $X_p(j\omega)|_{\omega=0}$ is

(a) 0.2
(b) 0.5
(c) 1.0
(d) 2.0
4. Let \( x(t) = \cos(\omega_0 t) \). Say we do impulse-train sampling at frequency \( \omega_s \) to obtain \( x_p(t) \) and then use a low-pass filter of cut-off frequency \( \omega_c = \omega_s / 2 \), attempting to recover \( x(t) \). If \( \omega_s = 3\omega_0 / 2 \) (undersampling), then in the following frequency range: \( -\omega_c < \omega < \omega_c \), \( X_p(j\omega) \), the Fourier transform of \( x_p(t) \), consists of two impulses at \( \omega_p > 0 \) and \( -\omega_p \), where \( \omega_p < \omega_c \). The value of \( \omega_p \) is

\[
\omega_c = \frac{\omega_0}{2}
\]

5. Let \( x(t) = \cos(\omega_0 t) \). Say we do impulse-train sampling at frequency \( \omega_s \) to obtain \( x_p(t) \) and then use a low-pass filter of cut-off frequency \( \omega_c = \omega_s / 2 \) in an attempt to recover \( x(t) \). If \( \omega_s = 3\omega_0 / 2 \) (undersampling), then \( x_r(t) \), the output signal of the low-pass filter, is

\[
X_r(j\omega) = \frac{1}{2} \left[ \delta(\omega - \omega_c) + \delta(\omega + \omega_c) \right]
\]

6. Given a continuous-time signal: \( x(t) = (9e^{-at} - 8e^{-5t})u(t) \). The ROC (Region of Convergence) of \( X(s) \), the Laplace transform of \( x(t) \), is

\[
\begin{align*}
(a) & \quad \Re\{s\} > -6 \\
(b) & \quad \Re\{s\} > -5 \\
(c) & \quad \Re\{s\} > -9 \\
(d) & \quad \Re\{s\} > +8
\end{align*}
\]

7. The Laplace transform of the impulse response function \( h(t) \) of a causal LTI system is given by

\[
H(s) = A \frac{(s - 1)^2}{(s + 1)(s - 2)},
\]

where \( A \neq 0 \) is a finite constant. Let \( H(j\omega) \) be the Fourier transform of \( h(t) \). Which of the following is true?

\[
\begin{align*}
(a) & \quad H(j\omega) \text{ exists if } A \text{ is small enough.} \\
(b) & \quad H(j\omega) \text{ exists regardless of the value of } A. \\
(c) & \quad H(j\omega) \text{ does not exist regardless of the value of } A. \\
(d) & \quad \text{It is impossible to construct such an LTI system in laboratory.}
\end{align*}
\]

8. Let \( x(t) = e^{-\alpha|t|} \) be a two-sided, continuous-time signal, where \( \alpha > 0 \) is a finite constant. The ROC of \( X(s) \) is

\[
\begin{align*}
(a) & \quad \Re\{s\} > 0 \\
(b) & \quad -\alpha < \Re\{s\} < 0 \\
(c) & \quad 0 < \Re\{s\} < +\alpha \\
(d) & \quad -\alpha < \Re\{s\} < +\alpha
\end{align*}
\]
9. The Laplace transform $X(s)$ of a continuous-time signal $x(t)$ has no pole and only a zero at $s = -a$, where $a > 0$ is a finite, real constant. Another continuous-time signal $x_1(t)$ satisfies the following conditions: (i) $x_1(t) \neq x(t)$ and (ii) $|X_1(j\omega)| = |X(j\omega)|$, where $X(j\omega)$ and $X_1(j\omega)$ are the Fourier transforms of $x(t)$ and $x_1(t)$, respectively. Then $X_1(s)$, the Laplace transform of $x_1(t)$, has

(a) a zero at $s = +a$.
(b) a zero at $s = -a$.
(c) a pole at $s = +a$.
(d) a pole at $s = -a$.

10. The Laplace transform $X(s)$ of a continuous-time signal $x(t)$ has no pole and only a zero at $s = -a$, where $a > 0$ is a finite, real constant. Another continuous-time signal $x_2(t)$ satisfies the following conditions: (i) $x_2(t) \neq x(t)$ and (ii) the phase angles of $X_2(j\omega)$ and $X(j\omega)$ are equal, where $X(j\omega)$ and $X_2(j\omega)$ are the Fourier transforms of $x(t)$ and $x_2(t)$, respectively. Then $X_2(s)$, the Laplace transform of $x_2(t)$, has

(a) a zero at $s = -a$.
(b) a zero at $s = +a$.
(c) a pole at $s = -a$.
(d) a pole at $s = +a$.

11. The Laplace transform $H(s)$ of the impulse response function $h(t)$ of a causal LTI system has two poles that constitute a complex-conjugate pair: one at $s_p = -3 + j2$ and another at $s_p^* = -3 - j2$. Now consider a new LTI system whose impulse response function is $e^{4t}h(t)$. Then,

(a) the new LTI system is stable.
(b) the new LTI system is unstable.
(c) the impulse response of the new LTI system does not have a Laplace transform.
(d) the impulse response of the new LTI system has a Fourier transform.

12. The Laplace transform of a continuous-time signal $x(t)$ is given by

$$X(s) = \frac{1}{(s + 1)(s + 2)}, \quad Re\{s\} > -1.$$  

The ROC of the Laplace transform of

$$y(t) = \int_{-\infty}^{\infty} x(\tau)d\tau$$

is

(a) $Re\{s\} > 0$
(b) $Re\{s\} > -1$
(c) $Re\{s\} < -1$
(d) $Re\{s\} > -2$
13. A first-order causal system is given by the following impulse response function:

\[ h(t) = \frac{1}{\tau} e^{-\tau t} u(t), \]

where \( \tau > 0 \) is a real constant. From a pole-zero plot of \( H(s) \), the Laplace transform of \( h(t) \), a competent EEE303 student can determine the phase angle of \( H(j\omega) \), the frequency response of the system, for sufficiently large value of \( \omega \): \( \omega \gg 1/\tau \). The angle is approximately

(a) \( \pi \)
(b) \( -\pi \)
(c) \( \pi/2 \)
(d) \( -\pi/2 \)

4 \( H(j\omega) = -\Theta \rightarrow -\pi \)
\[ \frac{\omega}{\tau} \gg \frac{1}{\tau} \]

14. A first-order causal system is given by the following impulse response function:

\[ h(t) = \frac{1}{\tau} e^{-\tau t} u(t), \]

where \( \tau > 0 \) is a real constant. From a pole-zero plot of \( H(s) \), the Laplace transform of \( h(t) \), a competent EEE303 student can determine the behavior of \( |H(j\omega)| \), the magnitude of the frequency response function of the system. If a dB scale (i.e., \( 20 \log_{10} |H(j\omega)| \)) is utilized, the student finds that the value of \( |H(j0)| \) is zero dB. Then, the value of \( |H(j\omega)| \) for \( \omega \gg \frac{1}{\tau} \) is approximately

(a) -10 dB.
(b) -20 dB.
(c) -30 dB.
(d) -40 dB.

15. The frequency response of an all-pass system satisfies: \( |H(j\omega)| = 1 \). The corresponding Laplace transform of the impulse response function has a pole at \( s = -a \) and a zero at \( s = a \), where \( a > 0 \) is a real constant. The phase angle of \( H(j\omega) \) for \( \omega = a \) is

(a) \( \pi/2 \).
(b) \( \pi/4 \).
(c) \( -\pi/4 \).
(d) \( -\pi/2 \).

16. Consider the following discrete-time signal \( x[n] \) of finite duration: \( x[n] = a^n \) for \( 0 \leq n \leq N-1 \) and \( x[n] = 0 \) otherwise, where \( a > 0 \) is a real constant. The ROC of the z-transform of \( x[n] \) is

(a) entire z-plane including \( z = 0 \) and \( z = +\infty \).
(b) entire z-plane including \( z = 0 \) but excluding \( z = +\infty \).
(c) \( |z| > a \)
(d) entire z-plane including \( z = +\infty \) but excluding \( z = 0 \).

\[ \sum_{n=0}^{N-1} (a z^{-1})^n = \frac{z^{N-1} - a^n}{z^{N-1} (z-a)} \]
\( z = a \) is not a pole but \( z = 0 \) is.
17. The ROC of the discrete-time signal \( x[n] = \delta[n+1] \) is
(a) entire \( z \)-plane including both \( z = 0 \) and \( z = +\infty \).
(b) entire \( z \)-plane including \( z = 0 \) but excluding \( z = +\infty \).
(c) entire \( z \)-plane excluding \( z = 0 \) but including \( z = +\infty \).
(d) entire \( z \)-plane excluding both \( z = 0 \) and \( z = +\infty \).

\[
\sum_{n=0}^{\infty} \delta[n+1] z^{-n} = \frac{z}{z - 1}
\]

18. A discrete-time LTI system is characterized by the following impulse response function:

\( h[n] = (1/2)^a u[n] \).

An EEE 303 student can figure out the \( z \)-transform of \( h[n] \) immediately. From that, by staring at the complex \( z \)-plane for two minutes, he or she gets a good idea about the behavior of \( H(e^{j\omega}) \), the frequency response function of the LTI system. At frequency \( \omega = \pi/3 \), the phase angle of \( H(e^{j\omega}) \) is

(a) \( \pi/3 \).
(b) \( -\pi/3 \).
(c) \( -\pi/6 \).
(d) \( -\pi/2 \).

\[
\frac{\pi}{3} - \frac{\pi}{6} = -\frac{\pi}{6}
\]

19. A discrete-time LTI system is characterized by the following impulse response function:

\( h[n] = (1/2)^a u[n] \).

An EEE 303 student can figure out the \( z \)-transform of \( h[n] \) immediately. From that, by staring at the complex \( z \)-plane for two minutes, he or she gets a good idea about the behavior of \( H(e^{j\omega}) \), the frequency response function of the LTI system. At what value of \( \omega \) does \( |H(e^{j\omega})| \), the magnitude of the frequency response function, reach minimum?

(a) \( \omega = \pi \).
(b) \( \omega = 3\pi/2 \).
(c) \( \omega = \pi/2 \).
(d) \( \omega = 0 \).

20. The \( z \)-transform of a discrete-time signal \( x[n] \) is given by

\[
X(z) = 8z^2 + 1 + 9z^{-3}.
\]

The signal \( x[n] \) consists of
(a) three impulses at \( n = -2, n = 0, \) and \( n = 3 \), respectively.
(b) three impulses at \( n = 2, n = 0, \) and \( n = -3 \), respectively.
(c) three impulses at \( n = 8, n = 1, \) and \( n = -9 \), respectively.
(d) three impulses at \( n = -8, n = -1, \) and \( n = 9 \), respectively.
II (4 points) The $z$-transform of a discrete-time signal $x[n]$ is given by

$$X(z) = \ln(1 + az^{-1}), \quad |z| > |a|.$$ 

Find $x[n]$. 

$$X(z) = \ln(1 + az^{-1}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} a^n z^{-n}$$ 

$$= \sum_{n=-\infty}^{\infty} \left( \frac{(-1)^{n+1}}{n} a^n u[n-1] \right) x[n]$$
III (6 points) The idea of sampling and reconstruction of a continuous-time signal $x(t)$ can be represented by the following diagram that you have seen many times:

$$
\begin{array}{c}
X(t) \\
\downarrow \Phi(t) \\
K \\
\downarrow \delta(t) \\
X_f(t) \\
\downarrow T \\
X_f(t) \\
\end{array}
$$

(1) (1 point) Write down the impulse response function $h(t)$ of the ideal low-pass filter in the diagram.

(2) (2 points) Obtain an explicit expression for $x_r(t) = x_p(t) * h(t)$ and then simplify it for the case of $\omega_c = \omega_s/2$, where $\omega_s$ is the sampling frequency.

(3) (3 points) For $\omega_c = \omega_s/2$, show that $x_r(kT) = x(kT)$ for any integer $k$, regardless of the choice of the sampling period $T$.

$$(1) \quad h(t) = T \frac{\sin \omega_c t}{\pi t}$$

$$(2) \quad X_f(t) = X_p(t) * h(t)$$

$$= \left[ \sum_{n=-\infty}^{\infty} X(nT) \delta(t-nT) \right] * h(t)$$

$$= \sum_{n=-\infty}^{\infty} X(nT) h(t-nT)$$

$$= \sum_{n=-\infty}^{\infty} X(nT) \frac{T \sin \left[ \frac{\pi \omega_c (t-nT)}{t} \right]}{\pi (t-nT)}$$

$$= \sum_{n=-\infty}^{\infty} X(nT) \frac{T \sin \left[ \frac{\pi \omega_c (t-nT)}{t} \right]}{\pi (t-nT)}$$

$$= X_r(kT) = \sum_{n=-\infty}^{\infty} X(nT) \frac{\sin \left( \frac{\pi \omega_c (k-nT)}{t} \right)}{(k-n) \pi} = x(kT)$$

$0, \quad k \neq n$

$1, \quad k = n$
IV (10 points) This problem concerns the actual design of a second-order Butterworth filter. The square magnitude of the frequency response function of such a filter is given by

\[ |B(j\omega)|^2 = \frac{1}{1 + (j\omega/j\omega_c)^{2N}}, \]

where \( N \) is the order of the filter (a positive integer). Assume the impulse response function \( b(t) \) is real. Let \( B(s) \) be the Laplace transform of the impulse response function \( b(t) \) of the filter.

1. (1 point) What does the condition that \( b(t) \) is real mean for \( B(j\omega) \) and \( B(s) \)?

2. (1 point) Let \( G(s) = B(s)B(-s) \). Obtain an explicit expression for \( G(s) \).

3. (2 points) Draw the pole-zero diagram of \( G(s) \) for \( N = 2 \).

4. (2 points) From the above diagram, obtain a pole-zero diagram of \( B(s) \) - you need to justify your choice.

5. (2 points) Write down an explicit expression of \( B(s) \) for \( N = 2 \).

6. (2 points) Obtain the ordinary differential equation that describes the second-order Butterworth filter as an LTI system.

1. \[ B^2(s) = B(S)B(S^*) = B(-j\omega) \]

2. \[ G(s) = \frac{1}{1 + (\frac{s}{j\omega_c})^{2N}} \]

3. Pole: \( s_p = \omega_c e^{j\frac{2\pi k}{2N} + j\frac{\pi}{4}} \)
   \[ k = 0, \pm 1, \ldots \]

4. ROC contains \( j\omega \) axis, \( \Rightarrow B(j\omega) \) exists.

5. \[ B(s) = \frac{\omega_c^2}{(s - \omega_c e^{j\frac{2\pi}{4}})(s - \omega_c e^{j\frac{4\pi}{4}})} \]

6. \[ B(s) = \frac{\omega_c^2}{(s + \omega_c e^{j\frac{2\pi}{4}})(s + \omega_c e^{j\frac{4\pi}{4}})} = \frac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2} \]

\[ \frac{d^2y}{dt^2} + \sqrt{2}\omega_c \frac{dy}{dt} + \omega_c^2 y(t) = \omega_c^2 x(t) \]